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SOUSLIN'S CONJECTURE AS A PROBLEM ON THE REAL LINE

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This paper is concerned with properties of real sets whose existence is related to Souslin's conjecture. One of these results is subsequently used to show that Souslin's conjecture is second order determined, i.e., $(\mathscr{X} \vdash_2 SC) \vee (\mathscr{X} \vdash_2 \sim SC)$.

By *Souslin's conjecture* (SC) we mean: every linearly ordered set with at most countably many pairwise disjoint intervals is separable. (A linearly ordered set L is *separable* if it has a countable subset such that between any two points of L there is a point of the subset). We first display a subset of the power set of the real line R whose existence is equivalent to $\sim SC$. Then we reformulate the conjecture geometrically as a question concerning a single subset of R of a certain type. Finally we point out that Souslin's conjecture is second order determined.

E. Miller [4] proved that $\sim SC$ is equivalent to the existence of a *Souslin tree,* i.e., an uncountable tree of countable height and countable width. A *tree* is a partially ordered set in which the set of all elements below any given element is a chain. The *height* of a partially ordered set P is the least cardinal $\mathfrak{m}$ such that no chain in P has cardinality greater than $\mathfrak{m}$. A is an *antichain* if no two elements of A are related. The *width* of P is the least cardinal $\mathfrak{n}$ such that no antichain in P has cardinality greater than $\mathfrak{n}$.

PROPOSITION 1.1. *The existence of a Souslin tree is equivalent to the existence of an uncountable collection of real sets such that*

1. *any two sets in the collection are either disjoint or one of them is a subset of the other, and*

2. *if $\mathscr{G}$ is any uncountable subcollection, then $\mathscr{G}$ has two disjoint members and two nondisjoint members.*

Proof. Assume there is a Souslin tree S. Let f be a one-to-one function from some uncountable subset of S into R. For each $x \in S$, let $U(x) = \{y: x \leqq y\}$, and let $\mathscr{F} = \{f(U(x)): x \in S\}$. Then $\mathscr{F}$ has the desired properties.

Conversely, if there is such a collection $\mathscr{F}$, let $A \leqq B$ mean $B \subseteqq A$, for $A, B \in \mathscr{F}$. Then $\mathscr{F}$ is a Souslin tree.

An application of Proposition 1.1 is found in §5. In the next section we show how a Souslin tree can be represented as a single subset of the line.

2. We first represent certain binary relations. For this purpose let $G \subset R$ and denote by G^* the set of all those points $x \in G$ which are midpoints of a nondegenerate segment whose endpoints are both in G. We shall call G^* the *set of midpoints in* G. Define a relation α on G^* by setting

$$\text{(2.1)} \qquad \begin{array}{l} x\alpha y \text{ iff } x \neq y \text{ and there exists } z \in G \text{ such that} \\ y \text{ is the midpoint of the segment } xz. \end{array}$$

Note that xz stands for $[x, z]$ or $[z, x]$ according as $x < z$ or $z < x$.

PROPOSITION 2.1. *α is a (strict) partial order for G^* iff for all elements $x, y, z \in G^*$ we have*

A. *(asymmetry) if x and y are the respective midpoints of yv and xu, and if $u \in G$, then $v \notin G$.*

B. *(transitivity) if y is the midpoint of xu and z is the midpoint of both yv and xw, and if $u, v \in G$, then also $w \in G$.*

The proof is immediate since no point is both midpoint and endpoint of the same nondegenerate segment.

THEOREM 2.2. *Let δ be any antireflexive relation on a set P of cardinality no larger than that of the continuum. Then there exists a subset G of the real line for which the relation α defined by* (2.1) *is isomorphic to δ.*

Proof. Let f be a one-to-one function mapping P into a Hamel basis for R. Let

$$\text{(2.2)} \qquad U = \{2f(q) - f(p) \colon p, q \in P, p\delta q\}$$

and

$$\text{(2.3)} \qquad G = U \cup 2f[P] \cup f[P] \cup \{0\} .$$

For each $p \in P$, $f(p)$ is the midpoint of the segment $2f(p)0$, whose endpoints belong to G. Thus $f[P] \subset G^*$. If $y \in G^*$, on the other hand, then there exist distinct points $x, z \in G$, such that $2y = z + x$. Also, $y \in G$. Writing $x = c_1a_1 + c_2a_2$ and $z = c_3a_3 + c_4a_4$, with $a_i \in f[P]$, we have

$$\text{(2.4)} \qquad \begin{array}{l} c_1 = 2 \text{ and } c_2 = -1 \text{ if } x \in U , \\ c_3 = 2 \text{ and } c_4 = -1 \text{ if } z \in U , \\ c_{2k-1} \in \{0, 1, 2\} \text{ and } c_{2k} = 0 \text{ otherwise} . \end{array}$$

Assuming now that $y \in U$, $y = 2a - b$, we have $4a - 2b = 2y = \sum c_ia_i$, and since $a \neq b$, (2.4) implies that only $c_1 = 2 = c_3$, $a_1 = a = a_3$

is possible. But this leads to $c_2 = -1 = c_4$, $a_2 = b = a_4$, and hence to $z = x$, which contradicts our assumption.

The cases $y \in 2f[P]$ and $y = 0$ similarly lead to the conclusion $z = x$. Thus by (2.3) we have $y \in f[P]$, and hence $G^* f[P]$.

To see that f is an isomorphism, let $p, q \in P$, $p\delta q$. Then $x = 2f(q) - f(p)$ is a member of $U \subset G$, and $f(q)$ is the midpoint of the nondegenerate segment $xf(p)$. Thus $f(p)\alpha f(q)$. Conversely, if $z\alpha y$ in $G^* = f[P]$, say $z = f(p), y = f(q)$, then $x = 2f(q) - f(p) \in G$ by (2.1). We use (2.3) and the independence of $f[P]$ to show that $x \in U$, and again the independence of $f[P]$ to see that $p\delta q$.

Comment 2.3. An obvious generalization of Theorem 2.2 permits us to represent an arbitrary antireflexive relation in a vector space of sufficiently large dimension over a field of characteristic 5 or larger. Here again "y is the midpoint of xz" means $2y = z + x, x \neq z$. For characteristic smaller than 5 we might mention that $f[P] \neq G^*$.

COROLLARY 2.4. *Let P be any partially ordered set of cardinal number no larger than that of the continuum. Then there exists a subset G of the real line such that P is isomorphic to the partially ordered set G^* of midpoints in G.*

This follows directly from Proposition 2.1 and Theorem 2.2.

We are now ready to apply Theorem 2.2 to trees. In a slight restatement of 2.1, A becomes: no segment with endpoints in G is trisected by points of G; B can be summarized by the phrase: G is *midpoint transitive*. Henceforth we assume that G has these two properties.

Chains in G^* are *generating* subsets of G^* in the sense that any two distinct points x, y of a chain generate a segment with endpoints in G, one of x and y acting as an endpoint of the segment, the other as the midpoint; i.e. if $u = 2y - x$, $v = 2x - y$, then $u \in G$ or $v \in G$. We call a subset X of G^* *segment free* (antichain) if every subset of X of cardinality $\geqq 2$ fails to be generating. X is *free* (from above) in G provided that for any two distict points $x, y \in X$ and any $u, v, z \in G$, z is not the midpoint of both the segments xu and yv.

Combining these notions with 2.1 we obtain our main result. Width bounds the cardinality of segment free sets and height that of generating sets in G^*.

THEOREM 2.5. *The existence of a Souslin tree is equivalent to the existence of a subset G of the real line whose set G^* of midpoints in G is uncountable and satisfies*

1. *no segment with endpoints in G is trisected by points of G,*
2. *G is midpoint transitive,*
3. *segment free subsets of G^* are free in G,*
4. *segment free subsets of G^* are countable,*
5. *generating subsets of G^* are countable.*

Proof. 1. and 2. imply that α is a partial order, by 2.1. 3. is the tree property, and 4. and 5. together with the fact that G^* is uncountable make the tree G^* into a Souslin tree. Thus the existence of G implies the existence of a Souslin tree, and if a Souslin tree exists, then G exists by 2.4.

4. In this section we conclude by applying a real line characterization of Souslin's conjecture to obtain a foundations result. In [2] and [3] the continuum hypothesis is shown to be second order determined, i.e.,

$$(\mathscr{Z} \vdash_2 CH) \vee (\mathscr{Z} \vdash_2 \sim CH)$$

where $\mathscr{Z}$ denotes Zermelo's axioms with the axiom of infinity and CH the continuum hypothesis. The reader is referred to Kreisel and Krivine [3] for a detailed discussion.

A modification of the proof in Kreisel and Krivine applies to Souslin's conjecture:

PROPOSITION 4.1. *Souslin's conjecture is second order determined, i.e.,*

$$(\mathscr{Z} \vdash_2 SC) \vee (\mathscr{Z} \vdash_2 \sim SC) .$$

Proof. Let C_ω be the collection of all hereditarily finite sets without individuals, and for $n \in \omega$, let $C_{\omega+n+1} = C_{\omega+n} \cup \mathscr{P}(C_{\omega+n})$, where $\mathscr{P}$ denotes the power set. From Proposition 1.1, Souslin's conjecture states that any collection of real sets which under set inclusion forms a tree of countable height and countable width is countable. We may thus canonically formulate Souslin's conjecture as follows:

$$\begin{aligned}&[X \subset \mathscr{P}(C_{\omega+1}) \wedge (x \in X \wedge y \in X \rightarrow x \cap y = \phi \vee x \subset y \vee y \subset x)\\ &\wedge ((Y \subset X \wedge ((x \in Y \wedge y \in Y \rightarrow x \cap y = \phi)\\ &\vee (x \in Y \wedge y \in Y \rightarrow x \subset y \vee y \subset x))) \rightarrow \bar{\bar{Y}} \leqq \bar{\bar{C}}_\omega)] \rightarrow \bar{\bar{X}} \leqq \bar{\bar{C}}_\omega .\end{aligned}$$

This is expressed by means of quantifiers over $C_{\omega+3}$, since one-to-one correspondences between subsets of $C_{\omega+2}$ are elements of $C_{\omega+3}$. Consequently [3; p. 192] we have $(\mathscr{Z} \vdash_2 SC) \vee (\mathscr{Z} \vdash_2 \sim SC)$.

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