PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 3, 1972

C_λ-GROUPS AND λ-BASIC SUBGROUPS

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The groups considered in this paper will be abelian primary groups. For λ a fixed but arbitrary countable limit ordinal, C. K. Megibben studied that class C_λ consisting of all p-groups G such that $G/p^\alpha G$ is a direct sum of countable groups for all $\alpha < \lambda$.

Fundamental to the development of C_λ-theory was the introduction of the concept of a λ-basic subgroup, which generalized the familiar concept of a basic subgroup, and the following existence theorem: A primary group G contains a λ-basic subgroup if and only if G is a C_λ-group. This paper extends, in a natural fashion, the concepts of "C_λ-group" and "λ-basic subgroup" to an arbitrary limit ordinal λ, and considers the analogous question of existence. This is used to examine the structure of p^λ-pure subgroups of C_λ-groups for limit ordinals λ such that $\lambda \neq \beta + \omega$ for any ordinal β. For an ordinal λ of this type, if H is a p^λ-pure subgroup of the C_λ-group G then both H and G/H are C_λ-groups.

The classical theory of torsion abelian groups corresponds to Megibben's C_ω-theory, in that the class of all p-groups coincides with C_ω.

1. Preliminaries. In this section we assemble the basic concepts which are crucial in the following development. For pertinent results related to these concepts, we refer the reader to [2].

A subgroup H of the p-group G is said to be a *p^α-pure subgroup* if $H \rightarrowtail G \twoheadrightarrow G/H$ represents an element of $p^\alpha \mathrm{Ext}(G/H, H)$. This notion is due to Nunke and shall assume the same role in our theory as that played by ordinary purity (i.e. p^ω-purity for p-groups) in the classical theory.

The subgroup H is said to be a *p^α-high subgroup* of G if H is maximal among the subgroups of G that intersect $p^\alpha G$ trivially. From [5] or [7], if H is a p^α-pure subgroup of G then $H \cap p^\beta G = p^\beta H$ for all $\beta \leqq \alpha$ and $p^\beta(G/H)[p] = (p^\beta G)[p] + H/H$ for all $\beta < \alpha$. Moreover, if G/H is divisible, where H is a p^α-pure subgroup of G and α is a limit ordinal, then $H + p^\beta G/p^\beta G = G/p^\beta G$ for all $\beta < \alpha$. If H is a p^α-high subgroup of G, then H is a $p^{\alpha+1}$-pure subgroup of G and $H + p^\alpha G/p^\alpha G$ is p^α-pure in $G/p^\alpha G$ (see [3]).

A subgroup H of the p-group G is *neat* if $pG \cap H = pH$. From [3], if H is a neat subgroup of the p-group G and

$$G[p] = H[p] + p^{\beta}G[p]$$

for each $\beta < \alpha$, then H is p^{α}-pure in G. Moreover, if A is a neat subgroup of $p^{\alpha}G$ and if $B \supseteqq A$ is maximal in G with respect to $B \cap p^{\alpha}G = A$, then B is $p^{\alpha+1}$-pure in G.

A subgroup H of the p-group G is *nice* in G if each coset $g+H$ contains an element $g+h$ that has maximal height in G. If g has maximal height in the coset $g+H$, we say g is *proper* with respect to H.

Totally projective groups as introduced by Nunke provide a generalization of the concept of a direct sum of countable reduced groups. A p-group G is *p^{α}-projective* if $p^{\alpha}\mathrm{Ext}\,(G, C) = 0$ for all groups C. A reduced p-group G is *totally projective* if $Gp^{\alpha}G$ is p^{α}-projective for every ordinal α. The following characterization of totally projective groups, given and utilized by Hill [4] to show that the Ulm invariants suffice to classify totally projective p-groups, is used extensively.

THEOREM A. *A reduced p-group G is totally projective if and only if G has a collection $\mathscr{C}_G$ of nice subgroups satisfying the following conditions*:

(0) *0 is a member of $\mathscr{C}_G$.*

(1) *$\mathscr{C}_G$ is closed with respect to group-theoretic union.*

(2) *If $A \in \mathscr{C}_G$ and H is a subgroup of G such that $(H+A)/A$ is countable, there exists $B \in \mathscr{C}_G$ such that $B \supseteqq H+A$ and B/A is countable.*

In the sequel, we shall refer to these conditions as the third axiom of countability and to condition (2) *as the countable extension property.*

An ordinal λ is said to be *confinal with ω* if λ is the limit of a countable ascending sequence of ordinals. From [7], if α is confinal with ω then every p^{α}-pure subgroup of a p^{α}-projective group is p^{α}-projective.

To extend the concepts of C_{λ}-group and λ-basic subgroup to an arbitrary limit ordinal, we introduce the following definitions. For a fixed but arbitrary limit ordinal λ, C_{λ} shall designate that class of all p-groups G such that $G/p^{\alpha}G$ is totally projective for all $\alpha < \lambda$. Groups in the class C_{λ} will be referred to as C_{λ}-groups. B is said to be a *λ-basic subgroup* of G if

(1) B is totally projective of length at most λ,

(2) B is a p^λ-pure subgroup of G, and

(3) G/B is divisible.

For B a λ-basic subgroup of G and $\alpha < \lambda$, a routine argument yields that the α-th Ulm invariant of B coincides with the α-th Ulm invariant of G. Hence, by Hill's version of Ulm's Theorem, we obtain the following analogue to a well-known property of ordinary basic subgroups

PROPOSITION 1.1. *If B and $\bar{B}$ are λ-basic subgroups of G then $B \cong \bar{B}$.*

We shall require for technical convenience the notion of a λ-high confinal tower. Let λ be an ordinal confinal with ω, and G an abelian p-group. A *λ-high confinal tower* of G is an ascending sequence $\{G_n\}$ of subgroups of G such that:

(1) For each positive integer n, G_n is a $p^{\alpha(n)}$-high subgroup of G;

(2) $\lambda = \sup \{\alpha(n)\}$, $\alpha(n) < \alpha(n+1)$;

(3) If $\lambda = \beta + \omega$ for some limit ordinal β, then $\alpha(n) = \beta + m$ for some positive integer m;

(4) If $\lambda \neq \beta + \omega$ for any ordinal β, then $\alpha(n) = \beta(n) + \omega$ for some limit ordinal $\beta(n)$.

2. The existence theorem.

In this section we determine, for an arbitrary but fixed limit ordinal λ, that class of all abelian p-groups G such that G contains a λ-basic subgroup (see Theorem 2.7).

LEMMA 2.1. *Suppose $G/p^\beta G$ is totally projective and B is a basic subgroup of $p^\beta G$. If H is a subgroup of G such that*

$$G/B = H/B \oplus p^\beta G/B$$

then H is totally projective.

Proof. If H is a subgroup of G such that $G/B = H/B \oplus p^\beta G/B$, then $G = H + p^\beta G$ and H is maximal in G with respect to $H \cap p^\beta G = B$. Thus H is $p^{\beta+1}$-pure in G. Consequently $p^\alpha H = p^\alpha G \cap H$ for all $\alpha \leqq \beta + 1$, and in particular $p^\beta H = p^\beta G \cap H = B$. We now observe that $H/p^\beta H$ is totally projective since

$$H/p^\beta H = H/p^\beta G \cap H \cong (H + p^\beta G)/p^\beta G = G/p^\beta G\ ,$$

and $p^\beta H = B$ is a direct sum of cyclic groups.

LEMMA 2.2. *Let λ be a limit ordinal confinal with ω such that $\lambda \neq \beta + \omega$ for any ordinal β. Suppose $G = \bigcup G_n$ with $\{G_n\}$ a λ-high confinal tower of G. If $A \subseteqq G$ satisfies the conditions:*

(1) A is the union of an ascending sequence of subgroups $A_1 \subseteqq A_2 \subseteqq \cdots$ such that A_n is nice in G_n for each n,

(2) $A \subseteqq p^\alpha G + A_n$ for all $\alpha < \alpha(n)$;

then A is nice in G.

Proof. We show that each coset $x + A$ contains an element $x + a$ that is proper with respect to A.

Let $x \in G - A$, and choose n such that $x \in G_n$. Let

$$\beta = h_G(x) < \alpha(n) .$$

For $k \geqq n$, there exists $a_k \in A_k$ such that $h_G(x + a_k) = h_{G_k}(x + a_k) \geqq h_{G_k}(x + a') = h_G(x + a')$ for any $a' \in A_k$. It suffices to show that the sequence $h_G(x + a_n) \leqq h_G(x + a_{n+1}) \leqq \cdots$ cannot be strictly increasing.

Suppose for some $m \geqq n$ that $h_G(x + a_m) > \beta = h_G(x)$. Then $h_G(a_{m+i}) = h_G(x)$ for $i = 1, 2, \cdots$. Let $\gamma = h_G(x + a_m)$ and observe $\gamma < \alpha(m)$ since $x + a_m \in G_m$. Moreover $\gamma + 1 < \alpha(m)$ since $\alpha(m)$ is a limit ordinal. We shall show that $x + a_m$ is proper with respect to A. Suppose $x + a_m$ is not proper with respect to A. Then for some k, $h_G(x + a_{m+k}) > h_G(x + a_m) = \gamma$ and $x + a_{m+k} \in p^{\gamma+1}G$. Since $A \subseteqq p^{\gamma+1}G + A_m$ we have $a_{m+k} = g_k + a_{m,k}$ with $g_k \in p^{\gamma+1}G$ and $a_{m,k} \in A_m$. Hence

$$x + a_{m,k} = x + g_k + a_{m,k} \in p^{\gamma+1}G$$

and $x + a_{m,k} \in p^{\gamma+1}G$. This however is absurd since

$$h_G(x + a_{m,k}) \leqq h_G(x + a_m) = \gamma .$$

Consequently $x + a_m$ is proper with respect to A and A is nice in G.

With λ and G as in Lemma 2.2, we shall now restrict our attention to the case where G_n is totally projective for each n. Let $\mathscr{C}_n$ denote a collection of nice subgroups of G_n satisfying the third axiom of countability. Let $\mathscr{C}$ be the collection of all subgroups A of G such that

(1) $A = \bigcup A_n$ with $A_1 \subseteqq A_2 \subseteqq \cdots$ and $A_n \in \mathscr{C}_n$ for each n,

(2) $A \subseteqq p^\alpha G + A_n$ for all $\alpha < \alpha(n)$.

The members of $\mathscr{C}$ are nice by Lemma 2.2.

LEMMA 2.3. *$\mathscr{C}$ has the countable extension property.*

Proof. For each n, we have $\alpha(n) = \beta(n) + \omega$ with $\beta(n)$ a limit ordinal. Thus $\lambda = \sup\{\alpha(n)\} = \sup\{\beta(n)\}$. We observe that to show

$B \subseteqq p^{\alpha}G + B_n$ for each ordinal $\alpha < \alpha(n)$, it suffices to show

$$B \subseteqq p^{\beta(n)+k}G + B_n \qquad \text{for each } k < \omega .$$

Let $A \in \mathscr{C}$ and H a subgroup of G such that H/A is countable. Let $S = \{x_i\}_{i<\omega}$ be such that $H = \langle A, S\rangle$ and let $S_n = S \cap G_n$. By induction, we shall construct, for each positive integer n, subgroups $B_1^{(n)} \subseteqq B_2^{(n)} \subseteqq \cdots \subseteqq B_n^{(n)}$ such that

(0) $A_i \subseteqq B_i^{(n)}$ for $i \leqq n$,
(1) $B_j^{(k)} \subseteqq B_i^{(n)}$ for $k \leqq n$,
(2) $B_i^{(n)} \in \mathscr{C}_i$,
(3) $B_{i+1}^{(n)} \subseteqq p^{\beta(i)+k}G + B_i^{(n)}$ for $k < \omega$,
(4) $B_i^{(n)} \supseteqq S_i$,
(5) $B_i^{(n)}/A_i$ is countable for $i \leqq n$.

We now show that the existence of subgroups $B_i^{(n)}$ satisfying the above conditions (0) – (5) will suffice to establish the lemma. For each $i < \omega$, let $B_i = \bigcup_{n \geqq i} B_i^{(n)}$ and observe that $B_i \in \mathscr{C}_i$ and $B_i \subseteqq B_{i+1}$. Moreover $B_{i+1} = \bigcup_{n \geqq i+1} B_{i+1}^{(n)} \subseteqq \bigcup_{n \geqq i+1}(p^{\beta(i)+k}G + B_i^{(n)}) = p^{\beta(i)+k}G + B_i$ for $k < \omega$, and by induction $B_{i+m} \subseteqq p^{\beta(i)+k}G + B_i$ for all $m < \omega$, $k < \omega$. Let $B = \bigcup_{i<\omega} B_i$. Clearly $B \supseteqq H$ and B/A is countable. Moreover $B \in \mathscr{C}$ since $B \subseteqq p^{\beta(i)+k}G + B_i$ for each i and k.

Suppose we have constructed $B_i^{(n)}$, $1 \leqq i \leqq n$, satisfying (0) – (5) above. We shall now construct $B_i^{(n+1)}$ for $1 \leqq i \leqq n + 1$.

For $1 \leqq i \leqq n$, let $B_{i,0} = A_i$ and $B_{i,1} = B_i^{(n)}$. Set $B_{n+1,0} = A_{n+1}$ and let $B_{n+1,1}$ be a member of $\mathscr{C}_{n+1}$ such that

$$B_{n+1,1} \supseteqq \langle B_n^{(n)} + A_{n+1}, S_{n+1}\rangle$$

and $B_{n+1,1}/A_{n+1}$ is countable. By induction, we shall construct a family of subgroups $B_{i,j}$, with $1 \leqq i \leqq n + 1$ and $j < \omega$, satisfying the conditions

(i) $B_{i,j} \subseteqq B_{i,k}$ for $j \leqq k$,
(ii) $B_{i,j} \in \mathscr{C}_i$,
(iii) $B_{i,j+1}/B_{i,j}$ is countable,
(iv) $B_{i+1,2j} \subseteqq p^{\beta(i)+k}G + B_{i,2j}$ for all $1 \leqq i \leqq n$ and j, $k < \omega$;
(v) $B_{i,2j+1} \subseteqq B_{i+1,2j+1}$ for all $1 \leqq i \leqq n$ and $j < \omega$.

We define $B_i^{(n+1)} = \bigcup_{j<\omega} B_{i,j}$ and observe that

$$\bigcup_{j<\omega} B_{i,2j} = B_i^{(n+1)} = \bigcup_{j<\omega} B_{i,2j+1} .$$

By (iv), we see that

$$B_{i+1}^{(n+1)} = \bigcup_{j<\omega} B_{i+1,2j} \subseteqq \bigcup_{j<\omega} (p^{\beta(i)+k}G + B_{i,2j}) = p^{\beta(i)+k}G + B_i^{(n+1)}$$

for all $k < \omega$, $1 \leqq i \leqq n$. By (v), $B_i^{(n+1)} = \bigcup_{j<\omega} B_{i,2j+1} \subseteqq \bigcup_{j<\omega} B_{i+1,2j+1} = B_{i+1}^{(n+1)}$ for all $1 \leqq i \leqq n$. It is now easy to see that conditions

$(0)-(4)$ are satisfied by the subgroups $B_i^{(j)}$, $1 \leqq i \leqq j \leqq n+1$. Since $B_{i,j+1}/B_{i,j}$ is countable for each

$$j < \omega,\ B_i^{(n+1)}/A_i = B_i^{(n+1)}/B_{i,0} = \bigcup_{j<\omega} B_{i,j+1}/B_{i,0}$$

is countable for all $1 \leqq i \leqq n+1$ and condition (5) is satisfied.

Suppose we have constructed $B_{i,j}$ satisfying (i) $-$ (v), for all $1 \leqq i \leqq n+1$ and all $j \leqq 2m+1$. We shall now construct $B_{i,2m+2}$ for $1 \leqq i \leqq n+1$. Define $B_{n+1,2m+2} = B_{n+1,2m+1}$. Assuming, for some positive integer $l \leqq n$, that $B_{i,2m+2}$ has been constructed for each $l+1 \leqq i \leqq n+1$, we let $\{x_j\}_{j<\omega} \subseteqq B_{l+1,2m+2}$ be such that $B_{l+1,2m+2} = \langle B_{l+1,2m}, \{x_j\}_{j<\omega}\rangle$. Since $G \subseteqq p^{\beta(l)+k}G + G_l$ we obtain decompositions $x_j = g_{j,k} + x_{j,k}$, with $g_{j,k} \in p^{\beta(l)+k}G$ and $x_{j,k} \in G_l$, for each $j, k < \omega$. Let $T_{l,2m+2} = \{x_{j,k}\}_{j,k<\omega} \subseteqq G_l$. Let $B_{l,2m+2}$ be a member of $\mathscr{C}_l$ such that $B_{l,2m+2} \supseteqq \langle B_{l,2m+1}, T_{l,2m+2}\rangle$ and $B_{l,2m+2}/B_{l,2m+1}$ is countable. Observe, for each $k < \omega$, $B_{l+1,2m+2} \subseteqq p^{\beta(l)+k}G + B_{l,2m+2}$ since

$$B_{l+1,2m} \subseteqq p^{\beta(l)+k}G + B_{l,2m} \subseteqq p^{\beta(l)+k}G + B_{l,2m+1} \subseteqq p^{\beta(l)+k}G + B_{l,2m+2}$$

and

$$\{x_j\}_{j<\omega} \subseteqq p^{\beta(l)+k}G + \langle T_{n,2m+2}\rangle \subseteqq p^{\beta(l)+k}G + B_{l,2m+2}\,.$$

To conclude the proof, it suffices to construct $B_{i,2m+3}$ for $1 \leqq i \leqq n+1$, having been given a collection $B_{i,j}$ satisfying (i) $-$ (v), for all $1 \leqq i \leqq n+1$ and all $j \leqq 2m+2$. Define $B_{1,2m+3} = B_{1,2m+2}$ and assume, for some positive integer $l \leqq n$, that $B_{i,2m+3}$ has been constructed for each $1 \leqq i \leqq l$. Since $B_{l,2m+3}/B_{l,2m+1}$ is countable and $B_{l,2m+1} \subseteqq B_{l+1,2m+1} \subseteqq B_{l+1,2m2}$, $(B_{l,2m+3} + B_{l+1,2m+2})/B_{l+1,2m+2}$ is countable. Thus there exists $B_{l+1,2m+3} \in \mathscr{C}_{l+1}$ such that $B_{l+1,2m+3} \supseteqq B_{l+1,2m+3} + B_{l+1,2m+2}$ and $B_{l+1,2m+3}/B_{l+1,2m+2}$ is countable. The collection of subgroups $B_{i,j}$, for $1 \leqq i \leqq n+1$ and $0 \leqq j \leqq 2m+3$, clearly satisfies conditions (i) $-$ (v).

LEMMA 2.4. *If α is confinal with ω and $G/p^\alpha G$ is totally projective, then every p^α-high subgroup of G is totally projective.*

Proof. Let α be an ordinal confinal with ω, and H a p^α-high subgroup of G. Since $H \cong (H + p^\alpha G)/p^\alpha G$ and $(H + p^\alpha G)/p^\alpha G$ is p^α-pure in the p^α-projective group $G/p^\alpha G$, H is p^α-projective. Since α is a limit ordinal, $H/p^\beta H \cong G/p^\beta G$ is p^β-projective for all $\beta < \alpha$. Consequently H is totally projective.

PROPOSITION 2.5. *Let λ be a limit ordinal confinal with ω, and*

$\{G_n\}$ a λ-high confinal tower of G. If G is a C_λ-group then $\bigcup G_n$ is totally projective of length at most λ.

Proof. Clearly $\bigcup G_n$ is an isotype subgroup of G and hence has length at most λ. The proof that $\bigcup G_n$ is totally projective shall consist of two cases.

Case 1. $\lambda = \beta + \omega$.

Consider the subgroup $(\bigcup G_n) \cap p^\beta G$ of $p^\beta C$, and observe that

$$G_n \cap p^{\beta+\omega} G = 0$$

for each n. Consequently $p^\omega((\bigcup G_n) \cap p^\beta G) = \bigcup (G_n \cap p^{\beta+\omega} G) = 0$ and thus $(\bigcup G_n) \cap p^\beta G$ is a p-group without elements of infinite height. Since

$$(\bigcup G_n) \cap p^\beta G = \bigcup (G_n \cap p^\beta G) = \bigcup p^\beta G_n$$

is the union of an ascending sequence of bounded subgroups, it follows, by the Kulikov criterion, that $(\bigcup G_n) \cap p^\beta G$ is a direct sum of cyclic groups. It is easy to see that $(\bigcup G_n) \cap p^\beta G$ is a pure subgroup of $p^\beta G$ and that $\bigcup G_n + p^\beta G = G$. Consequently $(\bigcup G_n) \cap p^\beta G$ is a basic subgroup of $p^\beta G$. Since G is a C_λ-group, $G/p^\beta G$ is totally projective and, by Lemma 2.1, it follows that $\bigcup G_n$ is totally projective.

Case 2. $\lambda \neq \beta + \omega$ for any ordinal β.

By Lemma 2.4, it follows that in this case G_n is totally projective for each n. To show that $\bigcup G_n$ contains a collection of nice subgroups satisfying the third axiom of countability, let $\mathscr{C}$ denote the collection of nice subgroups of $\bigcup G_n$ as defined preceding Lemma 2.3. Clearly $0 \in \mathscr{C}$. By Lemma 2.3, $\mathscr{C}$ has the countable extension property. Thus it suffices to show that $\mathscr{C}$ is closed with respect to group-theoretic union. Suppose $\{A_\gamma\}_{\gamma \in I} \subseteqq \mathscr{C}$ with $A_\gamma = \bigcup_{n<\omega} A_{n,\gamma}$ where

(1) $A_{n,\gamma} \subseteqq A_{k,\gamma}$ for $n \leqq k$,

(2) $A_{n,\gamma} \in \mathscr{C}_n$ for each n.

(3) For each n and $\alpha < \alpha(n)$, $A_\gamma \subseteqq p^\alpha G + A_{\gamma,n}$.

Then $\sum_{\gamma \in I} A_\gamma = \sum_{\gamma \in I}(\bigcup_{n<\omega} A_{n,\gamma}) = \bigcup_n(\sum_{\gamma \in I} A_{n,\gamma})$ with

$$\sum_{\gamma \in I} A_{n,\gamma} \subseteqq \sum_{\gamma \in I} A_{k,\gamma} \qquad \text{for } n \leqq k\,,$$

and $\sum_{\gamma \in I} A_{n,\gamma} \in \mathscr{C}_n$. Moreover, for each n and $\alpha < \alpha(n)$, we have

$$\sum_{\gamma \in I} A_\gamma \subseteqq \sum_{\gamma \in I} (p^\alpha G + A_{n,\gamma}) = p^\alpha G + (\sum_{\gamma \in I} A_{n,\gamma})\,.$$

Consequently $\sum_{\gamma \in I} A_\gamma \in \mathscr{C}$.

LEMMA 2.6. *Let $\{G_n\}$ be a λ-high confinal tower of G. If $H = \bigcup G_n$ then H is p^λ-pure in G.*

Proof. Let $\alpha < \lambda$, and recall that H is an isotype, and hence a neat, subgroup of G. There exists a positive integer n such that $\alpha < \alpha(n)$, and $G[p] = G_n[p] + (p^\alpha G)[p] = H[p] + (p^\alpha G)[p]$.

THEOREM 2.7. (a) *If G is a C_λ-group with λ confinal with ω then G contains a λ-basic subgroup.*

(b) *If G is a reduced p-group which contains a proper λ-basic subgroup then G is a C_λ-group and λ is confinal with ω.*

Proof. Part (a) follows from Proposition 2.5 and Lemma 2.6.

Conversely, suppose H is a proper λ-basic subgroup of the reduced p-group G. For $\alpha < \lambda$,

$$G/p^\alpha G = (H + p^\alpha G)/p^\alpha G \cong H/(H \cap p^\alpha G) = H/p^\alpha H$$

is totally projective. Thus G is a C_λ-group. That λ must be confinal with ω is immediate from (3.7) of [1] and (3.10) of [7].

3. C_λ-Groups for $\lambda \neq \beta + \omega$.

The purpose of this section is to examine the structure of p^λ-pure subgroups of C_λ-groups. We shall restrict our attention to ordinals that cannot be expressed in the form $\beta + \omega$ for any ordinal β. The techniques utilized are essentially those of Megibben in [6] and rely upon the existence of λ-basic subgroups as established in § 2.

The proofs given for Lemma 3 in [6] can, with the aid of § 2, be reproduced to yield the following lemmas.

LEMMA 3.1. *Let λ be an ordinal confinal with ω. Suppose H is a p^λ-pure subgroup of G and that $\{H_n\}$ is a λ-high confinal tower of H. Then there exists a λ-high confinal tower $\{G_n\}$ of G such that, for each n, $H_n \subseteq G_n$ and $H_n = H \cap G_n$.*

LEMMA 3.2. *Let λ be an ordinal confinal with ω such that $\lambda \neq \beta + \omega$ for any β. Suppose G is totally projective and that $G = \bigcup G_n$ where $\{G_n\}$ is a λ-high confinal tower. If H is a p^λ-pure subgroup of G such that, for each n, $H \cap G_n$ is a $p^{\alpha(n)}$-high subgroup of H, then H is a direct summand of G.*

THEOREM 3.3. *Let λ be any limit ordinal such that $\lambda \neq \beta + \omega$ for any β, and let G be a C_λ-group. If H is a p^λ-pure subgroup of G then H is a C_λ-group.*

Proof. It suffices to establish the proposition for ordinals λ such that λ is confinal with ω and $\lambda \neq \beta + \omega$ for and ordinal β. For such an ordinal λ, let $\{H_n\}$ be a λ-high confinal tower of H. By Lemma 3.1, there exists a λ-high confinal tower $\{G_n\}$ of G such that $H_n = H \cap G_n$ for each n. Since G is a C_λ-group, $\bigcup G_n$ is totally projective, by Proposition 2.5. By Lemma 3.2, it follows that $\bigcup H_n$ is a λ-basic subgroup of H and consequently, by Theorem 2.7, H is a C_λ-group.

LEMMA 3.4. *Let λ be confinal with ω, $\lambda \neq \beta + \omega$ for any β. Let A be a totally projective group of length at most λ and suppose A is a p^λ-pure subgroup of the C_λ-group G. Then there exists a subgroup C of G such that $A \oplus C$ is a λ-basic subgroup of G.*

Proof. Since A is a totally projective group of length at most λ, it follows from Proposition 1.1 that A is the union of a λ-high confinal tower $\{A_n\}$ of itself. By Lemma 3.1, there exists a λ-high confinal tower $\{G_n\}$ of G such that $A_n = A \cap G_n$ for each n. Let $B = \bigcup G_n$. By the proof of Theorem 2.7, B is a λ-basic subgroup of G. But $\{G_n\}$ is also a λ-high confinal tower of B and, by Lemma 3.2, we have the desired decomposition $B = A \oplus C$.

THEOREM 3.5. *Let λ be a limit ordinal such that $\lambda \neq \beta + \omega$ for any ordinal β, and let G be a C_λ-group. If H is a p^λ-pure subgroup of G then G/H is a C_λ-group.*

Proof. It suffices to establish the result for an arbitrary but fixed ordinal λ satisfying the conditions that λ is confinal with ω and $\lambda \neq \beta + \omega$ for any ordinal β. Let λ be such an ordinal and H a p^λ-pure subgroup of the C_λ-group G. By Theorem 3.3, H is a C_λ-group and thus, by Theorem 2.7, contains a λ-basic subgroup. Let A be a λ-basic subgroup of H and choose C, by Lemma 3.4, such that $A \oplus C$ is a λ-basic subgroup of G. If $x \in (H \cap C)[p]$, we can write, for each $\alpha < \lambda$, $x = a_\alpha + z_\alpha$ where $a_\alpha \in A[p]$ and $z_\alpha \in p^\alpha H$. Thus $-a_\alpha + x \in p^\alpha(A \oplus C) = p^\alpha A \oplus p^\alpha C$ and $x \in \bigcap p^\alpha C = p^\lambda C = 0$. We then have a direct decomposition $H \oplus C$. If $pg \in H \oplus C$, then

$$pg = a + ph + c$$

where $a \in A, h \in H$ and $c \in C$. Since $pG \cap (A \oplus C) = p(A \oplus C)$, we conclude that $pG \cap (H \oplus C) = p(H \oplus C)$ and $H \oplus C$ is neat in G. Moreover, $G[p] \subseteq (A \oplus C)[p] + p^{\alpha}G \subseteq (H \oplus C)[p] + p^{\alpha}G$ for all $\alpha < \lambda$ and therefore $H \oplus C$ is a p^{λ}-pure subgroup of G. Consequently, $(H \oplus C)/H$ is p^{λ}-pure in G/H. Also $(H \oplus C)/H \cong C$ is totally projective of length at most λ, and

$$(G/H)/(H \oplus C/H) \cong (G/A \oplus C)/(H \oplus C/A \oplus C)$$

is divisible. We have constructed a λ-basic subgroup of G/H and we conclude that G/H is indeed a C_{λ}-group.

As easy consequences of Theorem 3.5, we have the following analogues of familiar properties of pure subgroups.

COROLLARY 3.6. *Suppose λ is a limit ordinal such that*

$$\lambda \neq \beta + \omega$$

for any ordinal β. A subgroup H of a C_{λ}-group G is a p^{λ}-pure subgroup if and only if $(H + p^{\alpha}G)/p^{\alpha}G$ is a direct summand of $G/p^{\alpha}G$ for all $\alpha < \lambda$.

COROLLARY 3.7. *Suppose λ is a limit ordinal such that*

$$\lambda \neq \beta + \omega$$

for any ordinal β. If H is a p^{λ}-pure subgroup of the C_{λ}-group G and if $p^{\alpha}H = 0$ for some $\alpha < \lambda$, then H is a direct summand of G.

4. Remark. As noted above, we have not dealt with the problems of p^{λ}-pure subgroups of C_{λ}-groups where λ is a limit ordinal which may be expressed in the form $\lambda = \beta + \omega$. It would not be surprising, however, if the results of § 3 fail to hold for certain of such ordinals.

REFERENCES

1. P. Crawley and A. W. Hales, *The structure of abelian p-groups given by certain presentations*, J. of Algebra, **12** (1969), 10-23.
2. P. Griffith, *Infinite Abelian Group Theory*, Chicago, The University of Chicago Press, 1970.
3. P. Hill, *Isotype subgroups of direct sums of countable groups*, Illinois J. Math., **13** (1969), 281-290.
4. ———, *On the classification of abelian groups*, (to appear).
5. J. Irwin, C. Walker and E. Walker, *On p^{α}-pure Sequences of Abelian Groups*, Topics in Abelian Groups, Chicago, (1963), 69-119.
6. C. Megibben, *A generalization of the classical theory of primary groups*, Tôhoku Math. J., **22** (1970), 347-356.

7. R. Nunke, *Homology and direct sums of countable groups*, Math. Z., **101** (1967), 182–212.

Received July 13, 1971.

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