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THE CONVEX CONE OF n-MONOTONE FUNCTIONS

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A reformulation of the Krein-Milman Theorem is used to obtain an integral representation of each function in a certain class of real monotonic functions defined on [0, 1].

Let $\{i_1, i_2, i_3, \cdots\}$ denote a fixed sequence all of whose terms are either 0 or 1, and let M_1 be the set of real nonnegative functions f on [0, 1] such that

$$(-1)^{(i_1)}\Delta_h^1 f(x) = (-1)^{(i_1)}[f(x+h) - f(x)] \geqq 0 ,$$

$h > 0$, for $[x, x+h[\subset [0, 1]$. Let M_n, $n > 1$, be the set of functions belonging to M_{n-1} such that

$$(-1)^{(i_n)}\Delta_h^n f(x) = (-1)^{(i_n)}[\Delta_h^{n-1} f(x+h) - \Delta_h^{n-1} f(x)] \geqq 0$$

for $[x, x+nh] \subset [0, 1]$. If $f \in M_n$, then f is said to be an n-monotone function. Since the sum of two n-monotone functions is in M_n and since a nonnegative real multiple of an n-monotone function is an n-monotone function, the set M_n is a convex cone. It is the purpose of this paper to give the extremal elements (i.e., the generators of extreme rays) of this cone, and to show that for the n-monotone functions an integral representation in terms of extremal elements is possible.

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1. Extremal elements of M_n. Let f be a function in M_1 which assumes exactly one positive value in [0, 1]. If $f = f_1 + f_2$, where f_1 and $f_2 \in M_1$, then f_1 and f_2 are zero where f is zero and f_1 and f_2 are constant where f is constant. Therefore, f_1 and f_2 are proportional to f and f is an extremal element of M_1. On the other hand, if f assumes at least two positive values in [0, 1], then a nonproportional decomposition can be given by taking

$$f_1(x) = \min\{f(x), (1/2)[f(0) + f(1)]\}$$

and $f_2 = f - f_1$. Therefore, the extremal elements of M_1 are precisely the functions in M_1 which assume exactly one positive value in [0, 1].

Let $f \in M_n$, $n > 1$, and let $a_0 = 0$ if $i_1 = 0$ and $a_0 = 1$ if $i_1 = 1$. If $f(a_0) > 0$ and f is not constant, then take $f_1 = f(a_0)$ and $f_2 = f - f_1$.

In so doing, f_1 and $f_2 \in M_n$ and f_1 and f_2 are not proportional to f. Therefore, the only extremal elements f of M_n with $f(a_0) > 0$ are the positive constant functions.

Let $f \in M_n$, $n>1$, and define $a_0' = 1 - a_0$, if $i_2 = 0$ and $a_0' = a_0$ if $i_2 = 1$, where a_0 is defined above. It can be shown that if $f \in M_n$, then f must be continuous on $[0, 1]$ except at a_0' [9, p. 148]. It follows that the only extremal elements of M_1 that are in M_n are those which are continuous on $[0, 1]$ except, possibly, at a_0', and these functions are again extremal elements of M_n.

If $i_2 = 0, f \in M_n$, $n > 1$, f is not constant on $(0, 1)$ and f is discontinuous at $a_0' = 1 - a_0$, then take $f_1(x) = 0$ for $x \in [0, 1]$ and $x \neq a_0'$,

$$f_1(a_0') = f(a_0') - \operatorname*{limit}_{x \to a_0'} f(x) > 0$$

and $f_2 = f - f_1$. In so doing, f_1 and $f_2 \in M_n$ and f_1 and f_2 are not proportional to f. Hence, whenever $i_2 = 0$, the only extremal elements of M_n that are discontinuous at $a_0' = 1 - a_0$ are the functions which are positive at a_0' and zero elsewhere on $[0, 1]$.

On the other hand, if $i_2 = 1$, $f \in M_n$, $n > 1$, f is not constant on $(0, 1)$ and f is discontinuous at $a_0' = a_0$, then let

$$f_1(x) = \operatorname*{limit}_{x \to a_0'} f(x) > 0 \ ,$$

$x \in [0, 1]$ and $x \neq a_0', f_1(a_0') = 0$ and $f_2 = f - f_1$. Then f_1 and f_2 are in M_n and f_1 and f_2 are not proportional to f. Therefore, whenever $i_2 = 1$, the only extremal elements of M_n that are discontinuous at $a_0' = a_0$ are the functions which are zero at a_0' and equal to a positive constant elsewhere on $[0, 1]$.

Consequently, the extremal elements of M_n, $n > 1$, which are not extremal elements of M_1 must be zero at a_0 and continuous on $[0, 1]$. It will be shown that these extremal elements of M_n are indefinite integrals of the extremal elements of a cone which is similar to M_1. This cone is given in Definitions 1 and 2.

DEFINITION 1. If g is a real function monotonic on $(0, 1)$ and $n > 1$, then define the (possibly extended real-valued) function $I(g, n - 1; \cdot)$ by the equation

$$I(g, n-1; x) = \int_{a_0}^{x} \int_{a_1}^{t_1} \cdots \int_{a_{n-3}}^{t_{n-3}} \int_{a_{n-2}}^{t_{n-2}} g(t)\, dt\, dt_{n-2} \cdots dt_2\, dt_1$$

for $x \in (0, 1)$, where $a_0 = (1/2)\,[1 - (-1)^{(i_1)}]$ and

$$a_j = (1/2)\,[1 - (-1)^{(i_j + i_{j+1})}],\ 1 \leqq j \leqq n - 2\ .$$

DEFINITION 2. Let K_n, $n > 1$, denote the convex cone of real functions g on $(0, 1)$ such that

(a) g is right-continuous;

(b) $(-1)^{(i_{n-1})} g(x) \geqq 0$, for $x \in (0, 1)$;

(c) $(-1)^{(i_n)} \Delta_h^1 g(x) \geqq 0$, for $0 < x < x + h < 1$;

(d) $I(g, n-1; x)$ is finite, for $x \in (0, 1)$; and

(e) $\lim_{x \to 1 - a_0} I(g, n-1; x)$ exists and is finite.

Note. If $g \in K_n$, $n > 1$, then $I(g, n-1; \cdot)$ will denote the function which is the continuous extension to $[0, 1]$ of the function given in Definition 1.

DEFINITION 3. Let a and b be two distinct numbers in the interval $[0, 1]$ and define the function $\chi_{(a,b)}$ on $(0, 1)$ by

$$\chi_{(a,b)}(x) = 1, \text{ if } x \text{ is between } a \text{ and } b \text{ or } 0 < x = \min\{a, b\};$$
$$\chi_{(a,b)}(x) = 0, \text{ otherwise.}$$

DEFINITION 4. If m is a nonzero real number, $\xi \in [0, 1]$ and $n > 1$, then define the function $e(m, \xi, n-1; \cdot)$ by the equation

$$e(m, \xi, n-1; x) = mI(\chi_{(\xi, 1-a_{n-1})}, n-1; x)$$

for $0 \leqq x \leqq 1$, where $a_{n-1} = (1/2)[1 - (-1)^{(i_{n-1}+i_n)}]$.

The principal theorem of this section can now be stated and the remainder of the section will be devoted to its proof. The key results are Lemma 3 and Proposition 2.

THEOREM 1. *The extremal elements of M_1 are the functions in M_1 which assume exactly one positive value in $[0, 1]$. The positive constant functions and the extremal elements of M_1 which are discontinuous at $a_0' = (1/2)[1 + (-1)^{(i_1+i_2)}]$ are extremal elements of M_n, $n > 1$. The functions $e(m, \xi, n-1; \cdot)$, where $(-1)^{(i_{n-1})} m > 0$ and $\xi \in (0, 1)$ or $\xi = a_{n-1}$ are extremal elements of M_n, $n > 1$. There are no other extremal elements of M_2. The only other extremal elements of M_n, $n > 2$, are those functions $e(m, a_k, k; \cdot)$, where $(-1)^{(i_k)} m > 0$ and $1 \leqq k \leqq n-2$.*

In the same manner that the extremal elements of M_1 were found, it can be shown that the extremal elements of K_n are precisely those functions in K_n which assume exactly one nonzero value in $(0, 1)$. Before determining the extremal elements of M_n, it is shown in the following three lemmas how the n-monotone functions are related to the functions in K_n, where $n > 1$.

LEMMA 1. *If* $f \in M_n$, *then* $f_+^{(n-1)} \in K_n$, *where* $n > 1$.

Proof. Since $(-1)^{(i_n)} \Delta_h^n f(x) \geqq 0$ for $0 \leqq x < x + nh \leqq 1$, then $f^{(n-2)}$ exists and is continuous on $(0, 1)$ and $(-1)^{(i_n)} f^{(n-2)}$ is convex [1]. Therefore $(-1)^{(i_n)} f^{(n-2)}$ has a right-continuous, nondecreasing right-hand derivative [4, p. 10]. It follows that $(-1)^{(i_n)} \Delta_h^1 f_+^{(n-1)}(x) \geqq 0$ for $0 < x + h < 1$. If $f \in M_n$, then $(-1)^{(i_{n-1})} \Delta_h^{n-1} f(x) \geqq 0$ for $0 \leqq x < x + (n-1)h \leqq 1$, which implies that

$$(-1)^{(i_{n-1})} \Delta_{\delta_1}^1 \Delta_{\delta_2}^1 \cdots \Delta_{\delta_{n-1}}^1 f(x) \geqq 0$$

for $0 \leqq x < x + \delta_1 + \delta_2 + \cdots + \delta_{n-1} \leqq 1$ [1]. It then follows that $(-1)^{(i_{n-1})} f_+^{(n-1)}(x) \geqq 0$ for $0 < x < 1$, since $f_+^{(n-1)}$ exists on $(0, 1)$. It remains to show that

$$\operatorname*{limit}_{x \to 1 - a_0} I(f_+^{(n-1)}, n-1; x)$$

exists and is finite and this proof will be by induction on n.

If $f \in M_2$, then

$$f(x) = \int_{a_0}^{x} f'_+(t)\, dt + \operatorname*{limit}_{x \to a_0} f(x)\,,$$

which implies that

$$\operatorname*{limit}_{x \to 1 - a_0} I(f'_+, 1; x) = \operatorname*{limit}_{x \to 1 - a_0} f(x) - \operatorname*{limit}_{x \to a_0} f(x)$$

and this latter limit exists and is finite since f is monotonic on $[0, 1]$ [4, Theorem 1.1]. Now assume that $f \in M_n$ implies that

$$\operatorname*{limit}_{x \to 1 - a_0} I(f_+^{(n-1)}, n-1; x)$$

exists and is finite and let $f \in M_{n+1}$. Then $f \in M_n$ and it follows from the first part of the proof that $(-1)^{(i_{n-1})} f^{(n-1)}$ is nonnegative and monotonic on $(0, 1)$ and

$$\begin{aligned}(-1)^{(i_{n-1})} f^{(n-1)}(a_{n-1}) &= \operatorname*{limit}_{x \to a_{n-1}} (-1)^{(i_{n-1})} f^{(n-1)}(x)\\ &= \inf\{(-1)^{(i_{n-1})} f^{(n-1)}(x) \colon 0 < x < 1\}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\operatorname*{limit}_{x \to 1 - a_0} I(f_+^{(n)}, n; x)\\ &= \operatorname*{limit}_{x \to 1 - a_0} I(f^{(n-1)} - f^{(n-1)}(a_{n-1}), n-1; x)\\ &= \operatorname*{limit}_{x \to 1 - a_0} I(f^{(n-1)}, n-1; x) - f^{(n-1)}(a_{n-1})\, I(1, n-1; x)\end{aligned}$$

exists and is finite by the induction hypothesis.

LEMMA 2. *If* $g \in K_n$, *then* $I(g, n-1; \cdot) \in M_n$, *where* $n > 1$.

Proof. The proof will be by induction on n. If $g \in K_2$, then

$$I(g, 1; x) = \int_{a_0}^{x} g(t)\, dt$$

for $x \in [0, 1]$, and since $(-1)^{(i_1)} g(t) \geqq 0$, $t \in (0, 1)$, and

$$a_0 = (1/2)\,[1 - (-1)^{(i_1)}] ,$$

then $I(g, 1; x) \geqq 0$. If $0 \leqq x < x + h \leqq 1$, then

$$(-1)^{(i_1)} \Delta_h^1 I(g, 1; x) = \int_{x}^{x+h} (-1)^{(i_1)} g(t)\, dt \geqq 0 .$$

Since $(-1)^{(i_2)} g$ is nondecreasing, then $I((-1)^{(i_2)} g, 1; \cdot)$ is convex [4, p. 13]. It follows that $(-1)^{(i_2)} \Delta_h^2 I(g, 1; x) \geqq 0$ for $0 \leqq x < x + 2h \leqq 1$, and hence, $I(g, 1; \cdot) \in M_2$. Assume that $I(g, n-1; \cdot) \in M_n$ for $g \in K_n$ and $n > 1$. If $g \in K_{n+1}$, then let

$$f(x) = \int_{a_{n-1}}^{x} g(t)\, dt ,$$

for $x \in (0, 1)$. Since $(-1)^{(i_n)} g$ is nonnegative and

$$a_{n-1} = (1/2)\,[1 - (-1)^{(i_{n-1}+i_n)}] ,$$

it is easily seen that $f \in K_n$ and it follows from the induction hypothesis that $I(g, n; \cdot) = I(f, n-1; \cdot) \in M_n$. By a repeated application of the mean value theorem for a Riemann integral, it can be shown that

$$\Delta_h^{n-1} I(g, n; x) = h^{n-1} f(\xi)$$

for $0 \leqq x < \xi < x + (n-1)\,h \leqq 1$. Since $(-1)^{(i_{n+1})} g$ is nondecreasing, then $(-1)^{(i_{n+1})} f$ is convex on $(0, 1)$ [4, p. 13]. It follows that

$$\begin{aligned}(-1)^{(i_{n+1})} \Delta_h^{n+1} I(g, n; x) &= (-1)^{(i_{n+1})} \Delta_h^2 \Delta_h^{n-1} I(g, n; x) \\ &= (-1)^{(i_{n+1})} \Delta_h^2 f(\xi) \geqq 0\end{aligned}$$

for $0 \leqq x < x + (n+1)h \leqq 1$, and this inequality, together with the fact that $I(g, n; \cdot) \in M_n$ implies that $I(g, n; \cdot) \in M_{n+1}$.

In the proofs that follow, $f^{(k)}(a_k)$ should be interpreted as

$$f^{(k)}(a_k) = \operatorname*{limit}_{x \to a_k} f^{(k)}(x) ,$$

where $f \in M_n$, $n > 2$, and $1 \leqq k \leqq n-2$. Since $f^{(k)} \in K_{k+1}$, this limit will always exist and be finite. It is a consequence of Lemmas 1 and

2 that $f = I(f_+^{(n-1)}, n-1; \cdot)$ whenever $f \in M_n$, $n > 1$, and $f^{(k)}(a_k) = 0$ for $0 \leqq k \leqq n-2$. It is shown in the following lemma that extremal elements of M_n can be obtained directly from the extremal elements of K_n.

LEMMA 3. *If $g \in K_n$ and $f = I(g, n-1; \cdot)$, then f is an extremal element of M_n if, and only if, g is an extremal element of K_n, where $n > 1$.*

Proof. Suppose that f is an extremal element of M_n. If g_1 and $g_2 \in K_n$ such that $g = g_1 + g_2$, then

$$\begin{aligned} f = I(g, n-1; \cdot) &= I(g_1 + g_2, n-1; \cdot) \\ &= I(g_1, n-1; \cdot) + I(g_2, n-1; \cdot)\,. \end{aligned}$$

If $f_j = I(g_j, n-1; \cdot), j = 1, 2$, then f_1 and $f_2 \in M_n$ and $f = f_1 + f_2$. Since f is an extremal element of M_n, there are numbers $\lambda_j \geqq 0$ such that $f_j = \lambda_j f$, $j = 1, 2$, which implies that $g_j = \lambda_j f_+^{(n-1)} = \lambda_j g$, $j = 1, 2$, and g is therefore an extremal element of K_n.

Conversely, if g is an extremal element of K_n and f_1 and $f_2 \in M_n$ such that $f = f_1 + f_2$, then g_1 and $g_2 \in K_n$ and $g_1 + g_2 = f_+^{(n-1)} = g$, where g_j is the $(n-1)$th right derivative of f_j, $j = 1, 2$. This implies there are constants $\lambda_j \geqq 0$, $j = 1, 2$, such that $g_j = \lambda_j g$. It is evident from the definition of f that $f^{(k)}(a_k) = 0$, where $0 \leqq k \leqq n-2$. This, together with the fact that $f_j^{(k)} \in K_{k+1}$ for $1 \leqq k \leqq n-2$, implies that $f_j^{(k)}(a_k) = 0$, $j = 1, 2$ and $0 \leqq k \leqq n-2$.

Hence,

$$f_j = I(g_j, n-1; \cdot) = I(\lambda_j g, n-1; \cdot) = \lambda_j I(g, n-1; \cdot) = \lambda_j f$$

for $j = 1, 2$, and f is therefore an extremal element of M_n.

PROPOSITION 1. *The function $e(m, \xi, n-1; \cdot)$ is an extremal element of M_n, $n > 1$, where $(-1)^{(i_{n-1})} m > 0$ and $\xi \in (0, 1)$ or $\xi = a_{n-1}$.*

Proof. Since $m\chi_{(\xi, 1-a_{n-1})}$ is an extremal element of K_n whenever $(-1)^{(i_{n-1})} m > 0$ and $\xi \in (0, 1)$ or $\xi = a_{n-1}$, and

$$e(m, \xi, n-1; \cdot) = I\,(m\chi_{(\xi, 1-a_{n-1})}, n-1; \cdot)\ ,$$

the result follows immediately from Lemma 3.

PROPOSITION 2. *The function $e(m, a_k, k; \cdot)$ is an extremal element of M_n, $n > 2$, where $(-1)^{(i_k)} m > 0$ and $1 \leqq k \leqq n-2$.*

Proof. Since M_n is a subcone of M_{k+1} and $e(m, a_k, k; \cdot)$ is an extremal element of M_{k+1}, it is sufficient to show that

$$e(m, a_k, k; \cdot) \in M_n .$$

If $f = e(m, a_k, k; \cdot)$, then $f = I(f^{(k)}, k; \cdot)$, where

$$f^{(k)}(x) = m\chi_{(a_k, 1-a_k)}(x) = m\chi_{(0,1)}(x) = m$$

for $0 < x < 1$. Since $f^{(k)}$ is constant on $(0, 1)$, it follows from a repeated application of the mean value theorem for a Riemann integral that

$$\Delta_h^{k+1} f(x) = \Delta_h^1 \Delta_h^k f(x) = h^k \Delta_h^1 f^{(k)}(\xi) = 0$$

for $0 \leqq x < x + (k+1)h \leqq 1$, where $x < \xi < x + kh$ and thus, $\Delta_h^p f(x) = 0$ for $0 \leqq x < x + ph \leqq 1$ and $p \geqq k+1$. Hence, $f \in M_n$, for every n, which implies that f is an extremal element of M_p, for $p \geqq k+1$.

It will follow, as a consequence of the next three lemmas, that no other functions in M_n are extremal elements of M_n, $n > 2$.

LEMMA 4. *Let $f \in M_n$, $n > 2$, such that $f(a_0) = 0$, f is continuous on $[0, 1]$ and $f \neq e(m, a_k, k; \cdot)$ for $(-1)^{(i_k)} m > 0$ and $1 \leqq k \leqq n-2$. If there is an integer k such that $1 \leqq k \leqq n-2$ and $f^{(k)}(a_k) \neq 0$, then f is not an extremal element of M_n.*

Proof. Let k denote the smallest integer such that $f^{(k)}(a_k) \neq 0$. Then $f \in M_n \subset M_{k+2}$ implies that $f_+^{(k+1)} \in K_{k+2}$, and it follows from Lemma 2 that $I(f_+^{(k+1)}, k+1; \cdot) \in M_{k+2}$. Since $f(a_0) = 0$ and $f^{(p)}(a_p) = 0$ for $1 \leqq p < k$, then

$$I(f_+^{(k+1)}, k+1; \cdot) = I(f^{(k)}, k; \cdot) - f^{(k)}(a_k) I(1, k; \cdot) = f - e(m, a_k, k; \cdot)$$

where $m = f^{(k)}(a_k)$. Since

$$\Delta_h^p e(m, a_k, k; x) = 0$$

for $0 \leqq x < x + ph \leqq 1$ and $k+1 \leqq p \leqq n$ and $f \in M_n$, it follows that

$$(-1)^{(i_p)} \Delta_h^p I(f_+^{(k+1)}, k+1; x) = (-1)^{(i_p)} \Delta_h^p f(x) \geqq 0$$

for $0 \leqq x < x + ph \leqq 1$ and $k+1 \leqq p \leqq n$. Hence,

$$f - e(m, a_k, k; \cdot) \in M_n ,$$

where $m = f^{(k)}(a_k)$, and a nonproportional decomposition of f can be given by taking $f_1 = e(m, a_k, k; \cdot)$ and $f_2 = f - f_1$. Thus f is not an extremal element.

LEMMA 5. *Let $f \in M_n$, $n > 2$, such that $f \neq 0$, $f(a_0) = 0$, f is* *continuous on* $[0, 1]$ *and* $f \neq e(m, a_k, k; \cdot)$ *for* $(-1)^{(i_k)} m > 0$ *and* $1 \leqq k \leqq n-2$. *If* $f_+^{(n-1)} = 0$ *on* $(0, 1)$, *then* f *is not an extremal element of* M_n.

Proof. If $f_+^{(n-1)} = 0$, then there is a positive integer $k \leqq n-2$ such that $f^{(k)} \neq 0$ and $f^{(k)}$ is constant on $(0, 1)$. Thus, $f^{(k)}(a_k) \neq 0$ and it follows from Lemma 4 that f is not an extremal element.

It follows from Lemmas 4 and 5 that if f is an extremal element of M_n, $n > 2$ such that $f(a_0) = 0$, f is continuous on $[0, 1]$ and either $f_+^{(n-1)} = 0$ or $f^{(k)}(a_k) \neq 0$ for some k, $1 \leqq k \leqq n-2$, then $f = e(m, a_k, k; \cdot)$, where $(-1)^{(i_k)} m > 0$ and $1 \leqq k \leqq n-2$.

LEMMA 6. *Let* $f \in M_n$, $n \geqq 2$, *such that* f *is continuous on* $[0, 1]$, $f_+^{(n-1)} \neq 0$ *and* $f^{(k)}(a_k) = 0$ *for* $0 \leqq k \leqq n-2$. *If* f *is an extremal element of* M_n, *then* $f = e(m, \xi, n-1; \cdot)$, *where* $(-1)^{(i_{n-1})} m > 0$ *and* $\xi \in (0, 1)$ *or* $\xi = a_{n-1}$.

Proof. Since $f^{(k)}(a_k) = 0$ for $0 \leqq k \leqq n-2$, then

$$f = I(f_+^{(n-1)}, n-1; \cdot)$$

and it follows from Lemma 3 that $f_+^{(n-1)}$ is an extremal element of K_n. Thus, $f_+^{(n-1)} = m\chi_{(\xi, 1-a_{n-1})}$ for $(-1)^{(i_{n-1})} m > 0$ and $\xi \in (0, 1)$ or $\xi = a_{n-1}$, which implies that $f = I(f_+^{(n-1)}, n-1; \cdot) = e(m, \xi, n-1; \cdot)$. This completes the proof of Theorem 1.

2. **Integral representations.** The set of functions $M_n - M_n$, $n \geqq 1$, forms the smallest linear space containing the convex cone M_n. With the topology of simple convergence, $M_n - M_n$ is a Hausdorff locally convex space such that for each $x \in [0, 1]$, the linear functional L_x defined by $L_x(f) = f(x)$ is continuous.

PROPOSITION 3. *The set* M_n *is closed in* $M_n - M_n$ *for* $n \geqq 1$.

Proof. The linear functional F defined on $M_n - M_n$ by $F(f) = \Delta_h^n f(x)$, for $[x, x+nh] \subset [0, 1]$, is continuous in the topology of simple covergence. By definition, M_n is the intersection of a collection of closed half-spaces corresponding to such functionals.

Since M_n is closed and every n-monotone function f is nonnegative and bounded by $f(1-a_0)$, Tychonoff's theorem implies that the normalized n-monotone functions, namely

$$C_n = \{f \in M_n \colon f(1-a_0) = 1\} ,$$

form a compact base for M_n, $n \geqq 1$. Thus, every nonzero n-monotone function can be uniquely expressed as a positive multiple of some f in C_n and f is an extreme point of the convex set C_n if, and only if, f is an extremal element of M_n which lies in C_n.

DEFINITION 5. For $n \geqq 2$, let m_ξ denote the number which satisfies the equation $e(m_\xi, \xi, n-1; 1-a_0) = 1$, where $\xi \in (0, 1)$ or $\xi = a_{n-1}$. For $n > 2$, let m_k denote the constant which satisfies the equation $e(m_k, a_k, k; 1-a_0) = 1$, where $1 \leqq k \leqq n-2$. Let ext C_n denote the set of extreme points of C_n, $n \geqq 1$, and let $e(m_0, a_0, 0; \cdot)$ denote the unique function in ext C_n, $n \geqq 2$, which is discontinuous at $a_0' = (1/2)[1 + (-1)^{(i_1+i_2)}]$; that is, $e(m_0, a_0, 0; x) = (1/2)[1-(-1)^{(i_2)}]$ for $0 < x < 1$, $e(m_0, a_0, 0; a_0) = 0$ and $e(m_0, a_0, 0; 1-a_0) = 1$.

The principal theorem of this section can now be stated and the remainder of the section will be devoted to its proof.

THEOREM 2. *To each $f \in C_n$, $n \geqq 2$, there correspond unique non-negative regular Borel measures ν and μ on $[0, 1]$ and*

$$\{e(m_k, a_k, k; \cdot) : 0 \leqq k \leqq n-2\},$$

respectively, such that

$$\nu([0, 1]) + f(a_0) + \sum_{\substack{k=0 \\ k \neq k_0}}^{n-2} \mu[e(m_k, a_k, k; \cdot)] = 1$$

and

$$f(x) = \int_0^1 e(m_\xi, \xi, n-1; x)\, d\nu(\xi) + f(a_0) + \sum_{\substack{k=0 \\ k \neq k_0}}^{n-2} \alpha_k e(m, a_k, k; x)$$

for each $x \in [0, 1]$, where $\alpha_k = \mu[e(m_k, a_k, k; \cdot)]$ for each k and

$$e(m_{1-a_{n-1}}, 1-a_{n-1}, n-1; \cdot) = e(m_{k_0}, a_{k_0}, k_0; \cdot)$$

denotes the function which is the pointwise limit of the functions $e(m_\xi, \xi, n-1; \cdot)$ as ξ approaches $1-a_{n-1}$. Thus, each n-monotone function is a scalar multiple of such a representation.

Theorem 2 will be proved by using an integral reformulation of the Krein-Milman theorem. In order to apply this result, it must first be demonstrated that ext C_n is closed.

PROPOSITION 4. *The set of extreme points of C_n is closed in C_n, $n \geqq 2$.*

Proof. Since C_n with the relative topology is a subspace of a first countable space, it will suffice to show that if $\{f_i\}$ is a sequence of functions in ext C_n which converges pointwise to the function f, then $f \in \text{ext } C_n$ [3, p. 164]. Since all except a finite number of the functions in ext C_n are of the form $e(m_\xi, \xi, n-1; \cdot)$, where $\xi \in (0, 1)$ or $\xi = a_{n-1}$, it can be assumed without loss of generality that $f_i = e(m_{\xi_i}, \xi_i, n-1; \cdot)$ for each i.

If $a_0 = a_1 = \cdots = a_{n-1}$, then the function in C_n are convex and

$$f_i(x) = \left(\frac{x - \xi_i}{1 - a_0 - \xi_i}\right)^{n-1} \chi_{(\xi, 1-a_0)}(x)$$

for $x \in (0, 1)$. If the sequence $\{\xi_i\}$ of real numbers converges to $1 - a_0$, then it is easily seen that

$$\operatorname*{limit}_{i \to \infty} f_i(x) = 0$$

for $x \in (0, 1)$ or $x = a_0$. Since the topology of simple convergence is a Hausdorff topology, it follows that $f(1 - a_0) = 1$ and $f(x) = 0$, otherwise, which implies that $f = e(m_0, a_0, 0; \cdot)$ and $f \in \text{ext } C_n$. On the other hand, if $\{\xi_i\}$ does not converge to $1 - a_0$, then there is a real number $\xi_0 \neq 1 - a_0$ and a subsequence $\{\xi_j\}$ of $\{\xi_i\}$ such that $\{\xi_j\}$ converges to ξ_0. Hence,

$$\begin{aligned}
\operatorname*{limit}_{j \to \infty} f_j(x) &= \operatorname*{limit}_{j \to \infty} \left(\frac{x - \xi_j}{1 - a_0 - \xi_j}\right)^{n-1} \chi_{(\xi_j, 1-a_0)}(x) \\
&= \left(\frac{x - \xi_0}{1 - a_0 - \xi_0}\right)^{n-1} \chi_{(\xi_0, 1-a_0)}(x) \\
&= e(m_{\xi_0}, \xi_0, n-1; x)
\end{aligned}$$

for each $x \in (0, 1)$. Therefore, since the topology is a Hausdorff topology, $f = e(m_{\xi_0}, \xi_0, n-1; \cdot)$ and it follows that $f \in \text{ext } C_n$.

If $a_1 = a_2 = \cdots = a_{n-1}$ and $a_0 \neq a_{n-1}$, then the functions in C_n are concave and

$$f_i(x) = 1 - \left(\frac{x - \xi_i}{a_0 - \xi_i}\right)^{n-1} \chi_{(\xi_i, a_0)}(x)$$

for $x \in (0, 1)$. If the sequence $\{\xi_i\}$ converges to a_0, then

$$\operatorname*{limit}_{i \to \infty} f_i(x) = 1$$

for $x \in (0, 1)$ or $x = 1 - a_0$ and $f = e(m_0, a_0, 0; \cdot)$. On the other hand, if there is a subsequence $\{\xi_j\}$ of $\{\xi_i\}$ which converges to $\xi_0 \neq a_0$, then

$$\begin{aligned}\operatorname*{limit}_{j\to\infty} f_j(x) &= \operatorname*{limit}_{j\to\infty}\left[1 - \left(\frac{x-\xi_j}{a_0-\xi_j}\right)^{n-1}\chi_{(\xi_j,a_0)}(x)\right]\\ &= 1 - \left(\frac{x-\xi_0}{a_0-\xi_0}\right)^{n-1}\chi_{(\xi_0,a_0)}(x) = e(m_{\xi_0}, \xi_0, n-1; x)\end{aligned}$$

for each $x \in (0, 1)$ and $f = e(m_{\xi_0}, \xi_0, n-1; \cdot)$. In either case, it follows that $f \in \operatorname{ext} C_n$.

If there are exactly $p > 0$ integers $k_1, \cdots, k_p$ such that

$$1 \leqq k_1 < k_2 < \cdots < k_p \leqq n-2$$

and $a_{k_j} \neq a_{n-1}$, $1 \leqq j \leqq p$, and $a_0 = a_{n-1}$, then

$$\begin{aligned}f_i(x) = m_{\xi_i}\Bigg[&\frac{(x-\xi_i)^{n-1}}{(n-1)!}\chi_{(\xi_i,1-a_0)}(x)\\ &+ \sum_{r=1}^{p}(-1)^r\sum_{j_r=r}^{p}\cdots\sum_{j_1=1}^{j_2-1}\frac{(1-a_0-\xi_i)^{n-k_{J_r}-1}(1-2a_0)^{k_{J_r}-k_{J_1}}(x-a_0)^{k_{J_1}}}{(n-k_{j_1}-1)!\,(k_{j_r}-k_{j_{r-1}})!\cdots(k_{j_2}-k_{j_1})!\,(k_{j_1})!}\Bigg]\end{aligned}$$

for $x \in (0, 1)$, where

$$\begin{aligned}m_{\xi_i}^{-1} = &\frac{(1-a_0-\xi_i)^{n-1}}{(n-1)!}\\ &+ \sum_{r=1}^{p}(-1)^r\sum_{j_r=r}^{p}\cdots\sum_{j_1=1}^{j_2-1}\frac{(1-a_0-\xi_i)^{n-k_{J_r}-1}(1-2a_0)^{k_{J_r}}}{(n-k_{j_r}-1)!\,(k_{j_r}-k_{j_{r-1}})!\cdots(k_{j_2}-k_{j_1})!(k_{j_1})}\,.\end{aligned}$$

If there is a subsequence $\{\xi_j\}$ of $\{\xi_i\}$ which converges to $\xi_0 \neq 1 - a_0$, then it is easily seen that

$$f(x) = \operatorname*{limit}_{j\to\infty} f_j(x) = e(m_{\xi_0}, \xi_0, n-1; x)$$

for each $x \in (0, 1)$. On the other hand, if $\{\xi_i\}$ converges to $1 - a_0$, then

$$\begin{aligned}\operatorname*{limit}_{i\to\infty} f_i(x) = m_{k_p}\Bigg[&\frac{(x-a_0)^{(k_p)}}{(k_p)!}\\ &+ \sum_{r=1}^{p-1}(-1)^r\sum_{j_r=r}^{p-1}\cdots\sum_{j_1=1}^{j_2-1}\frac{(1-2a_0)^{k_p-k_{j_1}}(x-a_0)^{k_{j_1}}}{(k_p-k_{j_r})!\,(k_{j_r}-k_{j_{r-1}})!\cdots(k_{j_2}-k_{j_1})!\,(k_{j_1})!}\Bigg]\\ = e(m_{k_p}, a_{k_p}, k_p; x)&\end{aligned}$$

for $x \in (0, 1)$, where

$$\begin{aligned}m_{k_p}^{-1} = &\frac{(1-2a_0)^{(k_p)}}{(k_p)!}\\ &+ \sum_{r=1}^{p-1}(-1)^r\sum_{j_r=r}^{p-1}\cdots\sum_{j_1=1}^{j_2-1}\frac{(1-2a_0)^{(k_p)}}{(k_p-k_{j_r})!\,(k_{j_r}-k_{j_{r-1}})!\cdots(k_{j_2}-k_{j_1})!\,(k_{j_1})!}\,.\end{aligned}$$

In either case, it follows that $f \in \operatorname{ext} C_n$.

Finally if there are exactly $p > 0$ integers $k_1, \cdots, k_p$ such that $1 \leqq k_1 < k_2 < \cdots < k_p \leqq n-2$ and $a_{k_j} \neq a_{n-1}$, $1 \leqq j \leqq p$ and $a_0 \neq a_{n-1}$, then

$$\begin{aligned} &f_i(x) \\ &= m_{\xi_i}\Big[\frac{(a_0-\xi_i)^{n-1}}{(n-1)!} - \frac{(x-\xi_i)^{n-1}}{(n-1)!}\chi_{(\xi_i,a_0)}(x) \\ &\quad + \sum_{r=1}^{p}(-1)^r\sum_{j_r=r}^{p}\cdots\sum_{j_1=1}^{j_2-1}\frac{(a_0-\xi_i)^{n-k_{j_r}-1}(2a_0-1)^{k_{j_r}}}{(n-k_{j_r}-1)!\,(k_{j_r}-k_{j_{r-1}})!\cdots(k_{j_2}-k_{j_1})!\,(k_{j_1})!} \\ &\quad - \sum_{r=1}^{p}(-1)^r\sum_{j_r=r}^{p}\cdots\sum_{j_1=1}^{j_2-1}\frac{(a_0-\xi_i)^{n-k_{j_r}-1}(2a_0-1)^{k_{j_r}-k_{j_1}}(x-1+a_0)^{k_{j_1}}}{(n-k_{j_r}-1)!\,(k_{j_r}-k_{j_{r-1}})!\cdots(k_{j_2}-k_{j_1})!\,(k_{j_1})!}\Big] \end{aligned}$$

for $x \in (0, 1)$, where

$$\begin{aligned} &m_{\xi_i}^{-1} \\ &\quad= \frac{(a_0-\xi_i)^{n-1}}{(n-1)!} \\ &\quad\quad + \sum_{r=1}^{p}(-1)^r\sum_{j_r=r}^{p}\cdots\sum_{j_1=1}^{j_2-1}\frac{(a_0-\xi_i)^{n-k_{j_r}-1}(2a_0-1)^{k_{j_r}}}{(n-k_{j_r}-1)!\,(k_{j_r}-k_{j_{r-1}})!\cdots(k_{j_2}-k_{j_1})!\,(k_{j_1})!}\,. \end{aligned}$$

If there is a subsequence $\{\xi_j\}$ of $\{\xi_i\}$ which converges to $\xi_0 \neq a_0$, then it is evident that

$$f(x) = \operatorname*{limit}_{j\to\infty} f_j(x) = e(m_{\xi_0}, \xi_0, n-1; x)$$

for each $x \in (0, 1)$. On the other hand, if $\{\xi_i\}$ converges to a_0, then

$$\begin{aligned} &\operatorname*{limit}_{i\to\infty} f_i(x) \\ &= m_{k_p}\Big[\frac{(2a_0-1)^{(k_p)}}{(k_p)!} - \frac{(x-1+a_0)^{(k_p)}}{(k_p)!} \\ &\quad + \sum_{r=1}^{p-1}(-1)^r\sum_{j_r=r}^{p-1}\cdots\sum_{j_1=1}^{j_2-1}\frac{(2a_0-1)^{k_p-k_{j_1}}[(2a_0-1)^{k_{j_1}}-(x-1+a_0)^{k_{j_1}}]}{(k_p-k_{j_r})!\,(k_{j_r}-k_{j_{r-1}})!\cdots(k_{j_2}-k_{j_1})!\,(k_{j_1})!}\Big] \\ &= e(m_{k_p}, a_{k_p}, k_p; x) \end{aligned}$$

for $x \in (0, 1)$, where

$$\begin{aligned} &m_{k_p}^{-1} \\ &\quad= \frac{(2a_0-1)^{(k_p)}}{(k_p)!} \\ &\quad\quad + \sum_{r=1}^{p-1}(-1)^r\sum_{j_r=r}^{p-1}\cdots\sum_{j_1=1}^{j_2-1}\frac{(2a_0-1)^{(k_p)}}{(k_p-k_{j_r})!\,(k_{j_r}-k_{j_{r-1}})!\cdots(k_{j_2}-k_{j_1})!\,(k_{j_1})!}\,. \end{aligned}$$

In either case it follows that $f \in \operatorname{ext} C_n$ and this completes the proof.

DEFINITION 6. Let e_0 denote the function in $\operatorname{ext} C_n$ which is identically one and let $e(m_{1-a_{n-1}}, 1-a_{n-1}, n-1; \cdot)$ be the function defined by

$$e(m_{1-a_{n-1}}, 1-a_{n-1}, n-1; x) = \operatorname*{limit}_{\xi \to 1-a_{n-1}} e(m_\xi, \xi, n-1; x)$$

for $0 \leqq x \leqq 1$ and $n > 1$. Finally, let

$$e(m_{k_0}, a_{k_0}, k_0; \cdot) = e(m_{1-a_{n-1}}, 1-a_{n-1}, n-1; \cdot)$$

and notice that $k_0 = 0$ if $a_1 = a_2 = \cdots = a_{n-1}$ or k_0 is the largest positive integer such that $a_{k_0} \neq a_{n-1}$.

If the mapping ϕ: $[0, 1] \to \operatorname{ext} C_n$, $n \geqq 2$, is defined by

$$\phi(\xi) = e(m_\xi, \xi, n-1; \cdot) \qquad \text{for } 0 \leqq \xi \leqq 1 ,$$

then it follows from the proof of Proposition 4 that ϕ is continuous. If $E = \phi([0, 1])$, then ϕ is a homeomorphism from $[0, 1]$ onto E, since $[0, 1]$ is a compact space and E is a Hausdorff space. By the Krein-Milman representation theorem, to each f in C_n there corresponds a regular Borel probability measure μ on $\operatorname{ext} C_n$ such that

$$L(f) = \int_{\operatorname{ext} C_n} L\, d\mu$$

for each continuous linear functional L on $M_n - M_n$, since both C_n and $\operatorname{ext} C_n$ are compact subsets of $M_n - M_n$, $n \geqq 2$. For $0 \leqq x \leqq 1$, the evaluation functional L_x defined by $L_x(f) = f(x)$ is continuous on $M_n - M_n$, so that

$$\begin{aligned} f(x) &= \int_{\operatorname{ext} C_n} L_x d\mu \\ &= \int_E L_x\, d\mu + \mu(e_0) + \sum_{\substack{k=0 \\ k \neq k_0}}^{n-2} e(m_k, a_k, k; x)\mu[e(m_k, a_k, k; \cdot)] \end{aligned} \tag{1}$$

for each $x \in [0, 1]$. Define ν on each Borel subset B of $[0, 1]$ by

$$\nu(B) = \mu[\phi(B)];\ \text{i.e.,}\ \nu = \mu\phi .$$

Since $L_x[\phi(\xi)] = e(m_\xi, \xi, n-1; x)$, then

$$\int_E L_x\, d\mu = \int_{\phi^{-1}(E)} L_x\phi\, d(\mu\phi) = \int_0^1 e(m_\xi, \xi, n-1; x)\, d\nu(\xi)$$

for $0 \leqq x \leqq 1$. Finally, by observing that $\mu(e_0) = f(a_0)$, since e_0 is the only function in $\operatorname{ext} C_n$ which is positive at a_0, Equation (1) can be written as

$$f(x) = \int_0^1 e(m_\xi, \xi, n-1; x)\, d\nu(\xi) + f(a_0) + \sum_{\substack{k=0 \\ k \neq k_0}}^{n-2} e(m_k, a_k, k; x)\,\mu[e(m_k, a_k, k; \cdot)] \,.$$

It remains to prove that μ is unique. Since μ is supported by $\operatorname{ext} C_n$, then μ is a maximal measure in Choquet's ordering [6, pp. 24, 70]. Thus, by the Choquet-Meyer uniqueness theorem, it suffices to prove that C_n is a simplex [6, p. 66].

LEMMA 7. *Suppose $f \in M_n - M_n$ and $n \geqq 2$. Then there is a function $g \in K_n$ such that $g - f_+^{(n-1)} \in K_n$ and if h is any function in K_n such that $h - f_+^{(n-1)} \in K_n$, then it must follow that $h - g \in K_n$.*

Proof. First assume that $i_{n-1} = i_n = 0$. Since $f_+^{(n-1)} \in K_n - K_n$, then $f_+^{(n-1)}$ is of bounded variation on every interval $[0, x]$, where $0 < x < 1$. Define $g(x) = f_+^{(n-1)}(0) + P_0^x(f_+^{(n-1)})$, where $P_0^x(f_+^{(n-1)})$ denotes the positive variation of $f_+^{(n-1)}$ over $[0, x]$, $0 \leqq x < 1$ [8, p. 85]. Then both g and $g - f_+^{(n-1)}$ are nonnegative, nondecreasing and right-continuous on $[0, 1)$. If $h \in K_n$ such that $h - f_+^{(n-1)} \in K_n$, then it follows that $h - g$ is nonnegative, nondecreasing and right-continuous on $[0, 1)$. Therefore,

$$0 \leqq \operatorname*{limit}_{x \to 1 - a_0} I(h - g, n-1; x) \leqq \operatorname*{limit}_{x \to 1 - a_0} I(h, n-1; x) \,,$$

which implies that both g and $h - g$ are in K_n.

If i_{n-1} and i_n are not both zero, then define

$$y = (1/2)\,[1 - (-1)^{(i_{n-1} + i_n)}(1 - 2x)]$$

and

$$F(x) = (-1)^{(i_{n-1})} f_+^{(n-1)}(y) \qquad \text{for } 0 \leqq x < 1 \,.$$

Let $G(x) = F(0) + P_0^x(F)$ for $0 \leqq x < 1$ and define $g(x) = (-1)^{(i_{n-1})} G(y)$. Then g and $g - f_+^{(n-1)} \in K_n$ and it follows from the first part of the proof that if h and $h - f_+^{(n-1)} \in K_n$, then $h - g \in K_n$.

DEFINITION 7. If u is a function in $M_n - M_n$, $n \geqq 2$, then define the functions u_k, $0 \leqq k \leqq n-2$, by

$$u_0(x) = u(a_0) \qquad \text{and}$$
$$u_k(x) = I(u^{(k)}(a_k), k; x) \qquad \text{for } 1 \leqq k \leqq n-2$$

where $x \in [0, 1]$.

LEMMA 8. *Suppose $f \in M_n - M_n$ and $n \geqq 2$. Then there is a* *funtion $g \in M_n$ such that $g - f \in M_n$ and if h is any n-monotone function such that $h - f \in M_n$, then it must follow that $h - g \in M_n$.*

Proof. First assume that $f^{(k)}(a_k) = 0$ for $0 \leqq k \leqq n-2$ and let $g_+^{(n-1)}$ denote the function in K_n guaranteed by Lemma 7. Define $g = I(g_+^{(n-1)}, n-1; \cdot)$; then $g \in M_n$ and

$$g - f = I(g_+^{(n-1)} - f_+^{(n-1)}, n-1; \cdot) \in M_n .$$

If h is an n-monotone function such that $h - f \in M_n$, then $h_+^{(n-1)}$ and $h_+^{(n-1)} - f_+^{(n-1)} \in K_n$ and it follows that $h_+^{(n-1)} - g_+^{(n-1)} \in K_n$. If $h^{(k)}(a_k) = 0$ for $0 \leqq k \leqq n-2$, then

$$h - g = I(h_+^{(n-1)} - g_+^{(n-1)}, n-1; \cdot) \in M_n .$$

If there is some integer p such that $0 \leqq p \leqq n-2$ and $h^{(p)}(a_p) \neq 0$, then let

$$\bar{h} = h - \sum_{k=0}^{n-2} h_k ,$$

where $h_0 = h(a_0)$ and $h_k = I(h^{(k)}(a_k), k; \cdot)$ for $1 \leqq k \leqq n-2$. Then $\bar{h}^{(k)}(a_k) = 0$ for $0 \leqq k \leqq n-2$ and $\bar{h}$ and $\bar{h} - f \in M_n$, since h and $h - f \in M_n$ (cf. proof of Lemma 4). It follows that $\bar{h} - g \in M_n$ which implies that

$$h - g = \bar{h} - g + \sum_{k=0}^{n-2} h_k \in M_n$$

since h_k is an n-monotone function for $0 \leqq k \leqq n-2$.

On the other hand, if there is a nonnegative integer $p \leqq n-2$ such that $f^{(p)}(a_p) \neq 0$, then let

$$\bar{f} = f - \sum_{k=0}^{n-2} f_k$$

where f_k is given by Definition 7. Since $\bar{f} \in M_n - M_n$ and $\bar{f}^{(k)}(a_k) = 0$ for $0 \leqq k \leqq n-2$, it follows from the first part of the proof that there is an n-monotone function $\bar{g}$ such that $\bar{g} - \bar{f} \in M_n$ and if h is an n-monotone function such that $h - \bar{f} \in M_n$, then $h - \bar{g} \in M_n$. Let k_j, $0 \leqq j \leqq p < n-1$, denote those integers for which

$$(-1)^{i_{k_j}} f^{(k_j)}(a_{k_j}) > 0$$

and define

$$g = \bar{g} + \sum_{j=0}^{p} f_{k_j} .$$

Then $g \in M_n$ since

$$f_{k_j} = I(f^{(k_j)}(a_{k_j}), k_j; \cdot) = e(f^{(k_j)}(a_{k_j}), a_{k_j}, k_j; \cdot) \in M_n$$

for $0 \leqq j \leqq p$, and

$$g - f = \bar{g} + \sum_{j=0}^{p} f_{k_j} - f = \bar{g} - \bar{f} - \sum_{k \neq k_j} f_k \in M_n$$

since $-f_k \in M_n$ if $k \neq k_j$. Suppose that h is an n-monotone function such that $h - f \in M_n$. Then

$$h - f - \sum_{k=0}^{n-2} (h - f)_k \in M_n$$

which implies that

$$h - f - \sum_{k \neq k_j} (h - f)_k = h - f - \sum_{k=0}^{n-2} (h - f)_k + \sum_{j=0}^{p} (h - f)_{k_j} \in M_n$$

since $(h - f)_{k_j} \in M_n$(cf. proof of Lemma 4). Since h_k is an n-monotone function for $0 \leqq k \leqq n - 2$, then

$$\begin{aligned} h - f + \sum_{k \neq k_j} f_k &= h - f - \sum_{k \neq k_j} (h_k - f_k) + \sum_{k \neq k_j} h_k \\ &= h - f - \sum_{k \neq k_j} (h - f)_k + \sum_{k \neq k_j} h_k \in M_n . \end{aligned}$$

Therefore,

$$h - \sum_{j=0}^{p} f_{k_j} - \bar{f} = h - f + \sum_{k \neq k_j} f_k \in M_n$$

and $h - \sum_{j=0}^{p} f_{k_j} \in M_n$ since $h - \sum_{j=0}^{p} h_{k_j} \in M_n$ and

$$h - \sum_{j=0}^{p} f_{k_j} = h - \sum_{j=0}^{p} h_{k_j} + \sum_{j=0}^{p} (h_{k_j} - f_{k_j}) = h - \sum_{j=0}^{p} h_{k_j} + \sum_{j=0}^{p} (h - f)_{k_j} .$$

It follows that $h - \sum_{j=0}^{p} f_{k_j} - \bar{g} \in M_n$, which implies that $h - g \in M_n$.

If the function g of Lemma 8 is denoted by $f \vee 0$, then the least upper bound of two functions f_1 and $f_2 \in M_n - M_n$ can be given by $f_1 + (f_2 - f_1) \vee 0$ and therefore $M_n - M_n$ is a vector lattice. Thus, C_n is a simplex and the proof of Theorem 2 is complete.

3. REMARKS. If $i_2 = 0$, then C_2 is the set of functions f which are monotonic and convex on $[0, 1]$ such that $\max \{f(x): 0 \leqq x \leqq 1\} = 1$. If $i_1 = 0$, then the C_2 functions are nondecreasing and $e(m_\xi, \xi, 1; x) = 0$, $x \in [0, \xi]$ and $(x - \xi)/(1 - \xi)$ for $x \in [\xi, 1]$, where $0 \leqq \xi < 1$. Thus, to each $f \in C_2$ there corresponds a unique nonnegative regular Borel measure ν on $[0, 1]$ such that

$$f(x) = f(0) + \int_0^x \frac{x - \xi}{1 - \xi} d\nu(\xi)$$

for $0 < x < 1$. On the other hand, if $i_1 = 1$, then these functions are nonincreasing and $e(m_\xi, \xi, 1; x) = 1 - (x/\xi)$, $x \in [0, \xi]$ and 0 for $x \in [\xi, 1]$, where $0 < \xi \leqq 1$. It follows from Theorem 2 that to each f in C_2 there corresponds a unique nonnegative regular Borel measure ν on $[0, 1]$ such that

$$f(x) = f(1) + \int_x^1 [1 - (x/\xi)]\, d\nu(\xi)$$

for $0 < x < 1$.

If $i_k = 0$ for every $k \leqq n$, then $e(m_\xi, \xi, n - 1; x) = 0$, $x \in [0, \xi]$ and $[(x - \xi)/(1 - \xi)]^{n-1}$ for $x \in [\xi, 1]$, where $0 \leqq \xi < 1$, and

$$e(m_k, 0, k; x) = x^k$$

for x $[0, 1]$, where $1 \leqq k \leqq n - 2$. Thus, for each function f in C_n, there exist unique nonnegative real numbers $\alpha_1, \cdots, \alpha_{n-2}$ and a unique nonnegative regular Borel measure ν on $[0, 1]$ such that

$$f(x) = f(0) + \sum_{k=1}^{n-2} \alpha_k x^k + \int_0^x \left(\frac{x - \xi}{1 - \xi}\right)^{n-1} d\nu(\xi)$$

for $0 < x < 1$. In this case, the intersection of the M_n cones is the class of absolutely monotonic functions on $[0, 1]$. It is well known that if $f \in C_n$ for every n, then

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\, (x^n/n!)$$

for $0 \leqq x < 1$. For a discussion of these cones see [5].

Lastly, if $i_k = (1/2)[1 + (-1)^k]$ for $1 \leqq k \leqq n$, then

$$e(m_\xi, \xi, n - 1; x) = 1 - [1 - (x/\xi)]^{n-1} ,$$

$x \in [0, \xi]$ and 1 for $x \in [\xi, 1]$, where $0 < \xi \leqq 1$, and

$$e(m_k, 1, k; x) = 1 - (1 - x)^k$$

for $x \in [0, 1]$, where $1 \leqq k \leqq n - 2$. It follows from Theorem 2 that for each function f in C_n, there exist unique nonnegative real numbers $\alpha_1, \cdots, \alpha_{n-2}$ and a unique nonnegative regular Borel measure ν on $[0, 1]$ such that

$$f(x) = 1 - \sum_{k=1}^{n-2} \alpha_k (1 - x)^k - \int_x^1 [1 - (x/\xi)]^{n-1}\, d\nu(\xi)$$

for $0 < x < 1$. In this case, the C_n functions were called alternating of order n by Choquet [2, p. 170]. It can be shown that if $f \in C_n$ for every n, then

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(1)\, [(x - 1)^n/n!]$$

for $0 < x \leqq 1$. For a proof of this fact together with a discussion of these cones see [7].

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