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MATRIX REPRESENTATIONS FOR LINEAR TRANSFORMATIONS ON SERIES ANALYTIC IN THE UNIT DISC

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Let S be the space of all complex sequences A such that if z is a complex number and $|z| < 1$ then $\sum A_n z^n$ converges. We present three characterizations of the linear transformations from S to S which have matrix representations. We also characterize the linear transformations from S to the bounded sequences (or to the convergent sequences) which have matrix representations. The characterizations are in terms of natural topologies for the spaces.

These results are a blend of Köthe and Toeplitz' much quoted study [3] of complex sequence spaces, Haplanov's beautiful characterization [1] of those matrices which transform S to S, and some rather natural norms for S which have been used by V. Ganapathy Iyer [2] in his study of entire functions.

Köthe and Toeplitz study complex sequence spaces, matrices, and linear transformations having a kind of continuity which is independent of norms. A space is said to be *normal* provided that if x is in the space and $|y_n| \leqq |x_n|$, $n = 0, 1, \cdots$, then y is also in the space. Köthe and Toeplitz show that a continuous linear transformation from a normal space to a normal space has a matrix representation, and conversely, provided that each space contains all the "finite" sequences. Our space S is normal and the space of bounded sequences is also normal. The continuity criteria used in our theorems (statement (2) in each) are special cases of the continuity condition of Köthe and Toeplitz. It follows from their work that the existence of a matrix for a linear transformation L is necessary and sufficient for L to have "analytic" continuity (see definition below).

Given a matrix transformation from S to S and a norm N_r ($0 < r < 1$: if A is in S, $N_r(A) = \sum_{p=0}^{\infty} |A_p| r^p$) for S, Haplanov's theorem provides another such norm N_R such that the transformation is continuous from the normed linear space $\{S, N_r\}$ to $\{S, N_R\}$. Finally, to complete Theorem 1, each linear transformation which is continuous relative to some such pair of norms is represented by a matrix, even though S is complete with respect to neither of the norms.

Our second theorem is like the first: the transformations are from S to the bounded sequences (or convergent sequences).

In [6] Wilanski gives a result of a similar kind for convergence-preserving transformations.

For a topological approach to obtaining continuity for a transformation from the existence of a matrix representation, see Wilanski's *Functional Analysis* [5, p. 204].

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THEOREM 1. *Suppose that L is a linear transformation from S to S. These statements are equivalent:*

(1) *L has a matrix representation (there is a complex matrix M such that if A is in S and n is a nonnegative integer then $L(A)_n = \sum_{k=0}^{\infty} M_{nk}A_k$).*

(2) *If A is a sequence of sequences in S and A has limit 0 analytically (definition below), then the sequence $\{L(A_n)\}_{n=0}^{\infty}$ has limit 0 analytically also.*

(3) *If $0 < R < 1$ then there is a number r between 0 and 1 such that L is a continuous transformation from the normed linear space $\{S, N_r\}$ to $\{S, N_R\}$. (For ρ between 0 and 1 and A in S, $N_\rho(A) = \sum_{n=0}^{\infty} |A_n| \rho^n$.)*

(4) *There are numbers r and R between 0 and 1 such that L is a continuous linear transformation from $\{S, N_r\}$ to $\{S, N_R\}$.*

DEFINITION. If A is a sequence *of sequences in S* and f is a sequence of analytic functions such that if n is a nonnegative integer and $|z| < 1$ then

$$f_n(z) = \sum_{k=0}^{\infty} A_{nk} z^k$$

and f converges uniformly with limit 0 on each closed subset of the unit disc, then A is said to have limit 0 *analytically.*

HAPLANOV'S THEOREM. *Suppose that M is an infinite complex matrix. Then these statements are equivalent:*

(1) *If A is in S then $M \cdot A$ is in S. ($(M \cdot A)_n = \sum_{k=0}^{\infty} M_{nk}A_k$.)*

(2) *There are numbers r and d such that $0 < r < 1$ and d is a positive integer such that if j and k are nonnegative integers and $k > jd + d$ then $|M_{jk}| < r^k$, and there is a sequence s in S such that $|M_{jk}| \leqq s_j$ $(j, k = 0, 1, \cdots)$.*

THEOREM 2. *Suppose that L is a linear transformation from S to the bounded sequences (convergent sequences). Then these statements are equivalent:*

(1) *L has a matrix representation.*

(2) *If A is a sequence with values in S and has limit 0 analytically, then $\{L(A_n)\}_{n=0}^{\infty}$ is a sequence with limit 0 in the least upper bound norm.*

(3) *There is a number r between 0 and 1 such that if $0 < R < 1$*

then L is continuous from $\{S, N_r\}$ to $\{S, N_R\}$.

(4) *There is a number r between 0 and 1 such that L is a continuous transformation from $\{S, N_r\}$ to the space of bounded sequences (convergent sequences) under the least upper bound norm.*

The matrix transformations from S to the bounded sequences (convergent sequences) are characterized as follows [4]:

THEOREM A. *Suppose that M is an infinite complex matrix. Then these statements are equivalent:*

(1) *If A is in S, then $M \cdot A$ is a bounded (convergent) sequence.*

(2) *Each column of M is a bounded (convergent) sequence and there are numbers r and t such that $0 < r < 1$ and if each of j and k is a nonnegative integer then $|M_{jk}| \leqq tr^k$.*

The following notation and lemmas are useful in the proofs of our theorems.

NOTATION. l_1 is the space of all sequences x such that $\sum |x_k|$ converges, and N_1 is its usual norm: if x is in l_1 then $N_1(x) = \sum_{k=0}^{\infty} |x_k|$. If each of x and y is a sequence, then $x \cdot y$ is the sequence such that if n is a nonnegative integer then $(x \cdot y)_n = x_n \cdot y_n$.

Cauchy's Inequalities. Suppose that A is in S, $0 < r < 1$, and μ is a number such that if $|z| = r$ then $|\sum_{n=0}^{\infty} A_n z^n| \leqq \mu$. Then, for each nonnegative integer n, $|A_n| r^n \leqq \mu$.

LEMMA 0. *If A is a sequence of sequences in S which has limit 0 analytically and B is the sequence such that*

$$B(n)_k = B_{nk} = \max \{|A_{nj}|, j = 0, 1, \cdots, k\} \; (n, k = 0, 1, \cdots),$$

then B has limit 0 analytically.

Proof. Let r be a number between 0 and 1. Let R be a number between r and 1. Let ε be a positive number. Let m be a positive integer such that if n is an integer exceeding m and $|z| \leqq R$ then $|\sum_{k=0}^{\infty} A_{nk} z^k| < \varepsilon/(1 - r/R)$. Let n be an integer exceeding m. By Cauchy's inequalities, $|A_{nk}| < \varepsilon R^{-k}/(1 - r/R)$, $k = 0, 1, \cdots$. For each nonnegative integer k let j_k be a nonnegative integer such that $j_k \leqq k$ and $B_{nk} = |A(n, j_k)|$. If k is a nonnegative integer, $R^{-k} \geqq R^{-j_k}$. Consequently,

$$\sum_{k=0}^{\infty} B_{nk} r^k = \sum_{k=0}^{\infty} |A(n, j_k)| r^k$$
$$< \sum_{k=0}^{\infty} \frac{\varepsilon}{1 - r/R} R^{-k} r^k = \varepsilon .$$

LEMMA 1. *Suppose that A is a sequence (of sequences in S), A has limit 0 analytically, and C is the sequence such that if each of n and k is a nonnegative integer then $C(n)_k = C_{nk} = |A_{nk}|$. Then C has limit 0 analytically.*

Proof. This follows immediately from Lemma 0.

LEMMA 2. *Suppose that A is a sequence, A has limit 0 analytically, and s is in S. Let C be the sequence such that if each of n and k is a nonnegative integer then $C(n)_k = C_{nk} = A_{nk}s_k$. Then C has limit 0 analytically.*

Proof. Let r be a number between 0 and 1 and let R be a number between r and 1. Recall the Cauchy-Hadamard characterization for S: the sequence x belongs to S only in the case that $\overline{\lim\sup}_n |x_n|^{1/n} \leqq 1$. Therefore there is a number t such that if k is a nonnegative integer then $|s_k r^k| < tR^k$. Then, if n is a nonnegative integer and $|z| \leqq r$,

$$\left| \sum_{k=0}^{\infty} C_{nk} z^k \right| \leqq \sum_{k=0}^{\infty} |A_{nk} s_k| r^k$$
$$\leqq t \sum_{k=0}^{\infty} |A_{nk}| R^k ,$$

so that by Lemma 1, C has limit 0 analytically.

LEMMA 3. *If A is a sequence which has limit 0 analytically, d is a positive integer, and C is the sequence such that if each of n and k is a nonnegative integer then $C(n)_k = C_{nk} = A(n, kd + d)$, then C has limit 0 analytically. Furthermore, if $0 < r < 1$ and $R = r^{1/d}$ then $N_r(C_n) \leqq N_R(A_n)/r$, $n = 0, 1, \cdots$.*

Proof. Let r be a number between 0 and 1. Let R be $r^{1/d}$ and let ε be a positive number. By Lemma 1 there is is a positive integer m such that if n is an integer exceeding m then $\sum_{k=0}^{\infty} |A_{nk}| R^k < r \cdot \varepsilon$. Let n be an integer exceeding m. Then, if $|z| \leqq r$,

$$\left| \sum_{k=0}^{\infty} C_{nk} z^k \right| \leqq N_r(C_n) = \sum_{k=0}^{\infty} |A(n, kd + d)| r^k$$
$$= \sum_{k=0}^{\infty} |A(n, kd + d)| R^{-d} R^{kd+d} \leqq R^{-d} \sum_{k=0}^{\infty} |A_{nk}| R^k < \varepsilon ,$$

and

$$N_r(C_n) \leqq R^{-d} N_R(A_n) = \frac{1}{r} N_R(A_n) .$$

LEMMA 4. *Suppose that* $0 < a < 1, R = a^2$, s *is in* S, *and* $s_k \geqq 0, k = 0, 1, \cdots$. *Then* s^2 *in* S *and* $N_R(s^2) \leqq N_a(s)^2$.

Proof.

$$\begin{aligned} N_R(s^2) &= N_{a^2}(s^2) = \sum_{n=0}^{\infty} s_n^2 (a^2)^n = \sum_{n=0}^{\infty} s_{2n-n} s_n a^{2n} \\ &\leqq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{2n} s_{2n-k} s_k \right) a^{2n} \leqq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} s_{n-k} s_k \right) a^n \\ &= \left(\sum_{n=0}^{\infty} s_n a^n \right)^2 = N_a(s)^2 . \end{aligned}$$

LEMMA 5. *Suppose that* $0 < r < 1$ *and* A *is in* S *and* B *is the sequence such that if* n *is a nonnegative integer then* $B_n = \max\{|A_k|, k = 0, \cdots, n\}$. *Then*

$$N_r(B) \leqq N_r(A)/(1 - r) .$$

Proof.

$$\begin{aligned} N_r(B) &= \sum_{n=0}^{\infty} B_n r^n \leqq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} |A_k| \right) r^n \\ &= \sum_{k=0}^{\infty} |A_k| \sum_{n=k}^{\infty} r^n = \sum_{k=0}^{\infty} |A_k| r^k \sum_{n=0}^{\infty} r^n \\ &= N_r(A)/(1 - r). \end{aligned}$$

Proof of Theorem 1. $(1 \rightarrow 2)$ Statement (2) follows statement (1) as a consequence of Köthe and Toeplitz' *Satz* 2, §8 [3, p. 208]. Lemma 1 (above) is useful in showing that our "analytic" continuity is equivalent to their continuity. Alternately, one can cite a general theorem concerning *FK* spaces [5, Cor. 5, p. 204], or else one can use Haplanov's Theorem and Lemmas 0, 1, 2, and 3.

$(2 \rightarrow 1)$ Statement (1) follows from (2) as a consequence of *Satz* 7, §6 [3, p. 207]. This can also be done by modifying the argument under the heading $(4 \rightarrow 1)$ below.

$(1 \rightarrow 3)$ Let R be a number between 0 and 1, and let M be the matrix representation for L. Let s be the sequence such that if n is a nonnegative integer then $s_n = \sum_{k=0}^{\infty} |M_{nk}|$. By Haplanov's Theorem, s is in S, and there are numbers q and d such that $0 < q < 1$ and d

is a positive integer such that if j and k are nonnegative integers and $k > jd + d$ then $|M_{jk}| < q^k$. Let r be a number between q and 1 and between $R^{1/2d}$ and 1.

Let A be in S and suppose that each of B and C is a sequence such that

$$B_n = \max\{|A_j|, j = 0, 1, \cdots n\} \quad \text{and} \quad C_n = B_{nd+d}(n = 0, 1, \cdots) .$$

Each of B and C is in S, and, by Cauchy's inequality and Lemmas 4, 3, and 5,

$$\begin{aligned} N_R(L(A)) &= \sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty} M_{nk}A_k\right| R^n \\ &\leqq \sum_{n=0}^{\infty}\sum_{k=0}^{nd+d} |M_{nk}||A_k| R^n + \sum_{n=0}^{\infty}\sum_{k=nd+d+1}^{\infty} |M_{nk}||A_k| R^n \\ &\leqq \sum_{n=0}^{\infty} s_n B_{nd+d} R^n + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^k |A_k| R^n \\ &\leqq \sum_{n=0}^{\infty} s_n R^{n/2} C_n R^{n/2} + \frac{1}{1-R}\sum_{k=0}^{\infty} |A_k| q^k \\ &\leqq \left(\sum_{n=0}^{\infty} s_n^2 R^n\right)^{1/2}\left(\sum_{n=0}^{\infty} C_n^2 R^n\right)^{1/2} + N_q(A)/(1-R) \\ &\leqq N_{\sqrt{R}}(s) N_{\sqrt{R}}(C) + N_q(A)/(1-R) \\ &\leqq R^{-1/2} N_{\sqrt{R}}(s) N_r(B) + N_r(A)/(1-R) \\ &\leqq \frac{R^{-1/2}}{1-r} N_{\sqrt{R}}(s) N_r(A) + \frac{1}{1-R} N_r(A) . \end{aligned}$$

L is a continuous linear transformation from $\{S, N_r\}$ to $\{S, N_R\}$.

$(4 \rightarrow 1)$ For each nonnegative-integer pair $\{j, k\}$ let M_{jk} be $L(\delta_k)_j$. ($(\delta_k)_j = 1$ if $j = k$, $(\delta_k)_j = 0$ otherwise.) Let B be a member of S and let n be a nonnegative integer. Recall that l_1 is a subset of the ring S and let T be that linear transformation from l_1 to the complex plane such that if A is in l_1 then $T(A) = L(B \cdot A)_n$. If A is a sequence of sequences in l_1 and the sequence $\{N_1(A_j)\}_{j=0}^{\infty}$ has limit 0, then $\{N_r(B \cdot A_j)\}_{j=0}^{\infty}$ has limit 0, so that $\{N_R(L(B \cdot A_j))\}_{j=0}^{\infty}$ has limit 0 and $\{T(A_j)\}_{j=0}^{\infty}$ has limit 0 (by Cauchy's inequalities). T is a continuous linear transformation from l_1 to the plane. Consequently, there is a bounded complex sequence b such that if x is in l_1 then $T(x) = \sum_{k=0}^{\infty} b_k x_k$. Furthermore, if k is a nonnegative integer then

$$b_k = T(\delta_k) = L(B \cdot \delta_k)_n = L(B_k \cdot \delta_k)_n = B_k \cdot L(\delta_k)_n = M_{nk} B_k .$$

Let A be the sequence whose only value is 1. A is in S. Let x be the sequence with values in l_1 such that if each of j and k is a nonnegative integer then $x_{jk} = 1$ if $j \geqq k$ and $x_{jk} = 0$ if $j < k$. The sequence $\{N_r(B\cdot(x_j - A))\}_{j=0}^{\infty}$ has limit 0. So $\{N_R(L(B\cdot(x_j - A)))\}_{j=0}^{\infty}$ has limit 0, and, by Cauchy's inequalities, $\{L(B\cdot x_j)_n\}_{j=0}^{\infty}$ has limit $L(B\cdot A)_n = L(B)_n$: the sequence $\{T(x_j)\}_{j=0}^{\infty}$ has limit $L(B)_n$. But, if j is a nonnegative integer,

$$T(x_j) = \sum_{k=0}^{\infty} b_k x_{jk} = \sum_{k=0}^{j} b_k = \sum_{k=0}^{j} M_{nk}B_k ,$$

and

$$L(B)_n = \sum_{k=0}^{\infty} M_{nk}B_k ,$$

so L has a matrix representation.

Statement (4) follows immediately from (3). So we have proved Theorem 1.

Proof of Theorem 2. $(1 \to 2)$ One can again quote [3, *Satz* 2, p. 208] or quote [5, Cor. 5, p. 204] or simply apply Theorem A and Lemma 1.

$(1 \to 4)$ Let M be the infinite matrix such that if x is a bounded (convergent) sequence then $L(x)_n = \sum_{k=0}^{\infty} M_{nk}x_k$ $(n = 0, 1, \cdots)$. There are numbers r and t such that $|M_{jk}| \leqq tr^k$ $(j, k = 0, 1, \cdots)$. Let A be in S and let j be a nonnegative integer.

$$|L(A)_j| = \left|\sum_{k=0}^{\infty} M_{jk}A_k\right| \leqq t\sum_{k=0}^{\infty} r^k\,|A_k| = tN_r(A) .$$

Hence, L is a continuous transformation from $\{S, N_r\}$ to the space of bounded sequences (convergent sequences) with least upper bound norm.

$(2 \to 1)$ Suppose that A is a sequence having limit 0 analytically and $0 < r < 1$. Let ε be a positive number and let m be a positive integer such that if n is an integer exceeding m then

$$|L(A_n)_k| < \varepsilon(1 - r) \qquad (k = 0, 1, \cdots) .$$

Then, if $|z| \leqq r$ and n is an integer exceeding m,

$$\left|\sum_{k=0}^{\infty} L(A_n)_k z^k\right| \leqq \sum_{k=0}^{\infty} |L(A_n)_k|\, r^k < \varepsilon .$$

Consequently, the sequence $\{L(A_n)\}_{n=0}^{\infty}$ has limit 0 analytically, so that by Theorem 1, L has a matrix representation.

$(4 \to 3)$ Let r be a number between 0 and 1 such that L is a continuous transformation from $\{S, N_r\}$ to the space of bounded sequences (convergent sequences) under the least upper bound norm. Suppose that $0 < R < 1$.

There is a number K such that if A is in S then

$$|L(A)_j| \leqq KN_r(A) \qquad (j = 0, 1, \cdots) .$$

So, if A is in S,

$$N_R(L(A)) = \sum_{j=0}^{\infty} |L(A)_j| R^j \leqq \frac{K}{1-R} N_r(A) ,$$

and L is continuous from $\{S, N_r\}$ to $\{S, N_R\}$.

Statement (1) follows from statement (3) by Theorem 1, and so we are finished with the proof of Theorem 2.

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