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FIBER INTEGRATION IN SMOOTH BUNDLES

J. W. AUER

The purpose of this paper is to comment on the operations of fiber integration, or integration over the fiber, which arise in the study of the cohomology of bundles.

Let $\xi = (E, \Pi_E, B, F, G)$ be a smooth (C^∞) bundle, where as usual E is the total space, B the base, F the (connected) fiber, G the group, and $\Pi_E: E \to B$ the projection. Assume $H^k(F)$ to be finite dimensional for all k. A form ω on the total space is said to have fiber-compact support if, and only if, for all $x \in B$, there is a neighbourhood U_x of x, a trivialization $\phi: U_x \times F \cong \pi_E^{-1}(U_x)$, and a compact set $K \subset F$ such that (support $\phi^*\omega) \cap (U_x \times F) \subset U_x \times K$. Denote these k-forms by $A_F^k(E)$, and their de Rham cohomology by $H_F^k(E)$. When F is compact $A_F^k(E) = A^k(E)$, the algebra of all k-forms on E; if B is compact, $A_F(E)$ is the algebra $A_c(E)$ of forms on E with compact support. Now integration over fiber has been defined by various authors as a linear map

$$\Psi: H^k(E) \longrightarrow H^{k-m}(B; H^m(F)) , \quad k \geqq m ,$$

where m is the dimension of F. These definitions are essentially algebraic in nature; for example Ψ has been defined by a spectral sequence. Using this idea when ξ is orientable, a linear map

$$\Psi_1: H_F^k(E) \longrightarrow H^{k-m}(B)$$

is defined, and called algebraic fiber integration on account of the origin of the definition.

On the other hand, when ξ is orientable, there is the geometrically defined linear map $\int_F : A_F^k(E) \to A^{k-m}(B)$ given roughly speaking by

$$\left(\int_F \omega\right)(x) = \int_{F_x} \omega/F_x ,$$

where $\omega \in A_F^k(E)$, $x \in B$, and F_x denotes the fiber of ξ over x. The induced map Ψ_2 of cohomology is called geometric fiber integration: $\Psi_2: H_F^k(E) \to H^{k-m}(B)$. The main purpose of the paper is to show

THEOREM. $\Psi_1 = \Psi_2$.

1. The spectral sequence in $A_F(E)$.

(a) In this section we obtain results analogous to a theorem of Borel [6], which we use to obtain an expression for Ψ_1.

The action of G on F induces an action on $H^k(F)$ and on $H_c^k(F)$, $k = 0, 1, 2, \cdots, m$, where $H_c^k(F)$ denotes the de Rham cohomology of forms on F with compact support. Denote by h^k the total space of the bundle over B with fiber $H_c^k(F)$; because this bundle has a discrete group, the exterior derivative $\delta_B^{p,q}$ in $A_B^p(h^q)$ (the p-forms on B witn coefficients in h^q) is a differential operator. Denote $\operatorname{Ker} \delta_B^{p,q}/\operatorname{Im} \delta_B^{p-1,q}$ by $H^p(B; h^q)$.

(b) We filter $A_F^k(E)$ following *Hattori* [5] and *Borel* [6]: when $\{x_i\}$, $\{y_j\}$ are coordinates on neighbourhoods in B and F respectively, $i = 1, 2, \cdots, n-m$, $j = 1, 2, \cdots, m$, where $n = \dim E$, we use the same symbols to denote coordinates induced on sufficiently small open sets in E; then according to Hattori and Borel, I^p consists of those forms which involve at least p base differentials dx_i.

It will be convenient to define this filtration by a bigradation of $A_F(E)$. For this purpose, it is necessary to assume that a fixed connection ([7], p. 63) has been prescribed in ξ. Of course, by the remarks above, the filtration will be independent of the connection. If X is a C^∞ vector field on E, then by definition its horizontal and vertical parts, HX and VX respectively, induced by the connection, are again C^∞ vector fields. Whenever $\{x_i\}$, $\{y_j\}$ are coordinates on an open set $W \subset E$, as described above, we denote by dy_j^v, $j = 1, 2, \cdots, m$, the 1-forms on W defined by $dy_j^v(x) = dy_j(VX)$, where X is a vector field on W. Then $\{dx_i\} \cup \{dy_j^v\}$ generate $A^1(W)$ over $C^\infty(W)$.

DEFINITION 1. $C^{p,q} \equiv C^{p,q}(E) = \{\omega \in A_F^{p+q}(E) \mid i(X_1) \cdots i(X_{p+1})\omega = i(Y_1) \cdots i(Y_{q+1})\omega = 0$ for all horizontal vector fields X_i and vertical vector fields Y_j on $E\}$.

Here, as usual, $i(Z)$ is the substitution operator with respect to the vector field Z on E:

$$i(Z)\omega(Z_1, \cdots, Z_{p+q-1}) = \omega(Z, Z_1, \cdots, Z_{p+q-1})$$

when $Z_1, \cdots, Z_{p+q-1}$ are vector fields on E, $\omega \in A^{p+q}(E)$. Thus $i(Z)$: $A_F^{p+q}(E) \to A_F^{p+q-1}(E)$, because $A_F(E)$ is clearly stable under $i(Z)$.

Now we let Greek letters α, β, etc., represent sequences of positive integers of the form $(i_1, \cdots, i_p)$, for some positive integer p, with $i_1 < i_2 < \cdots < i_p$. Then dx_α will denote $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, and so on. We put $|\alpha| = p$ when $\alpha = (i_1, \cdots, i_p)$. Then if W is a coordinate neighbourhood in E, $\omega \in A^k(E)$, we may write ω/W as

$$\omega/W = \sum_{p+q=k} \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \omega^{\alpha\beta} dx_\alpha \wedge dy_\beta^v \tag{1}$$

where $\omega^{\alpha\beta} \in C^{\infty}(W)$, the C^{∞}-functions on W. Consequently any $\omega \in C^{p,q}(W)$ may be written

$$\omega = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \omega^{\alpha\beta} dx_{\alpha} \wedge dy_{\beta} \tag{2}$$

with $\omega^{\alpha\beta} \in C^{\infty}(W)$.

DEFINITION 2.

$$I^{p,q} = \bigoplus_{r \geqq p} C^{r,p+q-r}, \qquad I^p = \bigoplus_{q \geqq 0} I^{p,q}.$$

It is easily seen that

PROPOSITION 1. *$\{I^p\}_{p=0}^{\infty}$ is a decreasing filtration of $A_F(E)$. It is the filtration associated with the gradation $\{C^p\}_{p=0}^{\infty}$ of $A_F(E)$, where $C^p = \bigoplus_{q \geqq 0} C^{p,q}$, because $I^p = \bigoplus_{r \geqq p} C^r$.*

Note in particular that $I^0 = A_F(E)$, and $I^p = \{0\}$ for $p > \dim B = n - m$.

REMARK. If the above definitions are carried out for $A(E)$, one obtains the filtrations of *Hattori* [5] and *Borel* [6]. In particular, the *filtration* is independent of the choice of connection in ξ.

(c) Denote by $\{E_r^{p,q}\}$ the spectral sequence defined by the filtration $\{I^{p,q}\}$ of $A_F(E)$ defined above (see § 2). Then we will next show

PROPOSITION 2.

$$E_2^{p,q} \cong H^p(B; h^q), \qquad p, q \geqq 0.$$

Let $\mathscr{A}_B^p$ be the sheaf of germs of p-forms on B, and $\mathscr{C}_B^{\infty}$ the sheaf of germs of C^{∞}-functions on B.

DEFINITION 3. The sheaf $\mathscr{F}^q$ of fiber-compact q-forms along the fiber of ξ.

We first define the presheaf $\underline{\mathscr{F}}^q$ on B by $\underline{\mathscr{F}}^q(U) = C^{0,q}(\Pi_E^{-1}(U))$, when U is open in B. Then $\mathscr{F}^q$ is the sheaf induced by $\underline{\mathscr{F}}^q$.

LEMMA 1. *$E_0^{p,q} \cong \Gamma(\mathscr{A}_B^p \otimes \mathscr{F}^q)$ ($\Gamma(\mathscr{S})$ denotes the module of sections of any sheaf $\mathscr{S}$, and all tensor products are over the sheaf $\mathscr{C}_B^{\infty}$).*

Proof. From the definitions we have that

(3) $$E_0^{p,q} = C^{p,q}$$

DEFINITION 4.

$$\Omega\colon \Gamma(\mathscr{A}_B^P \otimes \mathscr{F}^q) \longrightarrow C^{p,q} .$$

Let $s \in \Gamma(\mathscr{A}_B^P \otimes \mathscr{F}^q)$; then if $U \subset B$ is sufficiently small, we may write s locally as a sum of terms of the form ω, where

$$\omega = \Big(\sum_{|\alpha|=p} \omega_1^\alpha\, dx_\alpha\Big) \otimes \Big(\sum_{|\beta|=q} \omega_2^\beta\, dy_\beta^v\Big)$$

with $\omega_1^\alpha \in C^\infty(U)$, $\omega_2^\beta \in C^\infty(\Pi_E^{-1}(U))$. Put

$$\Omega(\omega) = \sum_{\substack{|\alpha|=p\\|\beta|=q}} (\omega_1^\alpha \Pi_E)\omega_2^\beta\, dx_\alpha \wedge dy_\beta^v ,$$

and extend linearly to define $\Omega(s)$.

LEMMA 2. *Ω is well-defined, independent of the choice of coordinates.*

Proof. This follows from the fact that if $\{\bar{x}_i\}$, $\{\bar{y}_j\}$ are coordinates defined on an open set in E overlapping the domain of definition of the coordinates $\{x_i\}$, $\{y_j\}$, then

$$d\bar{x}_\alpha = \sum_{|\mu|=p} \frac{\partial \bar{x}_\alpha}{\partial x_\mu}\, dx_\mu$$

and

(4) $$d\bar{y}_\beta^v = \sum_{|\alpha|=q} \frac{\partial \bar{y}_\beta}{\partial y_\alpha}\, dy_\alpha^v$$

where $\partial\bar{x}_\alpha/\partial x_\mu$ represents the $p \times p$ sub-matrix with rows α, columns μ of the $(n-m) \times (n-m)$ matrix with entries $\partial\bar{x}_i/\partial x_j$, and analogously for $\partial\bar{y}_\beta/\partial y_\alpha$. Note that equation (4) is *not* the equation of transformation for the dy_β's; the latter equation also involves linear combinations of the dx_α's. Since Ω is easily seen to be an isomorphism, this completes the proof of Lemma 1.

DEFINITION 5. The homomorphism

$$\delta_F^{p,q}\colon \Gamma(\mathscr{A}_B^P \otimes \mathscr{F}^q) \longrightarrow \Gamma(\mathscr{A}_B^P \otimes \mathscr{F}^{q+1}) .$$

We first define the presheaf homomorphism $\underline{\delta}_F^q\colon \underline{\mathscr{F}}^q \to \underline{\mathscr{F}}^{q+1}$. Let $U \subset B$ be open, $\phi \in \underline{\mathscr{F}}^q(U) = C^{0,q}(\Pi_E^{-1}(U))$, and $\overline{Y_1, \cdots, Y_{q+1}}$ be vertical vector fields on $\Pi_E^{-1}(U)$. Then

$$\begin{aligned} &\delta^q_F(U)(\phi)(Y_1,\cdots,Y_{q+1}) \\ (5)\qquad &= \frac{1}{q+1}\sum_{i=1}^{q+1}(-1)^{i-1}Y_i(\phi(Y_1,\cdots,\hat{Y}_i,\cdots,Y_{q+1}) \\ &+ \frac{1}{q+1}\sum_{i<j}(-1)^{i+j}\phi([Y_i,Y_j],Y_1,\cdots,\hat{Y}_i,\cdots\hat{Y}_j,\cdots,Y_{q+1}) \end{aligned}$$

where the symbol $\hat{Y}_i$ indicates that Y_i is to be omitted. In terms of coordinates, if $\phi = \sum_{|\beta|=q}\omega^\beta\, dy^v_\beta$, with $\omega^\beta \in C^\infty(\Pi_E^{-1}(U))$, then

$$(6)\qquad \underline{\delta}^q_F(U)\,(\phi) = \sum_{|\beta|=q}\sum_k \frac{\partial\omega^\beta}{\partial y_k}\, dy^v_k \wedge dy^v_\beta$$

Now δ^q_F is the sheaf homomorphism induced by $\underline{\delta}^q_F$, and $\delta^{p,q}_F$ is the homomorphism of modules of sections induced by $(-1)^p\, 1_{\mathscr{A}^p_B}\otimes\delta^q_F$.

Let $\Delta^{p,q}_0\colon E^{p,q}_0 \longrightarrow E^{p,q+1}_0$ be the map induced by the exterior derivative on E; that is, $\Delta^{p,q}_0$ is defined by the diagram below, where $\rho^{p,q}_0$ is the canonical projection:

$$\begin{array}{ccc} I^{p,q} & \xrightarrow{\ \delta\ } & I^{p,q+1} \\ {\scriptstyle\rho^{p,q}_0}\big\downarrow & & \big\downarrow{\scriptstyle\rho^{p,q+1}_0} \\ E^{p,q}_0 & \xrightarrow[\ \Delta^{p,q}_0\ ]{} & E^{p,q+1}_0\ . \end{array}$$

Now in view of $E^{p,q}_0 = C^{p,q}$, $\Delta^{p,q}_0$ is exactly the differential operator induced in C^p by the exterior derivative on E:

$$\rho^{p,q+1}\delta = \Delta^{p,q}_0\colon\ C^{p,q} \longrightarrow C^{p,q+1}\ ,$$

where $\rho^{p,q+1}\colon A^{p+q+1}_F(E) \longrightarrow C^{p,q+1}$ is the canonical projection induced by $A_F(E) = \bigoplus_{p,q} C^{p,q}$.

LEMMA 3. *The diagram below is commutative*:

$$\begin{array}{ccc} \Gamma(\mathscr{A}_B{}^p\otimes\mathscr{F}^q) & \xrightarrow{\ \Omega\ } & C^{p,q} = E^{p,q}_0 \\ {\scriptstyle\delta^{p,q}_F}\big\downarrow & & \big\downarrow{\scriptstyle\Delta^{p,q}_0} \\ \Gamma(\mathscr{A}_B{}^p\otimes\mathscr{F}^{q+1}) & \xrightarrow{\ \Omega\ } & C^{p,q+1} = E^{p,q+1}_0\ . \end{array}$$

Proof. Any $s \in \Gamma(\mathscr{A}_B{}^P\otimes\mathscr{F}^q)$ is locally expressible on $W\subset E$ as a sum of terms of the form

$$\omega = \Big(\sum_{|\alpha|=p}\omega^\alpha_1\, dx_\alpha\Big)\otimes\Big(\sum_{|\beta|=q}\omega^\beta_1\, dy^v_\beta\Big)$$

so that

$$\Omega(\omega) = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} (\omega_1^\alpha \Pi_E)\, \omega_2^\beta\, dx_\alpha \wedge dy_\beta^v\ .$$

Now we may write on W

$$dy_j^v = dy_j + \sum_i \theta_j^i\, dx_i \tag{7}$$

$$dy_k = dy_k^v + \sum_j \zeta_k^j\, dx_j \tag{8}$$

where $\theta_j^i,\ \zeta_k^j \in C^\infty(W)$.

Replacing dy_β^v using equation (7) one obtains

$$\Omega(\omega) = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} (\omega_1^\alpha \Pi_E)\, \omega_2^\beta\, dx_\alpha \wedge dy^\beta + \sum_{\substack{|\alpha|>p \\ \mu}} (\cdots)^{\alpha\mu}\, dx_\alpha \wedge dy_\mu$$

(where $(\cdots)^{\alpha\mu}$ indicate coefficients of terms with $|\alpha| > p$) so that using the definition of $\rho^{p,q+1}$, one obtains on account of $\partial/\partial y_k(\omega_1^\alpha \Pi_E) = 0$,

$$\Delta_0^{p,q}\Omega(\omega) = \rho^{p,q+1}\delta\Omega(\omega) = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \sum_k \frac{\partial \omega_2^\beta}{\partial y_k} (\omega_1^\alpha \Pi_E)\, dy_k^v \wedge dx_\alpha \wedge dy_\beta^v\ . \tag{9}$$

One sees immediately that this is the same as the expression for $\Omega\delta_F^{p,q}(\omega)$, proving Lemma 3.

As a consequence we have

$$E_1^{p,q} = \mathrm{Ker}\ \Delta_0^{p,q}/\mathrm{Im}\ \Delta_0^{p,q-1} = \mathrm{Ker}\ \delta_F^{p,q}/\mathrm{Im}\ \delta_F^{p,q-1}\ .$$

Let $\mathscr{h}^q$ be the sheaf of germs of smooth sections of the bundle h^q. Now one can show by an argument identical to the one employed by Borel ([6], pp. 206, 207) that there is an isomorphism

$$\mathscr{h}^q = \mathrm{Ker}\ \delta_F^q/\mathrm{Im}\ \delta_F^{q-1}\ , \tag{10}$$

and hence also $\mathrm{Ker}\ \delta_F^{p,q}/\mathrm{Im}\ \delta_E^{p,q-1} \cong A_B^p(h^q)$. Thus we have

LEMMA 4. $E_1^{p,q} \cong A_B^p(h^q)$.

Proof of Proposition 2. Because $H^p(B; h^q) = \mathrm{Ker}\ \delta_B^{p,q}/\mathrm{Im}\ \delta_B^{p-1,q}$ (see § 1 (a) for the definition of $\delta_B^{p,q}$) and $E_2^{p,q} \cong \mathrm{Ker}\ \Delta_1^{p,q}/\mathrm{Im}\ \Delta_1^{p-1,q}$ it suffices to show that the diagram (11) below is commutative:

$$\begin{array}{ccc} A_B^p(h^q) & \xrightarrow{\ \Omega^1\ } & E_1^{p,q} \\ \downarrow{\scriptstyle \Delta_B^{p,q}} & & \downarrow{\scriptstyle \Delta_1^{p,q}} \\ A_B^{p+1}(h^q) & \xrightarrow{\ \Omega^1\ } & E_1^{p+1,q}\ . \end{array} \tag{11}$$

Here Ω^1 is the isomorphism induced by virture of Lemma 4.

Consider the exact sequence of cochain complexes

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^{p+1,q-1} & \xrightarrow{i} & C^{p,q} \oplus C^{p+1,q-1} & \xrightarrow{j} & C^{p,q} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Delta_0^{p+1,q-1}} & & \downarrow{\scriptstyle \partial^{p,q}} & & \downarrow{\scriptstyle \Delta_0^{p,q}} & & \\ 0 & \longrightarrow & C^{p+1,q} & \xrightarrow[i]{} & C^{p,q+1} \oplus C^{p+1,q} & \xrightarrow[j]{} & C^{p,q+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

where $\partial^{p,q}$ is induced by the exterior derivative δ: $A(E) \to A(E)$ and $i(d) = (0, d)$, $d \in C^{p+1,q-1}$, $j(c, d) = c$, $c \in C^{p,q}$.

Now it is known ([3], p. 85) that the differential operator $\Delta_1^{p,q}$ is the same as the connecting homomorphism $d^{p,q}$ induced by this exact sequence:

$$d^{p,q}\colon \operatorname{Ker} \Delta_0^{p,q}/\operatorname{Im} \Delta_0^{p,q-1} \longrightarrow \operatorname{Ker} \Delta_0^{p+1,q}/\operatorname{Im} \Delta_0^{p+1,p-1} .$$

Consequently, if $\omega \in \operatorname{Ker} \Delta_0^{p,q}$

$$\Delta_1^{p,q}([\omega]_{E_1^{p,q}}) = [i^{-1}\partial^{p,q}j^{-1}\omega]_{E_1^{p+1,q}} ,$$

where $[\omega]_{E_1^{p,q}}$ indicates the class of ω in $E_1^{p,q}$, and so on; that is, $\Delta_1^{p,q}([\omega]_{E_1^{p,q}})$ is $\partial^{p,q}\omega$ modulo $\operatorname{Im} \Delta_0^{p+1,q-1}$.

Now suppose that $\omega \in A_B^p(h^q)$; then for a coordinate neighbourhood $U \subset B$, ω/U may be written as

$$\omega/U(x) = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} [\omega^{\alpha\beta} dy_\beta^v]_{H_c^p(F_x)} dx_\alpha$$

where $x \in U \subset B$ and $\omega^{\alpha\beta} \in C^\infty(\Pi_E^{-1}(U))$.

A consideration of the isomorphism Ω^1 yields

$$\begin{aligned} \Omega^1(\omega/U) &= \sum_{\substack{|\alpha|=p \\ |\beta|=q}} [\omega^{\alpha\beta} dx_\alpha \wedge dy_\beta^v]_{E_1^{p,q}} \\ &= \Big[\sum_{\substack{|\alpha|=p \\ |\beta|=q}} \omega^{\alpha\beta} dx_\alpha \wedge dy_\beta + \sum_{\substack{|\alpha|>p \\ \mu}} (\cdots)^{\alpha\mu} dx_\alpha \wedge dy_\mu \Big]_{E_1^{p,q}} , \end{aligned}$$

where we have again replaced dy_β^v by means of equation (7).

A short computation now shows, when all quotients have been taken, and using (8), that

$$\Delta_1^{p,q}\Omega^1(\omega/U) = \Big[\sum_{\substack{|\alpha|=p\\|\beta|=q}} \sum_k \frac{\partial\omega^{\alpha\beta}}{\partial x_k} dx_k \wedge dx_\alpha \wedge dy_\beta^v\Big]_{E_1^{p+1,q}} ; \tag{12}$$

on the other hand,

$$\begin{aligned}\Omega^1\delta_B^{p,q}(\omega/U) &= \Omega^1\delta_B^{p,q}\Big(\sum_{\substack{|\alpha|=p\\|\beta|=q}} [\omega^{\alpha\beta}dy_\beta^v]_{H_c^q(F)}dx_\alpha\Big)\\ &= \Omega^1\Big(\sum_{\substack{|\alpha|=p\\|\beta|=q}}\sum_k \Big[\frac{\partial\omega^{\alpha\beta}}{\partial x_k}dy_\beta^v\Big]_{H_c^q(F)} dx_k \wedge dx_\alpha\Big)\end{aligned}$$

where $[\omega^{\alpha\beta}dy_\beta^v]_{H_c^q(F)}$ indicates the section of h^q over U defined by

$$[\omega^{\alpha\beta}dy_\beta^v]_{H_c^q(F)}(x) = [\omega^{\alpha\beta}dy_\beta^v]_{H_c^q(F_x)},\ x \in U$$

(recall that $(h^q)_x = H_c^q(F_x)$).

The expression above clearly yields the right hand side of equation (12), as required.

2. Fiber integration, algebraic definition. As mentioned earlier, algebraic fiber integration is defined by using the definition of Borel and Hirzebruch [1] applied to the spectral sequence $\{E_r\}$ arising from the filtration $\{I^p\}$ of fiber-compact forms on the total space of ξ.

For convenience we recall some definitions from the theory of (decreasing) spectral sequences:

$$\begin{aligned} Z_s^{p,q} &= A_F^{p,q}(E) \cap \{a \in I^p \mid \delta a \in I^{p+s}\}\\ D_s^{p,q} &= A_F^{p+q}(E) \cap I^p \cap \delta I^{p-s}\\ E_s^{p,q} &= Z_s^{p,q}/(Z_{s-1}^{p+1,q-1} \oplus D_{s-1}^{p,q})\end{aligned}$$

where $0 \leqq p, q, s \leqq \infty$.

Let $\Pi\colon Z_\infty^0 \to H_F(E)$ be the canonical projection; then

$$H^{p,q} = H_F^{p+q}(E) \cap \Pi(Z_\infty^p) \text{ filter } H_F(E), \text{ and } E_\infty^{p,q} = H_\infty^{p,q}/H_\infty^{p+1,q-1}. \tag{13}$$

Because $\dim F = m$ and $\dim B = n - m$, it follows that $E_r^{p,q} = 0$ for $q > m$, $p \geqq 0$, $r \geqq 0$ and that $I^p = 0$ for $p > n - m$.

LEMMA 5. (a) $E_r^{p,m} \subset E_{r-1}^{p,m}$, $r \geqq 3$, $p \geqq 0$.
(b) $E_r^{k-m,q} = E_\infty^{k-m,q}$, $r > \sup(n-k, k-m)$, $k \geqq m$, $q \geqq 0$.
(c) $E_\infty^{k-m,m} = H_F^k(E)/H_\infty^{k-m+1,m-1}$, $k \geqq m$.

As a consequence of Lemma 5 we now have an injection

$$h_1\colon E_\infty^{k-m,m} = E_{r_0}^{k-m,m} \subset \cdots \subset E_2^{k-m,m}$$

where $r_0 = \sup(n-k, k-m) + 1$, and a projection $h_2\colon H_F^k(E) \to E_\infty^{k-m,m}$. Let $\chi\colon E_2^{k-m,m} \cong H^{k-m}(B; h^m)$ be the isomorphism induced by Ω (Proposition 2). Then the definition of Borel, Hirzebruch yields $\chi h_1 h_2$:

$H_F^k(E) \to H^{k-m}(B; h^m)$. We now define σ: $H^{k-m}(B; h^m) \to H^{k-m}(B)$, under the assumption that ξ is *orientable*, in the following sense (see [4]):

DEFINITION 6. ξ is orientable if, and only if, there exists an m-form ψ on E such that for all $x \in B$, $i_x^* \psi$ is an orientation on F_x, the fiber over x; i_x: $F_x \subset E$. If such a ψ has been chosen, ξ is called oriented.

Clearly F is orientable when ξ is.

Recall that $\mathscr{h}^m = \operatorname{Ker} \delta_F^m / \operatorname{Im} \delta_F^{m-1} = \mathscr{F}^m / \operatorname{Im} \delta_F^{m-1}$.

We first define a map of presheaves, $\underline{\mathscr{I}}$: $\underline{\mathscr{F}}^m \to \underline{\mathscr{C}}_B^\infty$, where $\underline{\mathscr{C}}_B^\infty$ denotes the presheaf of germs of C^∞-functions on B: Let $x \in U$, U open in B, and $\omega \in \underline{\mathscr{F}}^m(U)$; then $(\underline{\mathscr{I}}(U)(\omega))(x) = \int_{F_x} \omega(x)$. To show this is well defined, let $\{U_j / j \in J\}$ be a covering of B by open sets such that E is trivial over each U_j, with ϕ_j: $U_j \times F \cong \Pi_E^{-1}(U_j)$, $j \in J$. Define $\psi_{j,x}$: $F \to F_x$ by $\psi_{j,x}(f) = \phi_j(x, f)$, $x \in U_j$, $f \in F$. Then

$$(\underline{\mathscr{I}}(U_j)(\omega))(x) = \deg \psi_{j,x} \int_F \psi_{j,x}^*(\omega(x)) \; ; \tag{14}$$

this shows that $\underline{\mathscr{I}}(U)(\omega)$ is C^∞ in x, because, since ω has fiber-compact support, the forms $\psi_{j,x}^*(\omega(x))$ on F have supports contained in a common compact set for x in sufficiently small open sets in B.

Thus there is a sheaf homomorphism $\mathscr{I}$: $\mathscr{F}^m \to \mathscr{C}_B^\infty$; if $\omega \in \operatorname{Im} \underline{\delta}_F^{m-1}(U)$, then by Stokes' Theorem, $(\underline{\mathscr{I}}(U)(\omega))(x) = 0$, for all $x \in U$. Consequently, $\mathscr{I}$ induces a sheaf homomorphism, also denoted by $\mathscr{I}$, $\mathscr{I}$: $\mathscr{F}^m / \operatorname{Im} \delta_F^{m-1} = \mathscr{h}^m \to \mathscr{C}_B^\infty$. Lastly,

$$\sigma\colon H^{k-m}(B; h^m) \longrightarrow H^{k-m}(B)$$

is canonically induced by $\mathscr{I}$ on account of the commutative diagram below:

$$\begin{array}{ccc}
A_B^{k-m}(h^m) \cong \Gamma(\mathscr{A}^{k-m} \otimes \mathscr{h}^m) & \xrightarrow{\Gamma(1 \otimes \mathscr{I})} & \Gamma(\mathscr{A}_B^{k-m}) = A_B^{k-m} \\
\Big\downarrow {\scriptstyle \delta_B^{k-m,m}} & & \Big\downarrow {\scriptstyle \delta_B^{k-m}} \\
A_B^{k-m+1}(h^m) \cong \Gamma(\mathscr{A}_B^{k-m+1} \otimes \mathscr{h}^m) & \xrightarrow{\Gamma(1 \otimes \mathscr{I})} & \Gamma(\mathscr{A}_B^{k-m+1}) = A_B^{k-m+1} \, .
\end{array}$$

Combining σ with the map $\chi h_1 h_2$ we obtain algebraic fiber integration $\Psi_1 = \sigma \chi h_1 h_2$; Ψ_1: $H_F^k(E) \to H^{k-m}(B)$.

3. Fiber integration, geometric definition ([4], chapter 7). For arbitrary manifolds B, F, and $x \in B$, $y \in F$, define

$$i_x\colon F \longrightarrow B \times F \text{ by } i_x(y) = (x, y),\ y \in F\ ,$$

$$i_y\colon B \longrightarrow B \times F \text{ by } i_y(x) = (x, y),\ x \in B\ .$$

When $\xi \in T_x(B)$, put $\hat{\xi} = (i_y)_*\xi \in T_{(x,y)}(B \times F)$ and when $\zeta \in T_y(F)$, let $\hat{\zeta} = (i_x)_*\zeta \in T_{(x,y)}(B \times F)$.

Define $\lambda_x = C^{p,q}(B \times F) \rightarrow A_c^q(F; \wedge^p T_x^*(B))$ (with the trivial connection in the product bundle $B \times F$) by

$$(\lambda_x \omega)(y; \zeta_1, \cdots, \zeta_q)(\xi_1, \cdots, \xi_p) = \omega((x, y); \hat{\xi}_1, \cdots, \hat{\xi}_p, \hat{\zeta}_1, \cdots, \hat{\zeta}_q))$$

where $x \in B$, $\omega \in C^{p,q}(B \times F)$, $\zeta_i \in T_y(F)$ and $\xi_j \in T_x(B)$. For the product bundle $B \times F$ geometric fiber integration is the linear map

$$\int_F\colon C^{p,r}(B \times F) \longrightarrow A^p(B),\ p \geqq 0\ ,$$

defined by $\left(\int_F \omega\right)(x) = \begin{cases} \int_F \lambda_x \omega,\ x \in B,\ r = m \\ 0,\ r \neq m, \end{cases}$ where $\omega \in C^{p,r}(B \times F)$.

For an arbitrary oriented bundle ξ, let $\{U_j, \phi_j\}$ be a family of trivializations as before with $\phi_j\colon U_j \times F \cong \Pi_E^{-1}(U_j)$. If $\omega \in A_F^k(E)$, $\phi_j^*\omega$ is a fiber-compact form on $U_j \times F$, so that ω_j defined by $\omega_j(x) = \deg \psi_{j,x}\left(\int_F \phi_j^*\omega\right)(x)$ is a $k-m$ form on U_j. Define $\int_F A_F^k(E) \rightarrow A^{k-m}(B)$ by $\left(\int_F(\omega)\right)(x) = \omega_j(x)$ when $x \in U_j$.

It is easily shown that this is independent of the choice of U_j, so that $\int_F$ is well defined ([4]).

Furthermore, $\int_F \delta_E = \delta_B \int_E$, so that there is an induced map

$$\Psi_2\colon H_F^k(E) \longrightarrow H^{k-m}(B)\ ,$$

$k \geqq m$, called geometric fiber integration.

4. *Proof of Theorem.* $\Psi_1 = \Psi_2$.

Let $[\omega] \in H_F^k(E)$ be represented by $\omega \in A_F^k(E)$.

If $W_j = \Pi_E^{-1}(U_j)$ is sufficiently small we may write

$$\omega / W_j = \sum_{|\alpha|+|\beta|=k} \omega^{\alpha\beta} dx_\alpha \wedge dy_\beta\ , \quad \omega^{\alpha\beta} \in C^\infty(W_j)\ ,$$

or, upon substitution of equation (8) for dy_β,

$$\omega / W_j = \sum_{\substack{|\alpha|=k-m \\ |\beta|=m}} \omega^{\alpha\beta} dx_\alpha \wedge dy_\beta^v + \sum_{\substack{|\alpha|>k-m \\ \beta}} (\cdots)^{\alpha\beta}\, dx_\alpha \wedge dy_\beta^v\ . \tag{15}$$

Since $h_2\colon H_F^k(E) \rightarrow H_F^k(E)/H_\infty^{k-m+1,m-1} = E_\infty^{k-m,m}$ is merely the projection,

$$h_2([\omega]) = \Big[\sum_{\substack{|\alpha|=k-m\\|\beta|=m}} \omega^{\alpha\beta} dx_\alpha \wedge dy^v_\beta\Big]_{E_\infty^{k-m,m}} .$$

Hence,

$$\chi h_1 h_2([\omega]) = \Big[\sum_{\substack{|\alpha|=k-m\\|\beta|=m}} [\omega^{\alpha\beta} dy^v_\beta]_{h^m}\, dx_\alpha\Big]_{H^{k-m}(B;\, h^m)}$$

and

$$\Psi_1([\omega]) = \sigma\chi h_1 h_2([\omega]) = [\mu]_{H^{k-m}(B)} ,$$

where $\mu \in A^{k-m}(B)$ with

$$\mu(x) = \sum_{|\alpha|=k-m} (\deg \psi_{j,x} \int \psi^*_{j,x}\Big(\sum_{|\beta|=m} \omega^{\alpha\beta} dy^v_\beta)\Big)\, dx^\alpha \tag{16}$$

when $x \in U_j$.

On the other hand,

$$\Psi_2([\omega]) = \Big[\int_F \omega\Big]_{H^{k-m}(B)}$$

and

$$\Big(\int_F \omega\Big)(x) = \deg \psi_{j,x} \int_F \lambda_x(\phi^*_j \omega) = \deg \psi_{j,x} \int_F \Big(\sum_{\substack{|\alpha|=k-m\\|\beta|=m}} \omega^{\alpha\beta}\, \psi_{j,x}\, dy_\beta\Big) dx_\alpha \tag{17}$$

as a short computation shows.

A comparison of (16), (17) shows that $\Psi_1 = \Psi_2$ as required.

REFERENCES

1. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958), 458-538.

2. S. S. Chern, *On the characteristic classes of complex sphere bundles and algebraic varieties*, Amer. J. Math., **75** (1953), 565-597.

3. R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, Hermann (1964).

4. W. H. Grueb, S. Halperin, and R. Vanstone, *Cohomology of Smooth Bundles I, II, III*, Academic Press (to appear).

5. A. Hattori, *A Spectral sequence in the de Rham cohomology of fibre bundles*, J. Fac. Sci. Univ. Tokyo Sect. I (1960), 289-331.

6. F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer Verlag (1966) appendix II.

7. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, volume I, Interscience (1963).

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BROCK UNIVERSITY