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A BIFURCATION THEOREM FOR k-SET CONTRACTIONS

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One of the the most often used results of bifurcation theory is the following theorem. Let C be a compact mapping from the Banach space X into itself. Suppose that C is such that $C(\theta) = \theta$ and the derivative of C exists at $x = \theta$. Then each characteristic value, μ_0, of odd multiplicity of $C'(\theta)$ is a bifurcation point of C, and to this bifurcation point there corresponds a continuous branch of eigenvectors of C. The main result in this paper will show that the above theorem can be extended to a class of non-compact mapping of the form $I - f$ where f is a k-set contraction.

In the process of proving the above mentioned theorem a characterization of topological degree for k-set contractions is obtained. This characterization can be used, and actually has been used, as an alternate definition. The theorem that yields this characterization also allows an axiomatic approach to topological degree for k-set contractions, which in turn yields a uniqueness theorem for this degree.

In § 2 we define k-set contractions and obtain several properties for k-set contractions. Some of these properties and other work concerning k-set contractions can be found in [4], [7] and [8].

Section 3 contains the main theorem, which extends the bifurcation theorem due to M. A. Krasnosel'skii as found on page 196 of [6]. Applications of this theorem will follow in a later paper. In the process of proving this theorem we obtain a characterization of topological degree that was used by Fenske as the definition. (See [5].) This definition is useful as a calculational device.

In § 4 we state axioms and use Corollary 5 to show that the topological degree of a k-set contraction is unique. The approach used here is much the same as was used in [3].

I would like to thank Robert F. Brown for his suggestions which let to the results obtained in § 4.

2. Preliminaries. Let X be a real Banach space. For any subset of X, say Ω, define the *measure of compactness* of Ω to be $\gamma(\Omega) = \inf\{d > 0 \mid \Omega$ can be covered by a finite number of sets of diameter less than or equal to $d\}$ (See [7], page 413). We shall say that g is a *k-set contraction* if given any bounded set $A \subseteqq X$, $g(A)$ is a bounded subset of X and $\gamma[g(A)] \leqq k\gamma[A]$ (See [4] and [8]). We shall consider only k-set contractions for which $k < 1$. An example of a k-set contraction is the sum of a compact map and a contraction.

Now suppose that G is an open bounded subset of X and that $f: \bar{G} \rightarrow X$ is a k-set contraction for which $a \notin (I-f)(\partial G)$. It is then possible to define the topological degree of $I - f$ at a with respect to G, denoted by $d(I - f, G, a)$, as is shown by Nussbaum in [8]. This definition of topological degree has all of the usual properties of topological degree and contains the usual definition of degree.

We next state a lemma giving some properties of k-set contractions that we shall need. Proofs of these properties can be found in [8].

LEMMA 1. *Suppose that f is a differentiable k-set contraction. Then*

(1) *$f'(x)$ is also a k-set contraction,*

(2) *the spectrum of $f'(x)$ is finite for $|\lambda| > k$, and*

(3) *$f'(x)$ is a Fredholm map of index* 0.

The above lemma is the key to the proofs given in this paper. These three properties are also satisfied by compact mappings and are, in fact, the properties that make possible much of the work done with compact mappings. It is for this reason that the class of k-set contractions seems to be a convenient setting in which to study much applied mathematics.

The last preliminary result that we shall need is that the product formula for topological degree holds for k-set contractions. We have the following proposition.

PROPOSITION 2. *Let X_1 and X_2 be real Banach spaces and $G_1 \subset X_1$ and $G_2 \subset X_2$ be bounded open sets. Suppose that $f_i: \bar{G}_i \rightarrow X_i$, $i = 1, 2$ are k-set contractions for which $\theta \notin (I_i - f_i)(\partial G_i)$, $i = 1, 2$ (where I_i denotes the identity on X_i). Then*

$$d(I - f_1 \times f_2, G_1 \times G_2, (\theta, \theta)) = d(I_1 - f_1, G_1, \theta)\, d(I_2 - f_2, G_2, \theta) ,$$

where I is the identity on $X_1 \times X_2$.

Proof. Nussbaum defined $d(I - f, G, \theta)$ to be the fixed point index of f restricted to a certain compact ANR [8]. Consequently, the the result follows from the product theorem for the fixed point index on compact ANR's [2].

3. Main result. Again let X denote a real Banach space, let f be a k-set contraction which is differentiable at $x = \theta$, and let F denote $f'(\theta)$. Without loss of generality let $f(\theta) = \theta$. (If $f(\theta) \neq \theta$, perform an appropriate translation.) We then state the following lemma.

LEMMA 3. *Let f and F be as above and suppose that μ_0, where $|\mu_0| > k$, is not a characteristic value of F. Then we can find a ball about θ, say $B \subset X$, in which there are no eigenvectors of f corresponding to eigenvalues close to μ_0.*

Proof. This theorem is proved for the case when f is compact on page 192 of [6]. Since the compactness is not used in that proof, the same proof holds for our lemma.

We next state a proposition that is helpful not only for obtaining our main theorem but also for obtaining Corollary 5, which provides a convenient method for calculating the topological degree of a k-set contraction.

PROPOSITION 4. *Let f and F be as above and suppose that 1 is not an eigenvalue of F. Then θ is an isolated solution of*

$$(I - f)(x) = \theta$$

and the index of θ is $(-1)^\beta$, where β is the sum of the multiplicities of the characteristic values of F in $(0, 1)$.

Proof. Since 1 is not an eigenvalue of F, there exists an $M > 0$ such that $\|(I-F)(y)\| \geqq M\|y\|$. Let R be such that

$$f(x) = F(x) + R(x)\,,$$

where $\|R(x)\|/\|x\| \to 0$ as $\|x\| \to 0$ ($f'(\theta) = F$ and $f(\theta) = \theta$). Choose ρ so that $x \in B_\rho$ (the ball of radius ρ about θ) implies that $\|R(x)\| \leqq M\|x\|/3$. Then

$$\begin{aligned}\|(I - f)(x)\| &\geqq \|(I - F)x\| - \|R(x)\| \\ &\geqq M\|x\| - M\|x\|/3 = (2M)/3\|x\|\,.\end{aligned}$$

Therefore, θ is an isolated solution.

If we let $H_1(x, t) = I - tf(x) - (1 - t)F(x)$, then a calculation much like the last one shows that H_1 is a homotopy such that $\theta \notin H_1(\partial B_\rho, t)$ for all t. Thus

$$d(I - f, B_\rho, \theta) = d(I - F,\ B_\rho,\ \theta)\,.$$

(Note that $d(I-F, B_\rho, \theta)$ is defined since by property (1) of Lemma 1, F is a k-set contraction.)

Now let $X_1 = \{$span of the eigenvectors of F associated with characteristic values in $(0, 1)\}$. X_1 is finite dimensional by property (2) of Lemma 1. Let β be the dimension of X_1, X^1 be the complementary space of X_1, P be the projection from X onto X_1 and set $B = B_\rho \cap X_1$ and $B^1 = B_\rho \cap X^1$. Then by the product formula for topological degree we have that

$$d(I - F, B_\rho, \theta) = d(I - f, B, \theta)\, d(I - F, B^1, \theta) .$$

Letting $H_2(x, t) = (I - tF)(x)$, we see that $H_2(x, t) = 0$ for some $t \in [0, 1]$ and $x \in \partial B^1$ if and only if F has a characteristic value in $(0, 1]$ (or $I(x) = 0$ on ∂B^1). But in X^1 this is clearly impossible. Thus

$$d(I - F, B^1, \theta) = d(I, B^1, \theta) = 1 .$$

If we let $H_3(x, t) = ((2t - 1)\, I - tF)\,(x)$, we see that $H_3(x, t) \neq \theta$ on ∂B and hence that

$$d(I - F, B, \theta) = d(-I, B, \theta) .$$

But $d(-I, B, \theta) = (-1)^\beta$. Thus $d(I - f, B_\rho, \theta) = (-1)^\beta$.

In [5] Fenske gives an alternate definition of the topological degree of a differentiable k-set contraction at a regular point and then proceeds to show that the new definition of topological degree satisfies all of the usual properties. It is not shown in [5] that the two definitions are equivalent. Using Proposition 4 we obtain this definition as a corollary.

COROLLARY 5. *Consider the open bounded subset G of the Banach space X. Suppose that $f\colon \bar{G} \to X$ is a differentiable k-set contraction. Given a point y in X, there is a regular point y' of f such that*

$$d(I - f, G, y) = \sum_{x \in (I-f)^{-1}(y')} (-1)^{\beta(f'(x))} ,$$

where $\beta(f'(x))$ denotes the sum of the multiplicities of the characteristic values of $f'(x)$ in $(0, 1)$.

Proof. By the Sard-Smale theorem, we may choose y' close enough to y so that $d(I - f, G, y) = d(I - f, G, y')$. Since $(I - f)^{-1}(y')$ is finite, the additivity property of degree (see § 4), a translation back to θ, and Proposition 4 then complete the proof.

We now proceed to state and prove our main result.

THEOREM 6. *Let f be a k-set contraction from the Banach space X into itself such that $f(\theta) = \theta$ and $f'(\theta)$ exists. Then each characteristic value, $|\mu_0| > k$, of odd multiplicity of $f'(\theta)$ is a bifurcation* *point of, and to this bifuraction point there corresponds a continuous branch of eigenvectors of f.*

Proof. For any $\varepsilon > 0$ there exists by Lemma 3 a ball with center at θ in which the only solution of the equations

$$(I - \mu_0 - \varepsilon)f)(x) = \theta$$

and $(I - (\mu_0 + \varepsilon)f)(x) = \theta$ is $x = \theta$. By Proposition 4 we can find a ball $B^* \subset B$ containing θ so that

$$d(I - (\mu_0 - \varepsilon)f, B^*, \theta) = -d(I - (\mu_0 + \varepsilon)f, B^*, \theta) .$$

Therefore, $I - (\mu_0 - \varepsilon)f$ and $I - (\mu_0 + \varepsilon)f$ are not homotopic, and hence there exists an $x \in \partial B^*$ and a $\mu \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon)$ such that $(I - \mu f)(x) = \theta$. This completes our proof.

It is sometimes convenient to consider a bifurcation problem in the form $(h - \mu f)(x) = \theta$, where h is a homeomorphism. We note that if we add the conditions that h is such that (1) $f \circ h^{-1}$ is a k'-set contraction for $k' < 1$ and (2) $h'(\theta)$ exists and is such that $h'(\theta)$ is a homeomorphism, then Theorem 6 will remain true.

As an application of Theorem 6 for which the case for compact maps will not apply we consider the Hartree equation for the Helium atom (See [10]). The Hartree equation is of the form

$$* \qquad -\frac{1}{2}\Delta u - \frac{2}{\|x\|}u + u\int\frac{u^2(t)}{|t - x|}\,dt = \lambda u,\ u \in L(R^3)$$

and can be represented in form $\lambda u = G(u)$ where G is not compact. Since G is of the form $L + H$ where L is compact and H satisfies $\|H(x) - H(y)\| \leqq M(x, y)\|x - y\|$ where $M(x, y) \to 0$ as $(x, y) \to (\theta, \theta)$, it is easy to see that there exists a neighborhood of the origin in which G is a k-set contraction. Thus we see that at the odd eigenvalues of equation * we obtain a continuous branch of eigenvectors.

4. The uniqueness of topological degree. While discussing the results obtained by Fenske in [5], it was suggested that it might be of interest to obtain a set of axioms for topological degree on Banach spaces out of which we could obtain a uniqueness theorem analogous to the result in [3]. In this section we shall obtain such a theorem for differentiable mappings on Banach spaces.

Suppose that G is a bounded, open subset of the Banach space X. Consider $I - f: \bar{G} \to X$ where f is a differentiable k-set contraction. We suppose that a degree of $I - f$ with respect to G at y (for $y \notin (I - f)(\partial G)$), denoted by $d(I - f, G, y)$, is defined and satisfies the following properties.

(1) (The Additivity Property). If all solutions of $(I-f)(x) = y$ in G are contained in $\bigcup_{i=1}^{n} G_i \subset G$, where the G_i are bounded, open and pairwise disjoint sets for which $y \notin \partial(I-f)(G_i)$, then

$$d(I-f, G, y) = \sum_{i=1}^{n} d(I-f, G_i, y) .$$

(2) (The Homotopy Property). If $H: \bar{G} \times [0, 1] \to X$ is such that H is a k-set contraction and such that $H(x, t) \neq y$ for $x \in \partial G$ for all $t \in [0, 1]$, then $d(I - H(\circ, t), G, y) =$ constant for all $t \in [0, 1]$.

(3) (The Normalization Property). $d(I, G, y) = 1$ $y \in G$.

(4) (The Product Property). If f_1, f_2, G_1, G_2, y_1 and y_2 are such that $d(I-f_1, G_1, y_1)$ and $d(I-f_2, G_2, y_2)$ are defined, then

$$d(I - f_1 \times f_2, G_1 \times G_2, (y_1, y_2))$$

is also defined and

$$d(I - f_1 \times f_2, G_1 \times G_2, (y_1, y_2)) = d(I - f_1, G_1, y_1) d(I - f_2, G_2, y_2) .$$

The next step is to show that

** $$d(I-f, G, y) = \sum_{x \in (I-f)^{-1}(y_1)} (-1)^{\beta(f'(x))} ,$$

where y_1 is any regular point of f in a sufficiently small neighborhood of y, follows from properties (1) – (4) stated above. If any topological degree satisfying properties (1) – (4) can be expressed as in **, then it is surely unique.

We note that the expression ** was obtained in Corollary 5. Thus we need only to verify that Proposition 4 and Corollary 5 follow from the properties (1) – (4) stated above. A review of these proofs shows that besides the steps that follow from the properties of k-set contractions, most of the steps are clearly consequences of properties (1) – (4). The major exception is the statement that $d(-I, B, \theta) = (-1)^{\beta}$ where B is a ball in β dimensional space. This can very easily be seen to be true considering the definition of finite dimensional topological degree via the Jacobian. However, for the purposes of this section, that approach is inadequate. For this reason we prove the following lemma.

LEMMA 7. *Let B and β be as above. Then*

$$d(-I, B, \theta) = (-1)^{\beta} .$$

Proof. We began by using the additivity property to reduce $d(-I, B, \theta)$ to $d(-I, C, \theta)$ where

$$C = \{(x_1, \cdots, x_\beta) \mid |x_i| < 1, i = 1, \cdots, \beta\} .$$

We then repeatedly use the product property to reduce our problem to that of showing that $d(-I, (-1, 1), \theta) = -1$. To prove this we consider the following function

$$g(x) = \begin{cases} -x & x \in (-1, 1) \\ x - 2 & x \in (1, 4) \,. \end{cases}$$

The additivity property yields

$$d(g, (-1, 4), \theta) = d(-I, (-1, 1), \theta) + d(x - 2, (1, 4), \theta) \,.$$

It is then easy to show that g is homotopic to the function $(x+6)/5$. Thus $d(g, (-1, 4), \theta) = d((x + 6)/5, (-1, 4), \theta)$.

We next need the following consequence of the additivity axiom: if $(I-f)(x) = y$ has no solutions in G, then $d(I-f, G, y) = 0$. This property is used to show that $d((x + 6)/5, (-1, 4), \theta) = 0$. Thus we have that $d(g, (-1, 4), \theta) = 0$.

We also note by the additivity property

$$d(x - 2, (-1, 4), \theta) = d(x - 2, (-1, 1), \theta) + d(x - 2, (1, 4), \theta) \,.$$

Again, as above, the additivity property implies that

$$d(x - 2, (-1, 1), \theta) = 0 \,.$$

On the other hand $x - 2$ is homotopic to I on $[-1, 4]$ without solutions on the boundary, so

$$d(x - 2, (-1, 4), \theta) = d(I, (-1, 4), \theta) = 1$$

by the homotopy property and property (3).

The above calculations then imply that

$$d(-I, (-1, 1), \theta) = -1$$

which is what we were to prove.

Thus using Lemma 7 along with the proofs of Proposition 4 and Corollary 5 we see that a topological degree for differentiable k-set contractions which satisfies properties (1) – (4) is unique. It is possible to remove the differentiability condition if we consider a Hilbert space by using the Weierstrauss approximation theorem which can be found in [9]. It is unknown whether this can be done for an arbitrary Banach space.

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