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INVERSE SYSTEMS OF GROUP-VALUED MEASURES

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In this paper a basic theory is developed for inverse (or projective) systems of group-valued measures. This theory parallels the one for nonnegative measures. However, many of the results are new even in the real case.

The main tools for dealing with group-valued measures are the concepts and results given by Sion in "Outer measures with values in a topological group", Proc. London Math. Soc., 19 (1969), 89–106. When dealing with inverse systems the point of view adopted is that of Mallory and Sion, "Limits of inverse systems of measures". Ann. Inst. Fourier, Tome 21, Fasc. 1 (1971) 25–57. This viewpoint involves finding a limit measure first on a large space A and then studying conditions under which this will yield a limit measure on some subset of A. By introducing the concept of almost-sequential maximality, this paper not only extends known results but is also able to indicate a connection between "abstract" and "topological" methods for producing a limit measure.

In the last section the results obtained are applied to cylinder measures. Here again the viewpoint adopted differs somewhat from the usual one, even for nonnegative measures, and enables one to study a variety of possibilities for a target space on which to place a limit measure.

0. Notation and basic notions. Throughout this paper, ω is the set of nonnegative integers, $\boldsymbol{R}$ is the real line, Γ is a commutative, complete, Hausdorff, topological group with identity o under the operations $+$ and $-$.

For any sets A and B,

$$A \sim B = \{x\colon x \in A \text{ and } x \notin B\}\ .$$

For any subsets A and B of Γ and $n \in w$,

$$A + B = \{x + y\colon x \in A \text{ and } y \in B\}\ ,$$

$$nA = A + \cdots + A \qquad (n \text{ terms})\ .$$

DEFINITION 0.1. For any family $\mathscr{H}$ of sets, $\mathscr{H}$ is *ω-compact* iff every countable subfamily of $\mathscr{H}$ with the finite intersection property has a nonempty intersection.

DEFINITIONS 0.2. For any function ξ on the family of all subsets of some space Ω to Γ,

(1) A is *ξ-measurable* iff $A \subset \Omega$ and, for every

$$T \subset \Omega,\ \xi(T) = \xi(T \cap A) + (T \sim A)\ .$$

(2) $M_\xi = \{A\colon A \text{ is } \xi\text{-measurable}\}$.

(3) A is *ξ-null* iff $A \subset \Omega$ and, for every $\alpha \subset A$, $\xi(\alpha) = o$.

(4) ξ is an *outer measure* on Ω iff.

  (i) M_ξ is a σ-field and ξ is σ-additive on M_ξ,

  (ii) for any $A \subset \Omega$, $\xi(A) =$ limit $\xi(\alpha)$ as α runs over $\{\alpha \in M_\xi\colon A \subset \alpha\}$ directed by $\supset$.

DEFINITION 0.3. For any function τ on a family $\mathscr{A}$ of sets to Γ and $\mathscr{H} \subset \mathscr{A}$, $\mathscr{H}$ is an *inner family* for τ iff, for every $A \in \mathscr{A}$ and neighborhood U of $\tau(A)$, there exists $H \in \mathscr{H}$ such that $H \subset A$ and, for every $\alpha \in \mathscr{A}$,

$$H \subset \alpha \subset A \Longrightarrow \tau(\alpha) \in U\ .$$

DEFINITION 0.4. For any topological space Ω, ξ is a *Radón outer measures* on Ω iff ξ is an outer measure on Ω such that

(i) closed sets are ξ-measurable,

(ii) the closed, compact sets form an inner family for (ξ/M_ξ).

In the sequel, we shall need the following theorems which generalize well-known results for real-valued measures.

THEOREM 0.5. *Let $\mathscr{A}$ be a field of subsets of a space Ω and τ be a σ-additive function on $\mathscr{A}$ to Γ such that, for any monotone sequence*[1] *α in $\mathscr{A}$, $\lim_n \tau(\alpha_n) \in \Gamma$. If*

$$\tau'(\cup \alpha_n) = \lim_n \tau(\alpha_n) \qquad \textit{for } \alpha_n \subset \alpha_{n+1} \in \mathscr{A}$$

and, for any $A \subset \Omega$, $\tau^*(A) =$ limit $\tau'(\beta)$ as β runs over

$$\{\beta \in \mathscr{A}_\sigma\colon A \subset \beta\} \qquad \textit{directed by } \supset$$

then τ^* is an outer measure on Ω such that $\mathscr{A} \subset M_{\tau^*}$ and $\tau^*/\mathscr{A} = \tau$.

Proof. See Sion [11] Theorem 3.3.

THEOREM 0.6. *Let Ω be a regular topological space, $\mathscr{K}$ be the family of closed, compact subsets of Ω, and τ be a σ-additive function on $\mathscr{K}$ to Γ such that*

(i) *for any ascending sequence C in $\mathscr{K}$, $\lim_n(C_n) \in \Gamma$,*

(ii) *for any neighborhood U of o and $C \in \mathscr{K}$, there exists an open G such that $C \subset G$ and, for every $C' \in \mathscr{K}$,*

[1] Depending on context, letters (α, E, etc.) may denote a single set or a sequence ($\{\alpha_n\}$, $\{E_n\}$) of sets.

$$C \subset C' \subset G \Longrightarrow \tau(C) - \tau(C') \in U .$$

If, for any open G, $\rho'(G) =$ limit $\tau(C)$ *as C runs over*

$\{C \in \mathscr{K}: C \subset G\}$ *directed by* $\subset$ *and, for any* $A \subset \Omega$,

$\rho(A) =$ limit $\rho'(G)$ *as G runs over*

$\{G$: *G is open and* $A \subset G\}$ *directed by* $\supset$,

then ρ *is a Radón outer measure on* Ω *and* $(\rho/\mathscr{K}) = \tau$.

Proof. See Sion [11] Theorem 6.3.

LEMMA 0.7. *For any outer measure* ξ, $A \in M_\xi$ *iff, for every neighborhood U of o, there exist* A', $A'' \in M_\xi$ *such that* $A' \subset A \subset A''$ *and* $(\alpha \subset A'' \sim A' \Rightarrow \xi(\alpha) \in U)$.

1. Inverse systems of abstract measures. Throughout this paper,
$\mathscr{F}$ is an index set directed by a relation $<$;
(S, r) is an inverse system of spaces indexed by $(\mathscr{F}, <)$, i.e.,
S_E is an abstract space for $E \in \mathscr{F}$,
$r_{E,F}$: $S_F \to S_E$ is surjective for $E < F$,
$r_{E,E}$ is the identity map,
$r_{EG} = r_{EF} \circ r_{FG}$ for $E < F < G$;
(Λ, p) is a limit of (S, r), i.e.,
Λ is an abstract space,
p_E: $\Lambda \to S_E$ is surjective for $E \in \mathscr{F}$,
$p_E = r_{EF} \circ p_F$ for $E < F$;
μ is a system of outer measures on S, i.e., for each $E \in \mathscr{F}$,
μ_E is a Γ-valued outer measure on S_E and $M_E = M_{\mu_E}$.

DEFINITIONS 1.1. (1) μ is an *inverse system of measures over* (S, r) iff, for $E, F \in \mathscr{F}$ with $E < F$ and $A \in M_E$,

$$r_{EF}^{-1}[A] \in M_F \quad \text{and} \quad \mu_F(r_{EF}^{-1}[A]) = \mu_E(A) .$$

(2) μ is *monotone* iff, for any ascending sequence E in $\mathscr{F}$ and any sequence A with

$$A_n \in M_{E_n} \quad \text{and} \quad r_{E_n E_{n+1}}^{-1}[A_n] \subset A_{n+1} \text{ for } n \in \omega ,$$

we have $\lim_n \mu_{E_n}(A) \in \Gamma$.

Note: Any inverse system of nonnegative measures is monotone.

DEFINITION 1.2. For any $\Omega \subset \Lambda$,
(1) ξ is a *limit* of μ on Ω iff ξ is a Γ-valued outer measure on

Ω such that, for every $E \in \mathscr{F}$ and $A \in M_E$,

$$(\Omega \cap p_E^{-1}[A]) \in M_\xi \qquad \text{and} \qquad \xi(\Omega \cap p_E^{-1}[A]) = \mu_E(A) .$$

(2) $\operatorname{Cyl} \Omega = \{\Omega \cap p_E^{-1}[A];\ E \in \mathscr{F} \text{ and } A \in M_E\}$.

(3) τ_Ω is the function τ on $\operatorname{Cyl} \Omega$ such that

$$\tau(\Omega \cap p_E^{-1}[A]) = \mu_E(A) \qquad \text{for } E \in \mathscr{F} \text{ and } A \in M_E .$$

(4) τ_Ω^* is the outer measure on Ω generated by τ_Ω as in Theorem 0.5.

REMARKS. (1) $\operatorname{Cyl} \Omega$ is a field of subsets of Ω.

(2) τ_Ω is well defined iff, for every $E \in \mathscr{F}$ and $A \in M_E$,

$$\Omega \cap p_E^{-1}[A] = \phi \Longrightarrow \mu_E(A) = o .$$

(This will clearly be the case if $p_E[\Omega] = S_E$ for $E \in \mathscr{F}$).

(3) When τ_Ω is well defined, it is finitely additive. Moreover, if μ is a monotone inverse system of measures then, for any monotone sequence α in $\operatorname{Cyl} \Omega$, $\lim_n \tau_\Omega(\alpha_n) \in \Gamma$. For such a μ therefore, by Theorem 0.5, we see that τ_Ω^* is well defined iff τ_Ω is σ-additive, in which case τ_Ω^* is an outer measure on Ω which extends τ_Ω.

Assumption 1.3. For the remainder of this section, we suppose μ is a monotone inverse system of measures over (S, r).

Remark (3) above then yields immediately the basic result.

LEMMA 1.4. *For any $\Omega \subset \Lambda$, there exists a limit of μ on Ω iff τ_Ω is σ-additive, in which case τ_Ω^* is such a limit.*

In view of the above lemma, all the known theorems about the existence of a limit of μ on Λ when $\Gamma = \boldsymbol{R}$ can be extended to the general situation discussed here with little or no difficulty. For the remainder of this section, we consider a problem which has received little attention in the literature, except in special cases, namely that of finding conditions under which the existence of a limit of μ on Λ implies the existence of a limit of μ on a subset of Λ.

We first note the following key lemma.

LEMMA 1.5. *Suppose τ_Λ is σ-additive, $\lambda = \tau_\Lambda^*$ and $\Omega \subset \Lambda$. Then τ_Ω is σ-additive iff $\lambda(\Omega \cap B) = \lambda(B)$ for $B \in \operatorname{Cyl} \Lambda$.*

Proof. (1) Suppose τ_Ω is σ-additive and $B \in \operatorname{Cyl} \Lambda$.

For any ascending sequence β in $\operatorname{Cyl} \Lambda$ with

$$\Omega \cap B \subset \bigcup_{n \in \omega} \beta_n \subset B$$

since $\Omega \cap \beta_n \in \mathrm{Cyl}\,\Omega$, we have

$$\lim_n \tau_\Lambda(\beta_n) = \lim_n \tau_\Omega(\Omega \cap \beta_n) = \tau_\Omega(\Omega \cap B) = \tau_\Lambda(B) = \lambda(B) .$$

We conclude therefore $\lambda(\Omega \cap B) = \lambda(B)$.

(2) If $\lambda(\Omega \cap B) = \lambda(B)$ for $B \in \mathrm{Cyl}\,\Lambda$ then the restriction of λ to the subsets of Ω is clearly a limit of μ on Ω and hence τ_Ω is σ-additive.

We then have the following.

THEOREM 1.6. *Suppose τ_Λ is σ-additive, $\lambda = \tau_\Lambda^*$ and $\Omega \subset \Lambda$. If, for any ascending sequence E in $\mathscr{F}$, the set*

$$N(E) = \{f \in \Lambda\colon \text{There does not exist } g \in \Omega \text{ with } p_{E_n}(g) = p_{E_n}(f) \text{ for all } n \in \omega\}$$

is λ-null then τ_Ω is σ-additive.

Proof. We shall show that the hypothesis implies that any sequence β in $\mathrm{Cyl}\,\Lambda$ which covers Ω must cover almost all of Λ. Indeed, let $\beta_n = p_{E_n}^{-1}[A_n]$ with $E_n < E_{n+1}$ and $A_n \in M_{E_n}$ for $n \in \omega$ and $\Omega \subset \bigcup_{n \in \omega} \beta_n$.

Then we must have

$$\Lambda \sim \bigcup_{n \in \omega} \beta_n \subset N(E)$$

for, if $f \in \Lambda$ and there exists $g \in \Omega$ with $p_{E_n}(g) = p_{E_n}(f)$ for all $n \in \omega$ then, since $g \in \beta_m$ for some $m \in \omega$, we have

$$p_{E_m}(f) = p_{E_m}(g) \in A_m , \quad \text{so} \quad f \in p_{E_m}^{-1}[A_m] = \beta_m .$$

Since $N(E)$ is λ-null, we conclude

$$\lambda(\bigcup_{n \in \omega} \beta_n) = \lambda(\Lambda)$$

and therefore, for any $B \in \mathrm{Cyl}\,\Lambda$, $\lambda(\Omega \cap B) = \lambda(B)$. Application of Lemma 1.5. then yields the desired conclusion.

We now state conditions on Ω in terms of the system μ, rather than in terms of the limit λ, which guarantee the existence of a limit of μ on Ω.

DEFINITION 1.7. For any neighborhood U of o, $E \in \mathscr{F}$ and $A \subset S_E$, A is *U-small* iff, for every $F \in \mathscr{F}$ with $E < F$,

$$\alpha \subset r_{EF}^{-1}[A] \Longrightarrow \mu_F(\alpha) \in U .$$

DEFINITIONS 1.8. For any $\Omega \subset \Lambda$,

(1) Ω is *sequentially maximal* iff, for every ascending sequence E in $\mathscr{F}$ and sequence f with

$$f_n \in S_{E_n} \quad \text{and} \quad r_{E_n E_{n+1}}(f_{n+1}) = f_n \qquad \text{for } n \in \omega ,$$

thers exists a $g \in \Omega$ with $p_{E_n}(g) = f_n$ for all $n \in \omega$.

(2) Ω is *almost sequentially maximal* iff, for every neighborhood U of o and ascending sequence E in $\mathscr{F}$, there exists a sequence A such that

(i) $A_n \in M_{E_n}$, $r^{-1}_{E_n E_{n+1}}[A_n] \subset A_{n+1}$ and A_n is U-small for $n \in \omega$,

(ii) for any sequence f with

$$f_n \in S_{E_n} \sim A_n \quad \text{and} \quad r_{E_n E_{n+1}}(f_{n+1}) = f_n \qquad \text{for } n \in \omega,$$

there exists a $g \in \Omega$ with $p_{E_n}(g) = f_n$ for all $n \in \omega$,

DEFINITIONS 1.9. For any family $\mathscr{C}$ of subsets of Λ,

(1) μ is *$\mathscr{C}$-tight* iff, for every neighborhood U of o, there exists $C \in \mathscr{C}$ such that $(S_E \sim p_E[C])$ is U-small for every $E \in \mathscr{F}$.[2]

(2) μ is *sequentially $\mathscr{C}$-tight* iff, for every nighborhood U of o and ascending sequence E in f, there exists $C \in \mathscr{C}$ such that $(S_{E_n} \sim p_{E_n}[C])$ is U-small for every $n \in \omega$.

REMARK. The condition of sequential maximality is extensively used in the literature in connection with finding a limit of μ, especially when dealing with abstract measures. (see, e.g. 2, 3, 4, 5).

However, as the observation at the end of § 3 shows, such a condition is much too strong to be useful in many applications. Hence, when dealing with topological measures, one finds the condition of tightness frequently used (see, e.g. 7, 8, 9, 10).

No relation seems to exist between the two approaches. By using the weaker concepts of almost sequential maximality and of sequential tightness, introduced above, we show the connection between the two notions while extending known results even when $\Gamma = \mathbf{R}$.

THEOREM 1.10. *If μ has a limit on Λ and $\Omega \subset \Lambda$ is almost sequentially maximal then μ has a limit on Ω.*

Proof. We shall show that the hypothesis of Theorem 1.6 is satisfied. Let $\lambda = \tau^*_\Lambda$, E be an ascending sequence in $\mathscr{F}$ and

$$\beta \subset \{f \in \Lambda\text{: there does not exist } g \in \Omega \text{ with } p_{E_n}(g) = p_{E_n}(f) \text{ for all } n \in \omega\} .$$

To see that $\lambda(\beta) = o$, given any neighborhood U of o, let A be

[2] See Ch. I of the forthcoming book by Laurent Schwartz "Radón measures on topological spaces." Tata Institute, Bombay.

a sequence satisfying Conditions (i) and (ii) of Definition 1.8.2. Then

$$\beta \subset \bigcup_{n \in \omega} p_{E_n}^{-1}[A_n] ,$$

for any $k \in \omega$,

$$\bigcup_{n=0}^{k} p_{E_k}^{-1}[A_n] = p_{E_k}^{-1}[A_k]$$

and for any $\alpha \in \mathrm{Cyl}\, \Lambda$,

$$\alpha \subset p_{E_k}^{-1}[A_k] \Longrightarrow \tau_\Lambda(\alpha) \in U .$$

Hence $\lambda(\beta) \in$ closure U. Since U is arbitrary, we conclude $\lambda(\beta) = o$.

THEOREM 1.11. *Suppose $\Omega \subset \Lambda$ and $\mathscr{C}$ is an ω-compact family of subsets of Ω such that, for $C \in \mathscr{C}$, $E \in \mathscr{F}$ and $f \in S_E$, $p_E[C] \in M_E$ and $(C \cap p_E^{-1}[f]) \in \mathscr{C}$. If μ is sequentially $\mathscr{C}$-tight then Ω is almost sequentially maximal.*

Proof. Given a neighborhood U of o and an ascending sequence E in $\mathscr{F}$, choose $C \in \mathscr{C}$ so that

$$S_{E_n} \sim p_{E_n}[C] \text{ is } U\text{-small} \qquad \text{for } n \in \omega ,$$

and let

$$A_n = S_{E_n} \sim p_{E_n}[C] .$$

Since

$$p_{E_n}[C] = r_{E_n E_{n+1}}[p_{E_{n+1}}[C]] ,$$

we have

$$r_{E_n E_{n+1}}^{-1}[A_n] \subset A_{n+1} .$$

Given any sequence f with

$$f_n \in S_{E_n} \sim A_n \qquad \text{and} \qquad r_{E_n E_{n+1}}(f_{n+1}) = f_n ,$$

we see that, for any $k \in \omega$,

$$\phi \neq C \cap p_{E_k}^{-1}[f_k] \subset C \cap \bigcap_{n=0}^{k} p_{E_n}^{-1}[f_n] .$$

Therefore

$$\bigcap_{n \in \omega} (C \cap p_{E_n}^{-1}[f_n]) \neq \phi$$

i.e., there exists $g \in C \subset \Omega$ with $p_{E_n}(g) = f_n$ for all $n \in \omega$.

COROLLARY 1.12. *Suppose μ has a limit on Λ, $\Omega \subset \Lambda$ and $\mathscr{C}$ is* *an ω-compact family of subsets of Ω such that, for $C \in \mathscr{C}$, $E \in \mathscr{F}$ and $f \in S_E$, $p_E[C] \in M_E$ and $C \cap p_E^{-1}[f] \in \mathscr{C}$. If μ is sequentially $\mathscr{C}$-tight then μ has a limit on Ω.*

2. Radón systems. We suppose now that, for each $E \in \mathscr{F}$, S_E is a topological space and $\mathscr{K}_E$ is the family of closed, compact subsets of S_E.

DEFINITION 2.1. μ is a *Radón system* iff, μ is a monotone, inverse system of measures such that, for each $E \in \mathscr{F}$,

(i) closed subsets of S_E are μ_E-measurable and

(ii) for every $A \in M_E$ and neighborhood U of o, there exists $C \in \mathscr{K}_E$ such that $C \subset A$ and $(A \sim C)$ is U-small.

For Radón systems, we have the following fundamental result.

THEOREM 2.2. *Let μ be a Radon system and r_{EF} be continuous for $E < F$. If Λ is almost sequentially maximal then τ_Λ is σ-additive.*

Proof. Let β be a descending sequence in Cyl Λ with

$$\lim_n \tau_\Lambda(\beta_n) \neq o .$$

We shall show that $\bigcap_{n \in \omega} \beta_n \neq \phi$. Choose a neighborhood U of o and $N \in \omega$ so that $\tau_\Lambda(\beta_n) \notin 3U$ for $n > N$ and let

$$\beta_{N+n} = p_{E_n}^{-1}[B_n] \quad \text{with} \quad E_n < E_{n+1} \quad \text{and} \quad B_n \in M_{E_n} .$$

Next, choose a sequence A satisfying Conditions (i) and (ii) of Definition 1.8.2 of almost sequential maximality. Thus,

$$\mu_{E_n}(B_n \sim A_n) \notin 2U \quad \text{and} \quad (B_{n+1} \sim A_{n+1}) \subset r_{E_n E_{n+1}}^{-1}[B_n \sim A_n] .$$

Let V be a sequence of neighborhoods of o with

$$\sum_{i=0}^{n} V_i \subset U \qquad \text{for } n \in \omega$$

and, by recursion, choose $C_n \in \mathscr{K}_{E_n}$ so that $C_n \subset B_n \sim A_n$,

$$C_{n+1} \subset r_{E_n E_{n+1}}^{-1}[C_n] ,$$

$$\mu_{E_n}(C_n) + \sum_{i=0}^{n} V_i \not\subset U \quad \text{so} \quad C_n \neq \varnothing .$$

Since $\{p_{E_n}^{-1}[C_n]; n \in \omega\}$ forms a filter base, let $\mathscr{H}$ be the ultra filter induced by it. Then $p_{E_n}[\mathscr{H}]$ is an ultrafilter in C_n and hence there exits $f_n \in C_n$ with $f_n =$ limit $p_{E_n}[\mathscr{H}]$. Since $r_{E_n E_{n+1}}$ is continuous, we have

$$r_{E_n E_{n+1}}(f_{n+1}) = f_n$$

hence there exists $g \in \Lambda$ such that $p_{E_n}(g) = f_n$ for $n \in \omega$, i.e.,

$$g \in \bigcap_{n \in \omega} p_{E_n}^{-1}[C_n] \subset \bigcap_{n \in \omega} \beta_n .$$

REMARKS. Even when $\Gamma = \boldsymbol{R}$, the above theorem extends known results in that it uses the weaker hypothesis of almost sequential maximality instead of sequential maximality. As we shall see, this is crucial in connecting "topological" and "abstract" methods for getting limits. Variations of it involving weaker conditions on the r_{EF} and replacing the K_E by more general families as in [3], [4], can also be obtained by using slightly different arguments than those used in the above proof.

We now turn our attention to some $\Omega \subset \Lambda$ on which a topology is given and try to determine when μ has a Radon limit measure on Ω.

Assnmptions 2.3. For the remainder of this section, we assume S_E is a Hausdorff, regular, topological space for $E \in \mathscr{F}$; r_{EF} is continuous for $E < F$;

$$f, g \in \Lambda \text{ and } f \neq g \Longrightarrow p_E(f) \neq p_E(g) \text{ for some } E \in \mathscr{F};$$

$\Omega \subset \Lambda$, Ω is a regular topological space and p_E/Ω is continuous for $E \in f$; μ is a Radon system.

Thus, Ω is a Hausdorff space and we let $\mathscr{K}_\Omega$ be the family of compact subsets of Ω.

Our main tool for constructing a Radon measure on Ω is Theorem 0.6. The following lemma enables us to check that the hypotheses of the theorem are satisfied.

LEMMA 2.4. *Suppose τ_Λ is σ-additive and $\lambda = \tau_\Lambda^*$. Then*

(1) *λ is σ-additive on $\mathscr{K}_\Omega$.*

(2) *for any $C \in \mathscr{K}_\Omega$ and neighborhood U of o, there exists an open $G \in \mathrm{Cyl}\,\Omega$ such that $C \subset G$ and*

$$C \subset A \subset G \Longrightarrow \lambda(C) - \lambda(A) \in U ,$$

(3) *for any $C \in \mathscr{K}_\Omega$,*

$$\lambda(C) = \textit{limit } \mu_F(p_F[C]) \textit{ as } F \textit{ runs over } \mathscr{F} .$$

Proof. Let $\mathscr{C}$ denote the family of subsets of Λ which are compact in the weakest topology on Λ under which p_E is continuous for all $E \in \mathscr{F}$. Then $\mathscr{K}_\Omega \subset \mathscr{C}$.

(1) We first check that λ is finitely additive on $\mathscr{C}$. Indeed, given disjoint $C, C' \in \mathscr{C}$, since the open elements in $\mathrm{Cyl}\,\Lambda$ form a base for the topology on Λ, we can find a $B \in \mathrm{Cyl}\,\Lambda$ with $C \subset B$ and $C' \cap B = \emptyset$. Since $B \in M_\lambda$, we get $\lambda(C \cup C') = \lambda(C) + \lambda(C')$. The σ-additivity of λ on K_Ω then follows from the fact that, for any ascending sequence α, $\lambda(\bigcup_{n \in \omega} \alpha_n) = \lim_n \lambda(\alpha_n)$ (see Sion [11] Th. 3.3]).

(2) Given $C \in \mathscr{C}$ and neighborhood U of o, from the definition of λ, there exists a sequence B in $\mathrm{Cyl}\,\Lambda$ which covers C and such that

$$C \subset A \subset \bigcup_{n \in \omega} B_n \Longrightarrow \lambda(A) - \lambda(C) \in U .$$

Let V be a sequence of neighborhoods of o with

$$\sum_{n=0}^{k} V_n \subset U \qquad \text{for } k \in \omega ,$$

$B_n = p_{E_n}^{-1}[\beta_n]$ with $E_n \in \mathscr{F}$ and $\beta_n \in M_{E_n}$.

Since μ is a Radon system, for each $n \in \omega$, there exists an open $\gamma_n \subset S_{E_n}$ such that $\beta_n \subset \gamma_n$ and $(\gamma_n \sim \beta_n)$ is V_n-small. Since $C \in \mathscr{C}$ there exists $k \in \omega$ such that if

$$G = \bigcup_{n=0}^{k} p_{E_n}^{-1}[\gamma_n]$$

then $C \subset G$. Moreover G is open, $G \in \mathrm{Cyl}\,\Lambda$ and

$$C \subset A \subset G \Longrightarrow \lambda(A) - \lambda(C) \in 2\,U .$$

Note that if $C \subset \Omega$ then $C \subset (G \cap \Omega)$, $(G \cap \Omega)$ is open in Ω, and $(G \cap \Omega) \in \mathrm{Cyl}\,\Omega$.

(3) Choose any $E \in \mathscr{F}$ with $E_n < E$ for $n = o, \cdots, k$ and let

$$\gamma = \bigcup_{n=0}^{k} r_{E_n E}^{-1}[\gamma_n] .$$

Then $G = p_E^{-1}[\gamma]$ and, for any $F \in \mathscr{F}$ with $E < F$ and any $\alpha \in M_F$,

$$\begin{aligned} p_F[C] \subset \alpha \subset r_{EF}^{-1}[\gamma] &\Longrightarrow C \subset p_F^{-1}[\alpha] \subset p_E^{-1}[\gamma] = G \\ &\Longrightarrow \lambda(p_F^{-1}[\alpha]) - \lambda(C) \in 2\,U \\ &\Longrightarrow \mu_F(\alpha) - \lambda(C) \in 2\,U . \end{aligned}$$

and therefore

$$\mu_F(p_F[C]) - \lambda(C) \in 3U .$$

REMARKS. When τ_Λ is σ-additive and $\lambda = \tau_\Lambda^*$, (1) in view of Lemma 2.4 and theorem 0.6, we see that $(\lambda/\mathscr{K}_\Omega)$ generates a Radon outer measure ρ on Ω.

(2) Lemma 2.4.3 gives a characterization of $(\lambda/\mathscr{K}_\Omega)$ directly in terms of μ, which poins out that, when $\Gamma = \boldsymbol{R}$, the processes for constructing a Radon limit measure followed respectively by Mallory and Sion in [4] and by C. Scheffer in [9] are essentially the same.

Putting all the pieces together, we get the following.

THEOREM 2.5. *μ has a Radon limit on Ω iff μ is $\mathscr{K}_\Omega$-tight.*[3]

Proof. (1) Suppose μ has a Radon limit ρ on Ω. Then, for any neighborhood U of o, there exists $C \in \mathscr{K}_\Omega$ such that

$$\alpha \subset \Omega \sim C \Longrightarrow \rho(\alpha) \in U .$$

For any $E \in \mathscr{F}$, since

$$\Omega \cap p_E^{-1}[S_E \sim p_E[C]] \subset \Omega \sim C ,$$

we see that $(S_E \sim p_E[C])$ is U-small. Thus, μ is $\mathscr{K}_\Omega$-tight.

(2) Suppose μ is $\mathscr{K}_\Omega$-tight. Then, by Theorem 1.11, Ω and hence a fortiori Λ is almost sequentially maximal so, by Theorem 2.2, τ_Λ is σ-additive. Let $\lambda = \tau_\Lambda^*$ and, using Lemma 2.4 and Theorem 0.6, let ρ be the Radon outer measure on Ω generated by $(\lambda/\mathscr{K}_\Omega)$. We shall check that ρ is a limit of μ on Ω. Let $E \in \mathscr{F}$, $A \in M_E$ and $\alpha = \Omega \cap p_E^{-1}[A]$.

(2a) To see that $\alpha \in M_\rho$, given any neighborhood U of o, choose a compact A' and open A'' such that $A' \subset A \subset A''$ and $(A'' \sim A')$ is U-small and let $\alpha' = \Omega \cap p_E^{-1}[A']$ and $\alpha'' = p_E^{-1}[A'']$. Then $\alpha', \alpha'' \in M_\rho$, $\alpha' \subset \alpha \subset \alpha''$ and, for any $C \in \mathscr{K}_\Omega$, by Lemma 2.4.3, $C \subset \alpha'' \sim \alpha' \Rightarrow \rho(C) = \lambda(C) \in U$. Hence, for any $\beta \subset \alpha'' \sim \alpha'$, we have $\rho(\beta) \in 2U$. Thus, by Lemma 0.7, $\alpha \in M_\rho$.

(2b) To see that $\rho(\alpha) = \mu_E(A)$, given any neighborhood U of o, choose $C_1, C_2 \in \mathscr{K}_\Omega$ and $K \in \mathscr{K}_E$ such that

(i) $(S_F \sim p_F[C_1])$ is U-small for all $F \in \mathscr{F}$,

(ii) $C_2 \subset \alpha$ and $\beta \subset \alpha \sim C_2 \Rightarrow \rho(\beta) \in U$,

(iii) $K \subset A$ and $(A \sim K)$ is U-small,

and let

$$C = (C_1 \cap p_E^{-1}[K]) \cup C_2 .$$

Then $C \in \mathscr{K}_\Omega$, $C_2 \subset C \subset \alpha$, so $\rho(\alpha) - \rho(C) \in U$ and, for any $F \in \mathscr{F}$ with $E < F$,

[3] The case of real-valued measures is treated in the forthcoming "Radón measures on topological spaces," (Tata Institute, Bombay) by Laurent Schwartz, Thms. I. 20, I. 21.

$$r_{EF}^{-1}[K] \sim p_F[C] \subset S_F \sim p_F[C_1]$$

$$p_F[C] \sim r_{EF}^{-1}[K] \subset r_{EF}^{-1}[A \sim K]$$

so

$$\mu_F(p_F[C]) - \mu_F(r_{EF}^{-1}[K]) \in 2U$$

and

$$\mu_F(p_F[C]) - \mu_E(A) \in 3U .$$

By Lemma 2.4.3 therefore, $\rho(C) - \mu_E(A) \in 4U$ and so

$$\rho(\alpha) - \mu_E(A) \in 5U .$$

REMARK. For $\Gamma = \boldsymbol{R}$, the above theorem was first given by Mourier [7] and later extended by Prohorov [8]. Minlos [6] attributes a similar theorem to V. Erohin.

3. **Cylinder measures.** We shall now apply the ideas of the previous sections to the study of cylinder measures.

Given any vector spaces X, Y over a field Φ, let $\mathscr{F}$ be the family of all finite dimensional subspaces of X directed by $\subset$;

S_E be the set of all linear functions on E to Y for $E \in \mathscr{F}$;

$$r_{EF}: f \in S_F \longrightarrow (f/E) \in S_E \quad \text{for } E, F \in \mathscr{F} \text{ with } E \subset F .$$

DEFINITIONS 3.1. (1) μ is a *cylinder measure* over (X, Y) iff μ is a monotone, inverse system of measures over (S, r).

(2) When the S_E are topological spaces for $E \in \mathscr{F}$, μ is a *Radon cylinder measure over* (X, Y) iff μ is a Radon system of measures over (S, r).

When $\Phi = Y = \boldsymbol{R}$, the S_E are finite dimensional spaces for $E \in \mathscr{F}$ and hence have a canonical locally convex topology with respect to which the r_{EF} are obviously continuous.

If, in addition, we let $\Gamma = \boldsymbol{R}$, μ be a Radon system, X be a *topological vector* space, Ω be the topological dual of X, and $p_E: f \in \Omega \rightarrow (f/E) \in S_E$ for $E \in \mathscr{F}$ then, in our terminology, τ_Ω is the function referred to as a cylinder measure by most workers. Even when the definition of cylinder measure is formulated as an inverse system of measures, the system is given in terms of Ω (see e.g., Minlos [6], Schwartz [10]).

Thus, besides allowing more general sets for Φ, Γ and Y, our definition of cylinder measure is free of any a priori choice of a target space Ω on which to place a limit measure and permits us therefore to consider a variety of sets Ω. Let Λ be the set of all linear functions on X to Y and p_E: $f \in \Lambda \rightarrow (f/E) \in S_E$ for $E \in \mathscr{F}$.

One justification for the choice of (Λ, p) a limit for (S, r) is that it can be identified in an obvious way with the canonical inverse limit (L, π) where

$$L = \left\{ f \in \prod_{E \in \mathscr{F}} S_E : r_{EF}(f_F) = f_E \text{ for } E, F \in f \text{ with } E \subset \mathscr{F} \right\}$$

and

$$\pi_E(f) = f_E \qquad \text{for } f \in L \text{ and } E \in \mathscr{F} .$$

Assumptions 3.2. For the remainder of this section, we assume S_E is a Hausdorff, regular topological space for $E \in \mathscr{F}$; r_{EF} is continuous for $E, F \in \mathscr{F}$ with $E \subset F$; μ is a Radon cylinder measure over (X, Y); $\Omega \subset \Lambda$.

The results of the previous sections then yield.

THEOREM 3.3. *μ has a limit on Λ.*

Proof. Λ is clearly sequentially maximal so Theorem 2.2 applies.

THEOREM 3.4. *If Ω is almost sequentially maximal then μ has a limit on Ω.*

Proof. Apply Theorem 1.10.

THEOREM 3.5. *If Ω is a regular topological space, (p_E/Ω) is continuous for $E \in \mathscr{F}$, and $\mathscr{K}$ is the family of compact subsets of Ω then*

(1) *μ has a Radon limit on Ω iff μ is $\mathscr{K}$-tight.*

(2) *If μ has a Radon limit on Ω then Ω is almost sequentially maximal.*

Proof. Apply Theorems 2.5 and 1.11.

We should point out that Theorem 3.4 would have very limited applicability if we replaced "almost sequential maximality" by "sequential maximality" in view of Theorem 3.5 and the following.

Observation. If X is a topological vector space over $\boldsymbol{R}$ with an infinite bounded linearly independent subset, $Y = \boldsymbol{R}$ and Ω is the topological dual of X then Ω is not sequentially maximal.

Proof. Let $\{e_n; n \in \omega\}$ be a bounded linearly independent subset of X, let X_0 denote its span and f be the linear functional on X_0 with $f(e_n) = n$ for $n \in \omega$. If Ω were sequentially maximal, there would exist $g \in \Omega$ with $g/X_0 = f$, which is impossible since f is not continuous on X_0.

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