PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 2, 1973

SOME REPRESENTATIONS OF THE AUTOMORPHISM GROUP OF AN INFINITE CONTINUOUS HOMOGENEOUS MEASURE ALGEBRA

ARTHUR LIEBERMAN

Let (X, τ, m) be an infinite continuous homogeneous measure space. Let A be the measure algebra of (X, τ, m) and G be the automorphism group of A. The canonical representation of G on the subspace of all elements of $\otimes^n L^2(X, \tau, m)$ of some fixed maximal symmetry type is irreducible. Two such representations are equivalent iff they correspond to the same $n \in N$ and to the same partition of n.

1. Introdution and notation. Let H be a Hilbert space. If $n \in N$, let $\bigotimes^n H$ denote the tensor product of H with itself n times. Let S_n be the symmetric group on the first n natural numbers. Let θ be the unique representation of S_n on $\bigotimes^n H$ such that $\theta(g)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{g(1)} \otimes v_{g(2)} \otimes \cdots \otimes v_{g(n)}$ if $g \in S_n$ and $v_i \in H$ for $1 \leqq i \leqq n$.

If S is a set of operators on a Hilbert space and S contains the adjoint of each of its members, then the commutant S' of S is a von Neumann algebra; the double commutant S'' of S is the smallest von Neumann algebra which contains S [1].

$\theta(S_n)''$ is generated by its mutually orthogonal minimal projections [9]. These projections are in a 1-1 correspondence with the maximal symmetry types. The vectors in the range of a minimal projection are said to be of the corresponding maximal symmetry type.

Let $U(H)$ be the unitary group on H. Let I be the canonical representation of $U(H)$ on H. If $n \in N$, let $\bigotimes^n I$ be the tensor product of I with itself n times. The restriction of $\bigotimes^n I$ to the subspace of all vectors of any fixed maximal symmetry type is irreducible. If H is finite dimensional, this result is classical [9]; if H is infinite dimensional, it is due to Segal [8]. The author has obtained similar results [3] for the symmetric groups on an infinite set S, where S is an orthonormal basis for H. In all three cases, these representations can be explicitly characterized.

Below we prove that the analogous representations of the automorphism group of an infinite homogeneous measure algebra are irreducible. No characterization of these representations is known.

DEFINITION. A continuous measure algebra $\langle M, \mu \rangle$ is homogeneous if and only if two principal ideals are isomorphic whenever their generators have equal and finite measure. A continuous measure space is homogeneous if and only if its measure algebra is homogeneous.

These definitions differ slightly from those of Maharam [4]. Note that we have no σ-finiteness requirement.

For the remainder of this paper, let (X, τ, m) be an infinite homogeneous measure space with measure algebra $\langle A, m\rangle$; Let G be the automorphism group of $\langle A, m\rangle$. Let $H = L^2(X, \tau, m)$. Let R be the canonical representation of G on H; $R(g)f = fg^{-1}$, where $g \in G$ and $f \in H$. If $n \in N$, $\bigotimes^n R$ will denote the tensor product or R with itself n times.

The remainder of this paper is devoted to proving the following theorem:

THEOREM 1. *Let (X, τ, m) be an infinite continuous homogeneous measure space with measure algebra $\langle A, m\rangle$. Let G be the automorphism group of $\langle A, m\rangle$. Let $n \in N$, and let y denote a maximal symmetry type. Let H_{ny} be the subspace of $\bigotimes^n H$ of all vectors of maximal symmetry type y. If $g \in G$, let $R_{ny}(g) = \bigotimes^n R(g) \mid H_{ny}$. Then R_{ny} is an irreducible representation of G.*

Let $m \in N$ and let k be a maximal symmetry type. Then $R_{ny} \cong R_{mk}$ if and only if $n = m$ and y corresponds to the same partition of n as k does.

2. Proof of the theorem.

The proof of the theorem requires two definitions and two lemmas.

DEFINITION. A decomposition of (X, τ, m) of type r, where r is a positive real number, is a subset $D = \{E_\lambda : \lambda \in \Lambda\}$ of τ, where Λ is an index set, such that $X = \bigcup_{\lambda \in \Lambda} E_\lambda$, $m(E_\lambda) = r$ if $\lambda \in \Lambda$, and $E_{\lambda_1} \cap E_{\lambda_2} = \phi$ if $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_1 \neq \lambda_2$.

DEFINITION. Let $D_1 = \{E_\lambda : \lambda \in \Lambda\}$ and $D_2 = \{F_\delta : \delta \in \Delta\}$ be decompositions of (X, τ, m), where Λ and Δ are index sets. We say that D_1 is subordinate to D_2 if every member of D_1 is a subset of a member of D_2.

LEMMA 1. *Let J be a Hilbert space. Let Γ be a unitary group on J. Let P be a projection on J. Let $\{P_q : q \in Q\}$ and $\{P_q^1 : q \in Q\}$, where Q is an index set, be sets of projections on J. Assume:*

(1) $PJ \cap P_q^1 J \neq 0$ *if* $q \in Q$.
(2) $P_q J \cap P_q^1 J \neq 0$ *if* $q \in Q$.
(3) $P \in \Gamma''$.
(4) $P_q \in \Gamma''$ *and* $P_q^1 \in \Gamma''$ *if* $q \in Q$.
(5) $I = \mathrm{lub}\{P_q : q \in Q\}$.
(6) $\{P\gamma P \mid PJ : \gamma \in \Gamma''\}$ *acts irreducibly on* PJ.

(7) $\{P_q\gamma P_q \mid P_qJ: \gamma \in \Gamma''\}$ *acts irreducibly on* P_qJ *if* $q \in Q$.
(8) $\{P_q^1\gamma P_q^1 \mid P_q^1J: \gamma \in \Gamma''\}$ *acts irreducibly on* P_q^1J *if* $q \in Q$.
Then Γ acts irreducibly on J.

Proof. Let $T \in \Gamma'$. Then $T \mid PJ = cI$ for some scalar c by Assumptions 3 and 6. Let $q \in Q$. By 4 and 8, $T \mid P_q^1J = c_q^1I$ for some scalar c_q^1. By 1, $c_q^1 = c$. By 4 and 7, $T \mid P_qJ = c_qI$ for some scalar c_q. By 2, $c_q = c_q^1 = c$. Therefore $T = cI$ on the subspaced spanned by $\bigcup_{q \in Q}(P_qJ)$. By 5, this subspace is dense in J.

LEMMA 2. *Let r be a positive real number. Let D be a decomposition of (X, τ, m) of type r. Let $E_i \in D$ for $1 \leqq i \leqq n$. Assume $E_i \neq E_j$ if $1 \leqq i, j \leqq n$ and $i \neq j$. Let Y be projection onto the subspace of $\bigotimes^n H$ spanned by $\theta(S_n)[C(E_1) \otimes C(E_2) \otimes \cdots \otimes C(E_n)]$. ($C(E_i)$ is the characteristic function of E_i.) Then $Y \in ((\bigotimes^n R)(G))''$.*

Proof. Let $M = \{1, 2, \cdots, n\}$ and 2^M be the power set of M. If $s \in 2^M$, let $G_s = \{g \in G: g(E_i) = E_i \text{ for } i \in s\}$. Let P_s be projection onto $\{v \in \bigotimes^n H: (\bigotimes^n R)(g)v = v \text{ for all } g \in G_s\}$. Then $P_s \in ((\bigotimes^n R)(G_s))'' \subseteqq ((\bigotimes^n R)(G))''$ by the double commutant theorem [1].

Homogeneity of a measure algebra implies that two principle ideals whose generators have equal and finite measure are carried onto one another by an automorphism of the algebra. To prove this, let x and y be generators of the principle ideals $\bar{x}$ and $\bar{y}$. Assume $m(x) = m(y) < \infty$. Let $x_1 = x - y$ and $y_1 = y - x$. By homogeneity there is an isomorphism δ of $\bar{x}_1$ onto $\bar{y}_1$. If $E \in A$, let $\Delta(E) = (E \cap \widetilde{(x_1 \cup y_1)}) \cup \delta(E \cap x_1) \cup \delta^{-1}(E \cap y_1)$. Δ is an isomorphism of A and $\Delta(\bar{x}) = \bar{y}$.

Homogeneity of a measure space implies that any measurable function on the space which is invariant under the automorphism group of the corresponding measure algebra must be a constant a.e. Consequently, P_s is projection on the subspace of $\bigotimes^n H$ spanned by vectors of the form $f_1 \otimes f_2 \otimes \cdots \otimes f_n$, where for each j, $1 \leqq j \leqq n$, $f_j = C(E_i)$ for some $i \in s$.

$Y = P_M \Pi_{s \in 2M - M}(I - P_s)$. (Note that the operators in this product all commute.) This follows from the description of the P_s and an easy consideration of various cases.

Proof of Theorem 1. Let D be a decomposition of (X, τ, m). If Z is a subset of D of cardinality n, let $Y(Z)$ be the projection associated with Z as in Lemma 2 and let $T(Z)$ be projection on $Y(Z)H_{ny}$. By Lemma 2, $Y(Z) \in (\bigotimes^n R(G))''$; consequently, $T(Z) \in R_{ny}(G)''$. Let $T_n(D) = \bigotimes T(D)$, where the sum is taken over all $Z \subseteqq D$ such that

Z has n members; $T_n(D) \in R_{ny}(G)''$. Let $P_n(D) = \text{lub}\{T_n(D'): D'$ is subordinate to $D\}$; $P_n(D) \in R_{ny}(G)''$.

Select a decomposition D of type 1. In Lemma 1, let $J = H_{ny}$, $\Gamma = R_{ny}(G)$, and $P = P_n(D)$. Let Q be the set of decompositions of (X, τ, m) of type 1. If $d \in Q$, let $\alpha(d)$ be any member of Q such that $\exists F_1 \in \alpha(d)$, $F_2 \in d$, and $F_3 \in D$ such that $m(F_1 \cap F_2) > 0$ and $m(F_1 \cap F_3) > 0$. Let $P_d = P_n(d)$ and $P_d^1 = P_n(\alpha(d))$.

The condition $m(F_1 \cap F_2) > 0$ leads to $P_dJ \cap P_d^1J \neq 0$; this is Hypothesis 2 of Lemma 1. To see this, let E_i, $1 \leqq i \leqq n$, be chosen so that $E_i \subseteqq (F_1 \cap F_2)$, $m(E_i) > 0$, and $E_i \cap E_j = \phi$ if $i \neq j$. Let $v = C(E_1) \otimes C(E_2) \otimes \cdots \otimes C(E_n)$. Let w be the projection of v onto H_{ny}. Then $w \neq 0$, $w \in P_dJ$, and $w \in P_d^1J$. Similarly, the condition $m(F_1 \cap F_3) > 0$ leads to $PJ \cap P_d^1J \neq 0$, this is Hypothesis 1 of Lemma 1.

Hypotheses 3 and 4 of Lemma 1 are clearly satisfied. The subspace of $\bigotimes^n H$ spanned by the characteristic functions of "rectangles" is dense in $\bigotimes^n H$; this is a property of product measures. Since the measure is continuous, the subspace of $\bigotimes^n H$ spanned by the characteristic functions of "rectangles" with disjoint sides is dense in $\bigotimes^n H$; any such characteristic function can be written as the sum of at most countably many functions of the same type, each of which is in the range of P_q for some $q \in Q$.

To prove irreducibility, it obviously suffices to prove that Hypothesis 6 is satisfied since Hypotheses 7 and 8 can be proved in an identical manner.

We do this by using Lemma 1. In Lemma 1, let $J = P_n(D)H_{ny}$, $\Gamma = R_{ny}(\{g \in G: R_{ny}(g)J = J\}) \mid J$ and $P = T_n(D)$. Let Q be the set of those decompositions of (X, τ, m) which are subordinate to D. If $q \in Q$, let $P_q = P_q^1 = T_n(q)$. The first four hypotheses of Lemma 1 are clearly satisfied since $PJ \subseteqq P_q^1J$ for each $q \in Q$. Hypothesis 5 follows from the definition of $P_n(D)$.

$T_n(D)H_{ny}$ is spanned by functions of the form

$$\Omega(C(E_1) \otimes C(E_2) \otimes \cdots \otimes C(E_n)) ,$$

where Ω is the projection of $\bigotimes^n H$ onto H_{ny}, $E_i \in D$, $1 \leqq i \leqq n$, and $E_i \neq E_j$ if $i \neq j$. $\{g \in G: g(E) \in D$ if $E \in D\}$ acts on $T_n(D)H_{ny}$ as an infinite permutation group; by [3, Lemma 5, part 3] this action is irreducible. Consequently, Hypothesis 6 of Lemma 1 is satisfied. Hypotheses 7 and 8 can be proved in an idential manner.

We now show that $\bigotimes^n R$ is disjoint from $\bigotimes^m R$ if $n > m$. Let the irreducible subrepresentation $R_{ny}(R_{mk})$ of $\bigotimes^n R(\bigotimes^m R)$ act on $H_n(H_m)$. Let $P_n(P_m)$ be the orthogonal projection with domain $\bigotimes^n H(\bigotimes^m H)$ and range $H_n(H_m)$.

Let $E_i \in D$, $1 \leqq i \leqq n$, with $E_i \neq E_j$ if $i \neq j$. Let $v = C(E_1) \otimes$

$C(E_2) \otimes \cdots \otimes C(E_n)$. $P_n v \neq 0$, and $R_{ny}(g)P_n v = P_n v$ if $g(E_i) = E_i$ for $1 \leqq i \leqq n$. Assume U implements an equivalence between R_{ny} and R_{mk}. Then $UP_n v$ is contained in the subspace of H_m spanned by vectors of the form $P_m w$, where $w = f_1 \otimes f_2 \otimes \cdots \otimes f_m$ and $\{f_i: 1 \leqq i \leqq m\} \subsetneqq \{C(E_i): 1 \leqq i \leqq n\}$. Assume $\{f_i: 1 \leqq i \leqq m\} = \{C(E_i): i \in s\}$, where s is some proper subset of $\{1, 2, \cdots, n\}$. Then $R_{mk}(g)P_m w = P_m w$ if $g(E_i) = E_i$ for $i \in s$. Consequently, $U^{-1}P_m w$ is in the subspace of H_n spanned by vectors of the form $P_n v_1$, where $v_1 = h_1 \otimes h_2 \otimes \cdots \otimes h_n$ and $\{h_i: 1 \leqq i \leqq n\} \subseteqq \{C(E_i): i \in s\}$. Then $v_1 \perp v$, so that $U^{-1}Uv \perp v$, which is a contradiction. The equivalence of R_{ny} with R_{nk} when and only when y and k correspond to the same partition of n is a consequence of the properties of the regular representation of S_n [9] and of the properties under equivalence of the set of common eigenvectors of eigenvalue 1 of a subset of a group.

Remarks. 1. Let G have the weakest topology such that R is weakly continuous. G is a topological group in this topology but is not locally compact. $\bigotimes^n R$ is continuous for $n \in N$.

Let $G_f = \{g \in G \mid \exists E \in \tau: m(E) < \infty, g \mid X - E = I \mid X - E\}$, where I is the identity map. G_f is a dense subgroup of G. Consequently, the restriction of $\bigotimes^n R$ to the subgroup G_f is irreducible.

2. Assume $(X, \tau, m) = \bigoplus_{\lambda \in \Lambda}(X_\lambda, \tau_\lambda, m_\lambda)$, where $\bigoplus$ means measure space direct sum [6] and $(X_\lambda, \tau_\lambda, m_\lambda)$ is a homogeneous measure space. Assume that $(X_{\lambda_1}, \tau_{\lambda_1}, m_{\lambda_1}) \oplus (X_{\lambda_2}, \tau_{\lambda_2}, m_{\lambda_2})$ is not homogeneous if $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_1 \neq \lambda_2$. Then $G = \Pi_{\lambda \in \Lambda} G_\lambda$, where G(resp. G_λ) is the automorphism group of the measure algebra of (X, τ, m) (resp. $(X_\lambda, \tau_\lambda, m_\lambda)$), and Π denotes complete direct product.

3. D. Maharam [4] has characterized continuous homogeneous measure spaces and has shown that any σ-finite continuous measure space is the measure space direct sum of homogeneous measure spaces.

4. A continuous measure space is pointwise homogeneous if and onlf if two principal ideals of its measure algebra are isomorphic via a measure preserving transformation whenever their generators have equal and finite measure. If (X, τ, m) is an infinite continuous pointwise homogeneous measure space, then the restriction of R_{ny} to the group of measure preserving transformations of (X, τ, m) is irreducible. The proof is identical to the proof of Theorem 1.

5. The theorem is not valid if m is a finite continuous homogeneous measure since then G acts irreducibly on the constant functions in $\bigotimes^n H$. The author does not know if the theorem holds with modifications. The difficulty is that Lemma 2 is false for finite measures.

6. If the measure m is not σ-finite, it is possible in the proof

of the theorem that $m(F_2 \cap F_3) = 0$ for all $F_2 \in d$ and $F_3 \in D$. This is the reason for the complexity of Lemma 1.

Let I be the unit interval. Let (I, τ, m) be the unit interval with Lebesgue measure; let (I, τ_1, m_1) be the unit interval with τ_1 the σ-algebra generated by all finite subsets of I and m_1 the counting measure. Let B be the measure algebra of $(I, \tau, m) \times (I, \tau_1, m_1)$.

Consider the measure space $(I \times I, \tau_2, m_2)$, where τ_2 is the σ-algebra generated by sets of the form $b_1 \times E_1$ and $E_2 \times b_2$, with $b_1, b_2 \in I$, $E_1, E_2 \in \tau$, $m_2(b_1 \times E_1) = m_1(E_1)$, and $m_2(E_2 \times b_2) = m_1(E_2)$; let A be the measure algebra of $(I \times I, \tau_2, m_2)$.

A is a continuous homogeneous measure algebra; any principle ideal of A whose generator has finite measure c is isomorphic to the measure algebra of the interval $[0, c]$. Note that A is isomorphic to $B \oplus B$.

Let $D_1 = \{b \times I: b \in I\}$ and $D_2 = \{I \times b: b \in I\}$. D_1 and D_2 are decompositions of $(I \times I, \tau_2, m_2)$ of type 1. $m_2((b_1 \times I) \cap (I \times b_2)) = 0$ for all $b_1, b_2 \in I$.

References

1. J. Dixmier, *Les Algèbres d'Opérateurs dans l'Espace Hilbertien*, Gauthier-Villars, Paris, 1957.

2. Michael Keane, *Contractibility of the automorphism group of a nonatomic measure space*, Proc. Amer. Math. Soc., **26** (1970), 420-422.

3. A. L. Lieberman, *The structure of certain unitary representations of infinite symmetric groups*, Trans. Amer. Math. Soc., **164** (1970), 189-198.

4. D. Maharam, *On Homogeneous Measure Algebras*, PNAS, **28** (1942), 108-111.

5. H. L. Royden, *Real Analysis*, MacMillan Company, New York, 1963.

6. I. E. Segal, *Equivalences of measure spaces*, Amer. J. Math., **73** (1951), 275-313.

7. ———, *Tensor Algebras over Hilbert Spaces, I*, Trans. Amer. Math. Soc., **81** (1956), 106-134.

8. ———, *The structure of a class of representations of the unitary group on a Hilbert space*, Proc. Amer. Math. Soc., **8** (1957), 197-203.

9. H. Weyl, *The Classical Groups*, Princeton University Press, Princeton, 1946.

Received April 19, 1971.

UNIVERSITY OF SOUTH FLORIDA