PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 2, 1973

STARLIKE AND CONVEX MAPPINGS IN SEVERAL COMPLEX VARIABLES

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In this paper, using the Bergman kernel function $K_D(z, \bar{z})$, we give necessary and sufficient conditions that a pseudo-conformal mapping $f(z)$ be starlike or convex in some bounded schlicht domain D for which the kernel function $K_D(z, \bar{z})$ becomes infinitely large when the point $z \in D$ approaches the boundary of D in any way. We also consider starlike and convex mappings from the polydisk or unit hypersphere into C^n.

Generalizing the results obtained by M. S. Robertson [10] using the principle of subordination, T. J. Suffridge has established necessary and sufficient conditions that a function be univalent and map the polydisk or

$$D_p = \left\{z: \left[\sum_{j=1}^{n} |z_j|^p\right]^{1/p} < 1,\ p \geqq 1\right\}$$

onto a starlike or convex domain [11].

Similar problems have been considered by T. Matsuno [8] for the hypershere. In this paper we deal with the same problems in terms of the Bergman kernel function $K_D(z, \bar{z})$, and show the results are equivalent to theorems of Suffridge in case of polydisk or hypersphere.

The author wishes to thank Professor S. Ozaki for helpful discussions on the preparation of the paper.

1. Preliminaries. We consider bounded schlicht domains D in C^n for which the kernel function becomes infinite everywhere on the boundary ∂D, i.e., it is the union of an increasing sequence of strictly pseudo-convex domains

$$D_t = [z: \varphi_t(z) \equiv K_D(z, \bar{z}) - t < 0,\ z \in D] \tag{1.1}$$

for some number $t > 0$, where $z = (z_1, \cdots, z_n)'$. (See [3]). First we have

LEMMA 1.1. *If D is a bounded domain, the Bergman kernel function $K_D(z, \bar{z})$ is strictly plurisubharmonic and*

$$1/\omega(D) \leqq K_D(z, \bar{z}) \leqq 1/\pi^n(l(z))^{2n}\ , \tag{1.2}$$

where $l(z) = \min_{\tau \in \partial D} \rho(\tau, z)$, $\rho(\tau, z) = \max_j \{|\tau_j - z_j|, j = 1, \cdots, n\}$ *and* $\omega(D)$ *signifies the euclidean volume of D.*

Proof. The minimum value of the integral $\|f\|_D^2 = \int_D |f(\zeta)|^2 dv_\zeta$ for functions $f(\zeta) \in \mathscr{L}^2(D)$ satisfying the condition $df(z)/d\zeta \cdot u = 1$, where $u = (u_1, \cdots, u_n)'$ is an arbitrary nonzero column vector, is

$$(1.3) \qquad 1/u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial \zeta^* \partial \zeta} u = \int_D \left| \frac{u^* \dfrac{\partial K_D(\zeta, \bar{z})}{\partial \zeta^*}}{u^* \dfrac{\partial^2 K_D(z, \bar{z})}{\partial \zeta^* \partial \zeta} u} \right|^2 dv_\zeta \, . \qquad \text{(See [1], [2].)}$$

Here we define partial derivatives of a function $g(\zeta, \bar{\tau})$ as

$$(1.4) \qquad \begin{aligned} \partial^2 g(\zeta, \bar{\tau})/\partial \tau^* \partial \zeta &= (\partial/\partial \bar{\tau}_1, \cdots, \partial/\partial \bar{\tau}_n)' \times (\partial/\partial \zeta_1, \cdots, \partial/\partial \zeta_n) \times g(\zeta, \bar{\tau}) \\ &= \begin{pmatrix} \partial^2/\partial \bar{\tau}_1 \partial \zeta_1, \cdots, \partial^2/\partial \bar{\tau}_1 \partial \zeta_n \\ \cdots\cdots\cdots \\ \partial^2/\partial \bar{\tau}_n \partial \zeta_1, \cdots, \partial^2/\partial \bar{\tau}_n \partial \zeta_n \end{pmatrix} \times g(\zeta, \bar{\tau}) \, , \end{aligned}$$

and if $g(\zeta)$ is a function of only ζ, we denote $dg(\zeta)/d\zeta = (\partial/\partial \zeta_1, \cdots, \partial/\partial \zeta_n) \times g(\zeta)$, where the sign $\times$ designates the Kronecker product and the sign $*$ denotes the transposed conjugate matrix. (Cf. [7].)

On the other hand, if we put $f(\zeta) = u^*(\zeta - z)/|u|^2$, then

$$\frac{df(z)}{d\zeta} u = u^* u / |u|^2 = 1 \, ,$$

therefore

$$(1.5) \qquad \begin{aligned} 1/u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial \zeta^* \partial \zeta} u &\leqq \int_D \left| \frac{u^*(\zeta - z)}{|u|^2} \right|^2 dv_\zeta \\ &\leqq \frac{1}{|u|^2} \int_D |\zeta - z|^2 dv_\zeta \leqq \frac{L^2 \omega(D)}{|u|^2} \, , \end{aligned}$$

where $L = \max_{\tau \in \partial D} |\tau - z|$ and $|u| = (\sum_{j=1}^n |u_j|^2)^{1/2}$.
Thus

$$u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial \zeta^* \partial \zeta} u > 0$$

for all $z \in D$, that is, $K_D(z, \bar{z})$ is strictly plurisubharmonic (see [3]). Next it is well known that the minimum value of the integral $\|f\|_D^2$ under the condition $f(z) = 1$, $z \in D$, becomes $1/K_D(z, \bar{z})$. Then, for the function $f(\zeta) \equiv 1$, we have

$$(1.6) \qquad 1/K_D(z, \bar{z}) = \int_D |K_D(\zeta, \bar{z})/K_D(z, \bar{z})|^2 dv_\zeta \leqq \int_D dv_\zeta = \omega(D) \, .$$

Also, using the Cauchy integral formula, we obtain

$$\left|\left(\frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})}\right)_{\zeta=z}\right| \tag{1.7}$$
$$\leqq \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{|K_D(\zeta, \bar{z})/K_D(z, \bar{z})|}{r_1\cdots r_n}r_1d\theta_1\cdots r_nd\theta_n\ ,$$

where $\zeta_j - z_j = r_je^{i\theta_j}$, $0 < r_j < l(z)$, $(j = 1, \cdots, n)$. We get therefore by the Schwarz integral inequality

$$l^{2n}/2^n \leqq \frac{1}{(2\pi)^n}\int\cdots\int_{\rho(\zeta,z)<l}\left|\frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})}\right|dv_\zeta \tag{1.8}$$
$$\leqq \frac{1}{(2\pi)^n}\left[(\pi l^2)^n\int\cdots\int_{\rho(\zeta,z)<l}\left|\frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})}\right|^2dv_\zeta\right]^{1/2}.$$

Then

$$\pi^{n/2}l^n \leqq \left[\int_D\left|\frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})}\right|^2dv_\zeta\right]^{1/2} = (1/K_D(z, \bar{z}))^{1/2}\ , \tag{1.9}$$

hence we have (1.2) from (1.6) and (1.9).

2. Convex mappings. We consider the above mentioned domains D and D_t, and suppose that $\partial K_D(z, \bar{z})/\partial z \neq 0$, $z \neq 0$, in D, and $K_D(0, 0) = \min_{z\in D} K_D(z, \bar{z})$ at only $z = 0$. For a holomorphic univalent function $w = f(z)$ of D, let

$$\varphi_t(z) = \varphi_t(f^{-1}(w)) \equiv \Phi_t(w),\ t > K_D(0, 0)\ , \tag{2.1}$$

and let $\Delta = f(D)$, $\Delta_t = f(D_t)$.
Then we have

$$\Delta_t = [w\colon \Phi_t(w) < 0,\ w \in \Delta] \tag{2.2}$$

corresponding to (1.1). On the boundary $\partial D_t\colon \varphi_t(z) = 0$, the total differential of $\varphi_t(z)$ becomes

$$d\varphi_t = \frac{\partial\varphi_t}{\partial z}dz + dz^*\frac{\partial\varphi_t}{\partial z^*} = 2\mathscr{R}\left[\frac{\partial\varphi_t}{\partial z}dz\right] = 0\ , \tag{2.3}$$

where $dz = (dz_1, \cdots, dz_n)'$. Consequently, since $\partial\varphi_t/\partial z^* = \partial K_D(z, \bar{z})/\partial z^*$ is perpendicular to all tangential vectors dz of the boundary ∂D_t at z, $\partial\varphi_t/\partial z^*$ is a normal vector of ∂D_t at z. And we can derive

$$\mathscr{R}\left[\frac{\partial\Phi_t}{\partial w}dw\right] = \mathscr{R}\left[\frac{\partial\Phi_t}{\partial z}\left(\frac{dz}{dw}\right)\left(\frac{dw}{dz}\right)dz\right] = \mathscr{R}\left[\frac{\partial\varphi_t}{\partial z}dz\right] = 0\ , \tag{2.4}$$

hence $\partial\Phi_t/\partial w^*$ is also a normal vector of the boundary $\partial\Delta_t\colon \Phi_t(w) = 0$ at $w = f(z)$. (See [5], [6].)

We can expand $\Phi_t(w + dw)$ into a Taylor series:

$$\begin{aligned}\Phi_t(w+dw) &= \Phi_t(w) + 2\mathscr{R}\left[\frac{\partial\Phi_t}{\partial w}dw\right] \\ &\quad + 2\mathscr{R}\left[\frac{\partial^2\Phi_t}{\partial w^2}dw^2 + dw^*\frac{\partial^2\Phi_t}{\partial w^*\partial w}dw\right] + 0(|dw|^2)\ ,\end{aligned} \tag{2.5}$$

where $dw^2 = (dw_1, \cdots, dw_n)' \times (dw_1, \cdots, dw_n)'$. (See [3], Chap. IX.) Since

$$\mathscr{R}\left[\frac{\partial\Phi_t}{\partial w}dw\right] = 0$$

at $w \in \partial\Delta_t$, it follows that

$$\Phi_t(w+dw) = 2\mathscr{R}\left[\frac{\partial^2\Phi_t}{\partial w^2}dw^2 + dw^*\frac{\partial^2\Phi_t}{\partial w^*\partial w}dw\right] + 0(|dw|^2)\ . \tag{2.6}$$

If the point $(w + dw)$ lie always the outside of Δ_t for all $w \in \partial\Delta_t$ and tangential vectors dw at w, i.e., $\Phi_t(w + dw) > 0$, then Δ_t is convex. From (2.6), we must have the following condition in order to consist always $\Phi_t(w + dw) > 0$:

$$\mathscr{R}\left[\frac{\partial^2\Phi_t}{\partial w^2}dw^2 + dw^*\frac{\partial^2\Phi_t}{\partial w^*\partial w}dw\right] > 0\ . \tag{2.7}$$

Now we can calculate as follows by formulas of matrix derivatives described in [7]:

$$\begin{aligned}\frac{\partial^2\Phi_t}{\partial w^2} &= \frac{\partial}{\partial w}\left(\frac{\partial\varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\right) = \frac{\partial}{\partial z}\left(\frac{\partial\varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\right)\left(\left(\frac{dw}{dz}\right)^{-1} \times E\right) \\ &= \frac{\partial^2\varphi_t}{\partial z^2}\left(\left(\frac{dw}{dz}\right)^{-1} \times \left(\frac{dw}{dz}\right)\right)^{-1} - \frac{\partial\varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\frac{d^2w}{dz^2}\left(\left(\frac{dw}{dz}\right)^{-1} \times \left(\frac{dw}{dz}\right)^{-1}\right),\end{aligned} \tag{2.8}$$

$$\frac{\partial^2\Phi_t}{\partial w^2}dw^2 = \left\{\frac{\partial^2\varphi_t}{\partial z^2} - \frac{\partial\varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\frac{d^2w}{dz^2}\right\}dz^2\ , \tag{2.9}$$

$$dw^*\frac{\partial^2\Phi_t}{\partial w^*\partial w}dw = dw^*\left\{\left(\frac{dw}{dz}\right)^{-1}{}^*\frac{\partial^2\varphi_t}{\partial z^*\partial z}\left(\frac{dw}{dz}\right)^{-1}\right\}dw = dz^*\frac{\partial^2\varphi_t}{\partial z^*\partial z}dz\ . \tag{2.10}$$

Then, substituting (2.9) and (2.10) into (2.7), we obtain

$$\mathscr{R}\left[\left\{\frac{\partial^2\varphi_t}{\partial z^2} - \frac{\partial\varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\frac{d^2w}{dz^2}\right\}dz^2 + dz^*\frac{\partial^2\varphi_t}{\partial z^*\partial z}dz\right] > 0\ . \tag{2.11}$$

Thus we have the following Lemma.

LEMMA 2.1. *For a fixed value t, a holomorphic univalent function $w = f(z)$ of D have convex image Δ_t of D_t defined by* (1.1) *if and only if at every point z on the boundary ∂D_t*

$$(2.12) \quad \mathscr{R}\left[\alpha^* \frac{\partial^2 K_D(z, \bar{z})}{\partial z^* \partial z}\alpha + \left\{\frac{\partial^2 K_D(z, \bar{z})}{\partial z^2} - \frac{\partial K_D(z, \bar{z})}{\partial z}\left(\frac{df}{dz}\right)^{-1}\frac{d^2 f}{dz^2}\right\}\alpha^2\right] > 0$$

for all unit vectors α *satisfying*

$$\mathscr{R}\left[\frac{\partial K_D(z, \bar{z})}{\partial z}\alpha\right] = 0 .$$

DEFINITION. We define the class $\mathscr{D}$ of bounded schlicht domains D for which the kernel function $K_D(z, \bar{z})$ becomes infinite everywhere on the boundary ∂D, $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$ only at $z = 0$, $\partial K_D(z, \bar{z})/\partial z \neq 0$, $z \neq 0$, in D, and there is the holomorphic mapping $g(z)$ of D into D satisfying $g(0) = 0$, for some one $z^{(1)}$ of two arbitrary points $z^{(1)}, z^{(2)} (\neq 0)$ in D $g(z^{(1)}) = z^{(2)}$, and $K_D(z, \bar{z}) \geqq K_D(g(z), \overline{g(z)})$.

For example, let D be a minimal domain or representative domain with center at the origin which is the image domain of $E = \{\zeta: |\zeta| = (\sum_{j=1}^{n} |\zeta_j|^2)^{1/2} < 1\}$ under the biholomorphic mapping $z = \varphi(\zeta)$ satisfying $0 = \varphi(0)$. Then $\det (d\varphi(\zeta)/d\zeta) \equiv$ const. when D is a minimal, domain and $d\varphi(\zeta)/d\zeta \equiv$ const. when D is a representative domain (see [4], Theorem 3.1). Hence, for any holomorphic mapping $g(z)$ of D into D satisfying $g(0) = 0$, we have $K_D(z, \bar{z}) \geqq K_D(g(z), \overline{g(z)})$ because $K_E(\zeta, \bar{\zeta}) \geqq K_E(\Phi(\zeta), \overline{\Phi(\zeta)})$ under the holomorphic mapping $\Phi(\zeta) \equiv \varphi^{-1}[g(\varphi(\zeta))]$, $\Phi(0) = 0$, of E into E. Also we have $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$ at only the origin. Moreover, for arbitrary points $z^{(1)}, z^{(2)} \in D$, if $|\varphi^{-1}(z^{(2)})| \leqq |\varphi^{-1}(z^{(1)})|$, then

$$g(z) \equiv \varphi\left(\frac{|\varphi^{-1}(z^{(2)})|}{|\varphi^{-1}(z^{(1)})|} U_2 U_1^* \varphi^{-1}(z)\right)$$

is a holomorphic mapping of D into D satisfying $g(0) = 0$ and $g(z^{(1)}) = z^{(2)}$ where

$$\varphi^{-1}(z^{(1)}) = U_1 \begin{pmatrix} |\varphi^{-1}(z^{(1)})| \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \varphi^{-1}(z^{(2)}) = U_2 \begin{pmatrix} |\varphi^{-1}(z^{(2)})| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and U_1, U_2 are unitary matrices. And we observe

$$\partial K_D(z, \bar{z})/\partial z = \partial K_E(\zeta, \bar{\zeta})/\partial \zeta \cdot (d\varphi(\zeta)/d\zeta)^{-1} \neq 0, \; z \neq 0 ,$$

because

$$\partial K_E(\zeta, \bar{\zeta})/\partial \zeta = (n + 1)\zeta^* K_E(\zeta, \bar{\zeta})/(1 - |\zeta|^2) \neq 0, \; \zeta \neq 0 .$$

THEOREM 2.1. *Let* D *be a bounded schlicht domain of the class* $\mathscr{D}$. *Suppose* $f: D \to C^n$ *is holomorphic,* $f(0) = 0$, *and* $\det (df/dz) \neq 0$ *for all* $z \in D$. *Then* f *is a univalent map of* D *onto a convex domain if* *and only if*

$$(2.13)\quad \mathscr{R}\left[\alpha^*\frac{\partial^2 K_D(z,\bar{z})}{\partial z^*\partial z}\alpha+\left\{\frac{\partial^2 K_D(z,\bar{z})}{\partial z^2}-\frac{\partial K_D(z,\bar{z})}{\partial z}\left(\frac{df}{dz}\right)^{-1}\frac{d^2 f}{dz^2}\right\}\alpha^2\right]>0$$

for all unit vectors α *satisfying*

$$\mathscr{R}\left[\frac{\partial K_D(z,\bar{z})}{\partial z}\alpha\right]=0\ .$$

Proof. The Bergman kernel function $K_D(z, \bar{z})$ of this domain D becomes infinite on ∂D. Then we define D_t and Δ_t by (1.1) and (2.2) respectively. If $\Delta = f(D)$ is schlicht and convex, then all Δ_t also become convex, i.e., for any $w^{(1)}, w^{(2)} \in \partial\Delta_t$,

$$(2.14)\qquad w^{(0)}=\tau w^{(2)}+(1-\tau)w^{(1)}\in\Delta_t,\quad 0<\tau<1\ .$$

In fact, if we put $z^{(1)} = f^{-1}(w^{(1)}), z^{(2)} = f^{-1}(w^{(2)})$, then $K_D(z^{(1)}, \overline{z^{(1)}}) = K_D(z^{(2)}, \overline{z^{(2)}}) = t$. Setting

$$(2.15)\qquad F(z)\equiv\tau f(g(z))+(1-\tau)f(z)$$

where $g(z)$ is a holomorphic mapping of D into D satisfying $g(0) = 0$ and $g(z^{(1)}) = z^{(2)}$, we observe that $F(0) = 0$ and $F(z) \prec f(z)$ because the mapping $f: D \to C^n$ is convex. Hence

$$(2.16)\qquad \psi(z)\equiv f^{-1}(F(z))$$

is a holomorphic mapping of D into D, so we have

$$K_D(z^{(1)},\overline{z^{(1)}})\geqq K_D(\psi(z^{(1)}),\overline{\psi(z^{(1)})})=K_D(f^{-1}(w^{(0)}),\overline{f^{-1}(w^{(0)})})\ .$$

Consequently $f^{-1}(w^{(0)}) \in D_t$, so $w^{(0)} \in \Delta_t$. Thus, by Lemma 2.1, (2.13) holds for all $z \in D$. Contrary, if (2.13) is realized for all $z \in$ D, every Δ_t is convex. Therefore we can conclude that the mapped domain Δ is convex.

Particularly if D is a unit hypersphere, then

$$K_D\ (z,\bar{z})=\frac{n!}{\pi^n(1-|z|^2)^{n+1}}\ .$$

Thus we have the following result by Theorem 2.1.

THEOREM 2.2. *Let* D *be the unit hypersphere and let* $f: D \to C^n$ *be holomorphic,* $f(0) = 0$ *and* $\det(df/dz) \neq 0$ *for all* $z \in D$. *Then* $f(D)$ *is convex if and only if*

$$(2.17)\qquad \mathscr{R}\left[|Az|^2+z^*\left(\frac{df}{dz}\right)^{-1}\frac{d^2 f}{dz^2}(Az\times Az)\right]\geqq 0\ ,$$

where

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix}, A_j \geqq 0, j = 1, \cdots, n ,$$

and the equality holds only if $Az = 0$.

Proof. We can compute as follows setting $K = K_D(z, \bar{z})$:

$$\partial K/\partial z = (n+1)\frac{z^*}{1 - |z|^2}K , \tag{2.18}$$

$$\partial^2 K/\partial z^2 = (n+1)(n+2)\frac{(z \times z)^*}{(1 - |z|^2)^2}K , \tag{2.19}$$

$$\partial^2 K/\partial z^* \partial z = (n+1)\frac{(1 - |z|^2)E + (n+2)zz^*}{(1 - |z|^2)^2}K . \tag{2.20}$$

Then, from (2.13), we have

$$\begin{aligned} \mathscr{R}\Big[&(n+2)\{|z^*\alpha|^2 + (z^*\alpha)^2\} \\ &+ (1 - |z|^2)\Big\{1 - z^*\Big(\frac{df}{dz}\Big)^{-1}\frac{d^2 f}{dz^2}\alpha^2\Big\}\Big] > 0 . \end{aligned} \tag{2.21}$$

Since

$$|z^*\alpha|^2 + \mathscr{R}(z^*\alpha)^2 = 0$$

from

$$\mathscr{R}\Big[\frac{\partial K}{\partial z}\alpha\Big] = 0, \text{ i.e., } \mathscr{R}[z^*\alpha] = 0 ,$$

we conclude

$$\mathscr{R}\Big[1 - z^*\Big(\frac{df}{dz}\Big)^{-1}\frac{d^2 f}{dz^2}\alpha^2\Big] > 0 . \tag{2.22}$$

Moreover, under the condition $\mathscr{R}[z^*\alpha] = 0$ it becomes that $z^*\alpha = ip(p \geqq 0, i = \sqrt{-1})$, because both α and $-\alpha$ are satisfy (2.22). Therefore we can put $\alpha = i(Az/|Az|)$ when $Az \neq 0$, where

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix}, A_j \geqq 0, (j = 1, \cdots, n) ,$$

are chosen arbitrarily. Thus we obtain (2.17) from (2.22).

REMARK 1. Suffridge's Theorem 5 [11] shows that

$$F = \frac{df}{dz}\left[A^2z + \left(\frac{df}{dz}\right)^{-1}\frac{d^2f}{dz^2}(Az \times Az)\right]\Big/2,\ w = \left(\frac{df}{dz}\right)^{-1}F \in \mathscr{P}_2\,,$$

i.e.,

$$\begin{aligned}\mathscr{R}\sum_{j=1}^{n} w_j|z_j|^2/z_j &= \mathscr{R}z^*\left[A^2z + \left(\frac{df}{dz}\right)^{-1}\frac{d^2f}{dz^2}(Az\times Az)\right]\Big/2\\ &= \mathscr{R}\left[|Az|^2 + z^*\left(\frac{df}{dz}\right)^{-1}\frac{d^2f}{dz^2}(Az \times Az)\right]\Big/2 \geqq 0\,,\end{aligned}$$

is the necessary and sufficient condition for convexity.

Next, if D is the polydisk $\{z \in C^n\colon |z_j| < 1, j = 1, \cdots, n\}$, the kernel function $K_D(z, \bar{z})$ becomes $1/\pi^n(1 - |z_1|^2)^2 \cdots (1 - |z_n|^2)^2$. Hence

$$\partial K/\partial z = 2K\cdot z^*Z\,, \tag{2.23}$$

$$\begin{aligned}\partial^2K/\partial z^2 &= 4K\cdot(z\times z)^*(Z \times Z)\\ &\quad + 2K\cdot(z \times z)^*(Z \times Z)\begin{pmatrix} \begin{matrix}1_0 & & \\ & \ddots & \\ & & 0\end{matrix} & & & 0\\ & \begin{matrix}0 & & & \\ & 1_0 & & \\ & & \ddots & \\ & & & 0\end{matrix} & & \\ 0 & & \ddots & \\ & & & \begin{matrix}0 & & \\ & \ddots & \\ & & 0_1\end{matrix}\end{pmatrix},\end{aligned} \tag{2.24}$$

$$\partial^2K/\partial z^*\partial z = 4K\cdot Zzz^*Z + 2K\cdot Z^2\,, \tag{2.25}$$

where

$$Z = \begin{pmatrix}1/(1 - |z_1|^2) & & 0\\ & \ddots & \\ 0 & & 1/(1 - |z_n|^2)\end{pmatrix}.$$

Substituting formally (2.23), (2.24), and (2.25) into (2.13) and setting

$$\mathscr{R}(z^*Z\alpha)^2 + |z^*Z\alpha|^2 = 0 \text{ and } \alpha = i\frac{Z^{-1/2}Az}{|Z^{-1/2}Az|}$$

where

$$Z^{-1/2} = \begin{pmatrix}\sqrt{1 - |z_1|^2} & & 0\\ & \ddots & \\ 0 & & \sqrt{1 - |z_n|^2}\end{pmatrix},$$

in place of the condition

$$\mathscr{R}\left[\frac{\partial K_D(z, \bar{z})}{\partial z}\alpha\right] = 2K\cdot\mathscr{R}[z^*Z\alpha] = 0 ,$$

we arrive at

$$\mathscr{R}\left[|Az|^2 + z^*Z\left(\frac{df}{dz}\right)^{-1}\frac{d^2f}{dz^2}(Z \times Z)^{-1/2}(Az \times Az)\right] \geqq 0 , \tag{2.26}$$

where the equality holds only if $Az = 0$.

THEOREM 2.3. *Let D be the polydisk and let $f\colon D \to C^n$ be holomorphic, $f(0) = 0$ and $\det(df/dz) \neq 0$ for all $z \in D$. Then f is a univalent map of D onto a convex domain if and only if the condition* (2.26) *is fulfilled.*

Proof. If f is a convex mapping, then by Suffridge's Theorem 3 [11] $f = T(\varphi_1(z_1), \cdots, \varphi_n(z_n))'$ where T is a nonsingular linear transformation and each $\varphi_j(z_j)$ is a univalent mapping from the unit disk in the plane onto convex domain in the plane. Then we have

$$\left(\frac{df}{dz}\right)^{-1}\frac{d^2f}{dz^2} = \begin{pmatrix} \varphi_1''(z_1)/\varphi_1'(z_1) 0 \cdots 0 & & & 0 \\ & 0\varphi_2''(z_2)/\varphi_2'(z_2) 0 \cdots 0 & & \\ & & \ddots & \\ 0 & & & 0 \cdots 0\, \varphi_n''(z_n)/\varphi_n'(z_n) \end{pmatrix}. \tag{2.27}$$

Substituting this into the left side of (2.26), we get

$$\mathscr{R}\left[\sum_{j=1}^{n} A_j^2 |z_j|^2\{1 + z_j\varphi_j''(z_j)/\varphi_j'(z_j)\}\right]. \tag{2.28}$$

Hence from the hypothesis $\mathscr{R}[1 + z_j\varphi_j''(z_j)/\varphi_j'(z_j)] > 0, j = 1, \cdots, n$, we get the inequality (2.26).

We will prove the converse. Fix $k, 1 \leqq k \leqq n$ and choose $A_k = 1, A_h = 0, h \neq k, 1 \leqq h \leqq n$. From (2.26)

$$\mathscr{R}\left[|z_k|^2 + \frac{z_k^2(1 - |z_k|^2)}{\det J}\sum_{j=1}^{n}\frac{\bar{z}_j}{1 - |z_j|^2}C_j^{k^2}\right] \geqq 0 , \tag{2.29}$$

where $J = df/dz$ and $G_j^{k^2}$ is obtained from $\det J$ by replacing the jth column by the column $\partial^2 f/\partial z_k^2 = (\partial^2 f_1/\partial z_k^1, \cdots, \partial^2 f_n/\partial z_k^2)'$. For $l, 1 \leqq l \leqq n, l \neq k$, setting $|z_j| < 1/2, j \neq l, 1 \leqq j \leqq n$, $(1 - |z_k|^2)/(1 - |z_l|^2)$ tends to infinity when $|z_l| \to 1$. Then we must have always

$$\mathscr{R}\left[\frac{1}{\det J}\frac{z_k^2}{z_l}G_l^{k^2}\right] \geqq 0 \tag{2.30}$$

from the condition (2.29). Here, since it becomes 0 at $z_k = 0$, we see that $G_l^{k^2} \equiv 0$ for each $l, l \neq k, 1 \leqq l \leqq n$. Next, if we set $A_k = A_l = 1, A_m = 0, m \neq k, l$, then (2.26) becomes as follows from the above results:

$$\begin{aligned}\mathscr{R}\Bigg[&|z_k|^2 + |z_l|^2 + \frac{|z_k|^2 z_k G_k^{k^2}}{\det J} + \frac{|z_l|^2 z_l G_l^{l^2}}{\det J} \\ &+ 2\frac{z_k z_l\sqrt{(1-|z_k|^2)(1-|z_l|^2)}}{\det J}\sum_{j=1}^{n}\frac{\bar{z}_j G_j^{kl}}{(1-|z_j|^2)}\Bigg] \geqq 0 .\end{aligned} \tag{2.31}$$

For $s, 1 \leqq s \leqq n$, setting

$$|z_h| < 1/2, h \neq s, 1 \leqq h \leqq n, \ \frac{\sqrt{(1-|z_k|^2)(1-|z_l|^2)}}{1-|z_s|^2}$$

tends to infinity when $|z_s| \to 1$. Then we must have always

$$\mathscr{R}\left[\frac{1}{\det J}\frac{z_k z_l}{z_s}G_s^{kl}\right] \geqq 0 . \tag{2.32}$$

Since it attains to the minimum value 0 at $z_k z_l = 0$, we must have $G_s^{kl} \equiv 0$ for each s. Thus we arrive at the conditions of the Theorem 3 of Suffridge following his methods. So we can conclude that f is a convex mapping.

3. Starlike mappings.

We now consider univalent functions of D which map D onto a starlike domain with respect to 0. First we set up the definition of starlikeness following Suffridge:

DEFINITION. A holomorphic mapping $f: D \to C^n$ is starlike if f is univalent, $f(0) = 0$ and $(1-\tau)f \prec f$ for all $\tau \in I = [0, 1]$.

THEOREM 3.1. *Let D be a bounded schlicht domain for which the kernel function $K_D(z, \bar{z})$ becomes infinite everywhere on the boundary, $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$ at only the origin, and $K_D(z, \bar{z}) \geqq K_D(g(z), \overline{g(z)})$ for any holomorphic mapping $g(z)$ of D into D satisfying $g(0) = 0$. Suppose $f: D \to C^n$ is holomorphic, $f(0) = 0$ and* $\det(df/dz) \neq 0$ *for all $z \in D$. Then f is starlike if and only if*

$$\mathscr{R}\left[\frac{\partial K_D(z, \bar{z})}{\partial z}\left(\frac{df}{dz}\right)^{-1} f\right] > 0 \tag{3.1}$$

for all $z \in D, z \neq 0$.

REMARK 2. Domains which belong to the above mentioned class $\mathscr{D}$ satisfy the conditions of this Theorem.

Proof. If f is starlike, then all image $\varDelta_t$ are starlike, that is, for all $w^{(1)} \in \partial\varDelta_t$ we have $w^{(0)} = (1-\tau)w^{(1)} \in \varDelta_t, \tau \in I$. In fact, if we set $z^{(1)} = f^{-1}(w^{(1)}), K_D(z^{(1)}, \overline{z^{(1)}}) = t$ and $\psi(z) \equiv f^{-1}((1-\tau)f(z))$, then we obtain

$$K_D(z^{(1)}, \overline{z^{(1)}}) \geqq K_D(\psi(z^{(1)}), \overline{\psi(z^{(1)})}) = K_D(f^{-1}(w^{(0)}), \overline{f^{-1}(w^{(0)})}) , \tag{3.2}$$

because $\psi(z)$ is a mapping of D into D and $\psi(0) = 0$. Then it holds that $f^{-1}(w^{(0)}) \in D_t$ which yields $w^{(0)} \in \varDelta_t$. Now, since

$$\varPhi_t\left(w + \varepsilon\frac{\partial\varPhi_t}{\partial w^*}\right) = 2\varepsilon\left|\frac{\partial\varPhi_t}{\partial w^*}\right|^2 + 0(\varepsilon^2) > 0$$

when $\varepsilon > 0$ is sufficiently small and $w \in \partial\varDelta_t, N_w \equiv \partial\varPhi_t/\partial w^*$ is the outward normal vector at the boundary point $w \in \partial\varDelta_t$. Hence $(1-\tau)w \in \varDelta_t (w \in \partial\varDelta_t, 0 < \tau \leqq 1)$ implies

$$\cos(-N_w, -w) = \mathscr{R}\left[\frac{\partial\varPhi_t}{\partial w}w\right]\Big/\left|\frac{\partial\varPhi_t}{\partial w^*}\right||w| > 0 \tag{3.3}$$

which yields (3.1) by virtue of

$$\frac{\partial\varPhi_t}{\partial w}w = \frac{\partial K}{\partial z}\left(\frac{df}{dz}\right)^{-1}f(z) .$$

Conversely, if (3.1) holds, then we conclude $(1-\tau)w \in \varDelta_t, w \in \partial\varDelta_t, 0 < \tau < \varepsilon(<1)$ for some $\varepsilon > 0$ by (3.3). Moreover, we can conclude $(1-\tau)w \in \varDelta_t, w \in \partial\varDelta_t, 0 < \tau \leqq 1$, because, if $(1-\tau_1)w \equiv w^{(1)} \in \partial\varDelta_t$ and $(1-\tau)w \in \varDelta_t, 0 < \tau < \tau_1$ for some $\tau_1 < 1$, then $(1-\tau)w^{(1)} \notin \varDelta_t, w^{(1)} \in \partial\varDelta_t$ which is a contradiction. Then the image domain $\varDelta$ of D becomes starlike.

COROLLARY 3.1. *Let D be the unit hypersphere, and let $f: D \rightarrow C^n$ be holomorphic, $f(0) = 0$ and* $\det(df/dz) \neq 0$ *for all $z \in D$. Then $f(z)$ is starlike if and only if*

$$\mathscr{R}\left[z^*\left(\frac{df}{dz}\right)^{-1}f\right] > 0 \tag{3.4}$$

for all $z \in D, z \neq 0$.

Proof. Substituting (2.18) into (3.1), we obtain the required result.

REMARK 3. The conditions of Suffridge's Theorem 4 [11]: $f = Jw, w \in \mathscr{P}_2$ are the same as (3.4).

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Received October 21, 1972.

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