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SUMMABILITY OF SUBSEQUENCES AND STRETCHINGS OF SEQUENCES

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In 1943 R. C. Buck gave a summability characterization of real convergent sequences by showing that a real sequence x is convergent if there exists a regular matrix summability method which sums every subsequence of x. In 1944 R. P. Agnew generalized Buck's result by showing that if x is a bounded complex sequence and A is a regular matrix, then there exists a subsequence y of x such that every limit point of x is a limit point of Ay. In the present paper a theorem concerning "stretchings" of sequences is proved; and from this theorem, summability characterizations of several classes of sequences are obtained, together with an extension of Agnew's result.

DEFINITION. The sequence $y = \{y_p\}_{p=1}^{\infty}$ is a stretching of $x = \{x_p\}_{p=1}^{\infty}$ provided there exists an increasing sequence $\{m_p\}_{p=0}^{\infty}$ of integers such that $m_0 = 1$ and $y_q = x_p$ if $m_{p-1} \leqq q < m_p$, $p = 1, 2, 3, \cdots$. Under these conditions, we shall say that y is the stretching of x induced by $\{m_p\}$.

Conditions which are necessary and sufficient for a matrix $A = (a_{pq})$ to be a regular summability method are

(1) $\{a_{pq}\}_{p=1}^{\infty}$ converges to $0, q = 1, 2, 3, \cdots$,

(2) $\{\sum_{q=1}^{\infty} a_{pq}\}_{p=1}^{\infty}$ converges to 1,

(3) $\sup_p \sum_{q=1}^{\infty} |a_{pq}| < \infty$.

Let $\mathscr{P}$ denote the set of all matrices P such that for all x, Px is a subsequence of x, and let $\mathscr{Q}$ denote the set of all matrices Q such that for all x, Qx is a stretching of x.

We note that any $P = (p_{ij}) \in \mathscr{P}$ is determined by an increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers as follows: $p_{ij} = 1$ if $j = n_i$, $p_{ij} = 0$ otherwise. Similarly, any $Q = (q_{ij}) \in \mathscr{Q}$ is determined by an increasing sequence $\{n_i\}_{i=0}^{\infty}$, $n_0 = 1$, of integers as follows: $q_{ij} = 1$ if $n_{j-1} \leqq i < n_j$, $q_{ij} = 0$ otherwise.

Clearly all $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$ are regular.

If x is the sequence of partial sums of a series $\sum c_p$, then any subsequence of x is the sequence of partial sums of a series obtained by bracketing the terms of $\sum c_p$ appropriately. On the other hand, any stretching of x is the sequence of partial sums of a series obtained by inserting zero terms in $\sum c_p$ appropriately.

A series $\sum c_p$ with partial sums $\{x_p\}$ converges absolutely if and only if $\sum |x_p - x_{p+1}| < \infty$. Thus a sequence $\{t_p\}$ is said to converge absolutely (or to be of bounded variation) provided $\sum |t_p - t_{p+1}| < \infty$.

We will let BV denote the set of all complex sequences which converge absolutely. These sequences have the property that the sequence of real (imaginary) parts of the terms is the difference of convergent nondecreasing sequences.

In light of Buck's result [2], [3], one might conjecture that a real sequence $x \in BV$ if there exists a regular matrix A such that $Ay \in BV$ for every subsequence y of x. This is not true, as shown by the following example. Let x be a null sequence such that $x \notin BV$. Let $\{n_q\}_{q=1}^{\infty}$ be an increasing sequence of positive integers such that $|x_p| < 2^{-q}$ if $p \geqq n_q$. Let $A = (a_{pq})$ be defined by $a_{pq} = 1$ if $q = n_p$, $a_{pq} = 0$ otherwise. Clearly A is regular and if y is a subsequence of x, then $Ay \in BV$. The conjecture is valid, however, if we replace "subsequence" with "stretching," as shown by Theorem 2.

Note that (3) of the regularity conditions is not assumed in any of our results which follow.

THEOREM 1. *If x is a complex sequence, A is a matrix satisfying* (1) *and* (2) *of the regularity conditions, and ε is a positive term null sequence, then there exist $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$ such that $PAQx = x + u$, where $|u_n| < \varepsilon_n$, $n = 1, 2, 3, \cdots$.*

Proof. Let $w_n^{-1} = 1 + |x_1| + \cdots + |x_n|$, $n = 1, 2, 3, \cdots$. Take n_1 and m_1 such that

$$\left|\sum_{q=1}^{m_1-1} a_{n_1q} - 1\right| < \varepsilon_1 2^{-2} w_1, \left|\sum_{q=s}^{t} a_{n_1q}\right| < \varepsilon_1 2^{-3} w_2,\ m_1 \leqq s \leqq t\ .$$

Using (1), we can find a positive integer N such that if $n > N$, then

$$\sum_{q=1}^{m_1-1} |a_{nq}| < \varepsilon_2 2^{-2} w_1\ .$$

Take $n_2 > n_1 + N$ and $m_2 > m_1$ such that

$$\left|\sum_{q=m_1}^{m_2-1} a_{n_2q} - 1\right| < \varepsilon_2 2^{-2} w_2$$

and

$$\left|\sum_{q=s}^{t} a_{n_jq}\right| < \varepsilon_j 2^{j-5} w_3,\ m_2 \leqq s \leqq t,\ j = 1, 2\ .$$

Take $n_3 > n_2$ and $m_3 > m_2$ such that

$$\sum_{q=1}^{m_2-1} |a_{n_3q}| < \varepsilon_3 2^{-2} w_2,\ \left|\sum_{q=m_2}^{m_3-1} a_{n_3q} - 1\right| < \varepsilon_3 2^{-2} w_3\ ,$$

$$\left|\sum_{q=s}^{t} a_{n_jq}\right| < \varepsilon_j 2^{j-6} w_4,\ m_3 \leqq s \leqq t,\ j = 1, 2, 3\ .$$

Continue the process. Let y be the stretching of x induced by $\{m_p\}_{p=0}^{\infty}$, where $m_0 = 1$. It is trivial to apply the Cauchy condition in order to show that $\sum_{q=1}^{\infty} a_{n_r q} y_q$ converges, $r = 1, 2, 3, \cdots$, and we omit the proof. If r is a positive integer, we have

$$\begin{aligned}
\left|\sum_{q=1}^{\infty} a_{n_r q} y_q - x_r\right| &\leqq \left|\sum_{q=m_{r-1}}^{m_r - 1} a_{n_r q} y_q - x_r\right| + w_{r-1}^{-1} \sum_{q=1}^{m_{r-1}-1} |a_{n_r q}| \\
&\quad + \sum_{p=r}^{\infty} \left|\sum_{q=m_p}^{m_{p+1}-1} a_{n_r q} y_q\right| \\
&\leqq |x_r| \left|\sum_{q=m_{r-1}}^{m_r - 1} a_{n_r q} - 1\right| + w_{r-1}^{-1} \varepsilon_r 2^{-2} w_{r-1} \\
&\quad + \sum_{p=r}^{\infty} |x_{p+1}| \left|\sum_{q=m_p}^{m_{p+1}-1} a_{n_r q}\right| \\
&\leqq |x_r| \varepsilon_r 2^{-2} w_r + \varepsilon_r 2^{-2} + \sum_{p=r}^{\infty} |x_{p+1}| \varepsilon_r 2^{r-3-p} w_{p+1} \\
&< \varepsilon_r .
\end{aligned}$$

Thus if $P \in \mathscr{P}$ is determined by $\{n_p\}$ and $Q \in \mathscr{Q}$ is determined by $\{m_p\}$, then $PAQx = x + u$, where $|u_n| < \varepsilon_n$, $n = 1, 2, 3, \cdots$. This completes the proof.

THEOREM 2. *A complex sequence x is convergent (absolutely convergent) [bounded] $\{$divergent to $\infty\}$ if there exists a matrix A satisfying* (1) *and* (2) *of the regularity conditions such that Ay is convergent (absolutely convergent) [bounded] $\{$divergent to $\infty\}$ for every stretching y of x.*

Proof. Take $\varepsilon_p = 2^{-p}$, $p = 1, 2, 3, \cdots$, and apply Theorem 1 to obtain $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$ such that $PAQx = x + u$, where $|u_n| < \varepsilon_n$, $n = 1, 2, 3, \cdots$. If Ay converges for every stretching y of x, then AQx converges and so $PAQx$ converges since P is regular. Thus $x = PAQx - u$ converges since u converges. If $Ay \in BV$ for every stretching y of x, then $AQx \in BV$ and so $PAQx \in BV$ since P is super regular, i.e., preserves absolute convergence. Hence $x = PAQx - u \in BV$ since $u \in BV$. The statements involving boundedness and divergence to ∞ follow similarly. This completes the proof.

Next we use Theorem 1 to prove an extension of the result of Agnew [1] previously mentioned. Specifically, we obtain Agnew's conclusion (in a sense) after weakening his hypothesis in two ways. Besides dropping the assumption that A satisfy (3) of the regularity conditions, we replace the boundedness of x with the assumption that x have a finite limit point.

THEOREM 3. *If x is a complex sequence having a finite limit point and A is a matrix satisfying* (1) *and* (2) *of the regularity conditions, then there exist $P_1, P_2 \in \mathscr{P}$ such that every finite limit point of x is a limit point of P_1AP_2x.*

Proof. Following Agnew [1, p. 596], we use the separability of the complex plane to obtain an infinite sequence $u_1, u_2, u_3, \cdots$, such that each u_j is a finite limit point of x and every finite limit point of x is either a term of $u_1, u_2, u_3, \cdots$ or a limit point of this sequence. Still following Agnew, we form the sequence

$$u_1; u_1, u_2; u_1, u_2, u_3; \cdots ,$$

which is then relabeled $v_1, v_2, v_3, \cdots$. Let A be a matrix satisfying (1) and (2), and let $\{\varepsilon_p\}$ be a positive term null sequence. Apply Theorem 1 to obtain an increasing sequence $\{n_p\}$ of positive integers and a stretching z of v such that

$$\left|\sum_{q=1}^{\infty} a_{n_pq}z_q - v_p\right| < \varepsilon_p,\ p = 1, 2, 3, \cdots .$$

Then every finite limit point of x is a limit point of $\{(Az)_{n_p}\}_{p=1}^{\infty}$. Let $\{m_p\}_{p=0}^{\infty}$ be an increasing sequence of positive integers such that z is the stretching of v induced by $\{m_p\}$. If p is a positive integer, then, since each of the sequences $\{a_{jq}\}_{j=1}^{\infty}$, $m_{p-1} \leqq q < m_p - 1$, is convergent, there exists a number $L_p > 1$ such that $|a_{st}| < L_p$, $s \geqq 1$, $m_{p-1} \leqq t < m_p - 1$. We now construct a subsequence y of x as follows. Let $y_1, \cdots, y_{m_1-1}$ be a finite subsequence of x such that

$$\sum_{p=1}^{m_1-1} |y_p - v_1| < (2L_1)^{-1} .$$

Let $y_{m_1}, \cdots, y_{m_2-1}$ be such that $y_1, \cdots, y_{m_2-1}$ is a finite subsequence of x and

$$\sum_{p=m_1}^{m_2-1} |y_p - v_2| < (2^2L_2)^{-1} .$$

Continue the process. Using the convergence of $\sum_{q=1}^{\infty} a_{n_pq}z_q$ and the Cauchy condition, it is easy to show that $\sum_{q=1}^{\infty} a_{n_pq}y_q$ converges, $p = 1, 2, 3, \cdots$. The details will not be given. Let $\lambda > 0$. Take k to be a positive integer such that $2^{-k} < \lambda$. Take N so that if $n > N$, then $|a_{nq}| < \lambda$, $1 \leqq q < m_k$. For $n_p > N$, we have

$$\begin{aligned}\left|\sum_{q=1}^{\infty} a_{n_p q} y_q - v_p\right| &\leqq \left|\sum_{q=1}^{\infty} a_{n_p q} y_q - \sum_{q=1}^{\infty} a_{n_p q} z_q\right| + \left|\sum_{q=1}^{\infty} a_{n_p q} z_q - v_p\right| \\ &\leqq \sum_{j=1}^{k} \sum_{q=m_{j-1}}^{m_j - 1} |a_{n_p q}| |y_q - v_j| \\ &\quad + \sum_{j=k+1}^{\infty} \sum_{q=m_{j-1}}^{m_j - 1} |a_{n_p q}| |y_q - v_j| + \varepsilon_p \\ &< \lambda \sum_{j=1}^{k} 2^{-j} + \sum_{k=k+1}^{\infty} L_j (2^j L_j)^{-1} + \varepsilon_p \\ &< \lambda + 2^{-k} + \varepsilon_p \\ &< 2\lambda + \varepsilon_p .\end{aligned}$$

Hence every finite limit point of x is a limit point of $\{(Ay)_{n_p}\}_{p=1}^{\infty}$. Clearly the conclusion follows.

COROLLARY. *A complex sequence x diverges to ∞ if there exists a matrix A satisfying* (1) *and* (2) *of the regularity conditions such that Ay diverges to ∞ for every subsequence y of x.*

Proof. Suppose A satisfies (1) and (2) and Ay diverges to ∞ for every subsequence y of x, but x has a bounded subsequence. Then x has a finite limit point P, and by the theorem, there exists a subsequence z of x such that P is a limit point of Az. But this is a contradiction. Thus x cannot have a bounded subsequence. Hence x diverges to ∞, and the proof is complete.

It is interesting to note that in Theorem 3 we cannot in general prove that every limit point of x (finite or infinite) is a limit point of $\{(Ay)_{n_p}\}_{p=1}^{\infty}$. For example, let $x = \{1, 0, 4, 0, 16, 0, \cdots\}$ and let A be defined by $a_{pq} = 2^{p-q-1}$ if $q \geqq p$, $a_{pq} = 0$ otherwise. Clearly, if a subsequence y of x contains at most a finite number of nonzero terms, then $(Ay)_n \to 0$ as $n \to \infty$. But if y contains infinitely many nonzero terms of x, then $\sum_{q=1}^{\infty} a_{pq} y_q$ diverges, $p = 1, 2, 3, \cdots$.

On the other hand, we have the following modification of Theorem 3.

THEOREM 4. *If x is a complex sequence and A is a row-finite matrix satisfying* (1) *and* (2) *of the regularity conditions, then there exists a subsequence y of x such that every limit point of x* (*finite or infinite*) *is a limit point of Ay.*

The proof of Theorem 4 involves only minor changes in the proof of Theorem 3, and will be omitted.

COROLLARY. *A complex sequence x is bounded if there exists a matrix A satisfying* (1) *and* (2) *of the regularity conditions such that Ay is bounded for every subsequence y of x.*

Proof. Suppose A satisfies (1) and (2) and Ay is bounded for every subsequence y of x, but x is not bounded. Then A is row-finite, for otherwise we could construct a subsequence z of x such that Az is not defined. Thus by the theorem, there exists a subsequence w of x such that ∞ is a limit point of Aw. But this is a contradiction. Hence x is bounded.

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