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HAUSDORFF DIMENSIONS FOR COMPACT SETS IN R^n

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A general Hausdorff dimension of sets in R^n is studied by considering the dependence of the dimension upon the size and shape, relative to the convex measure, of the elements in the covering family. The Hausdorff dimension of compact sets is related to the behavior of distribution functions of finite measures of compact support in R^n. A comparison of dimensions using diameter and Lebesgue measure is given in terms of the regularity of the shape of elements in the covering family.

1. Introduction. Eggleston [3] defined the Hausdorff dimension of sets E in R^n as follows: Let C denote the collection of all convex sets in R^n; and, for each positive number β, write

$$C^\beta(E) = \inf\{\Sigma(\delta(C_i))^\beta \colon \bigcup C_i \supseteqq E, \{C_i\} \subseteqq C\}\ ,$$

where $\delta(A)$ denotes the diameter of A. The Hausdorff dimension of E, denoted by $C(E)$, is then the supremum over all values β where $C^\beta(E) > 0$. This notion of dimension has been generalized in various ways in R^1, e.g., [1], [2], [5], [6]; and it is the intent of this paper to study the situation in R^n, where apparently deeper problems are involved than those studied in [2].

In particular, let τ be a nonnegative, monotone, translation invariant set function, defined and sub-additive on the convex subsets of R^n in the sense that if $\{A_i\}$ is a convex covering of the convex set A, then $\tau(A) \leqq \Sigma\tau(A_i)$. If, in addition, $\tau(A)$ tends to zero with $\delta(A)$, then τ is said to be a *convex measure* on R^n. Let K be an arbitrary collection of n-dimensional rectangles (hereafter referred to as rectangles) which have edges parallel to the coordinate axes and uniformly bounded diameters. If K is closed under translations, and contains a sequence of rectangles $\{R_i\}$ for which $\delta(R_i) \to 0$, then K is called a *covering class*. If K is a covering class, τ a convex measure on R^n, β a positive number, and E a subset of R^n, put

$$K_\tau^\beta(E) = \inf\{\Sigma\tau(A_i)^\beta \colon \bigcup A_i \supseteqq E, \{A_i\} \subseteqq K\}\ .$$

The *Hausdorff dimension of E relative to the convex measure τ and the covering class K* is the number

$$K_{\tau(E)} = \sup\{\beta \colon K_\tau^\beta(E) > 0\}\ .$$

The remainder of this work is concerned with the dependence of $K_\tau(E)$

upon the choices of K and τ. The nature of this dependence is more interesting in $R^n (n \geqq 2)$ than in R^1 for various reasons. One reason is that the usual choices for the convex measure τ, diameter δ and Lebesgue measure m, coincide in R^1. Another is that covering classes in R^1 are completely determined by the length of their members, while in higher dimensions, *shape* as well as *size* plays a key role.

Theorem 1 relates the Hausdorff dimension of compact subsets of R^n to the behavior of distribution functions of finite measures supported by such sets. The theorem yields a sufficient condition for the relation

$$K_\tau(E) \geqq M_\tau(E)$$

for all compact sets E, in terms of the *shape* and *size* of elements in the covering classes K and M. A second result, Theorem 2, relates the dimensions $K_\delta(E)$ and $K_m(E)$ by establishing a necessary and sufficient condition for the relation

$$K_m(E) = \frac{1}{n} K_\delta(E)$$

to hold for all compact sets E.

2. **The Hausdorff dimension of compact sets.** If K is a covering class, then K can be completely described by a set of points in R^n; namely by those points $\boldsymbol{x}$ whose i^{th} component, x_i, is the length of the edge of the given rectangle which is parallel to the i^{th} coordinate axis. Accordingly, a set of points K in R^n, with positive coordinates, is a covering class, if and only if it is bounded and contains a sequence converging to the origin. In the following, elements of a covering class will be referred to either as points or as rectangles, as convenience dictates. Now let E be a compact subset of R^n, and denote by $\mathscr{M}(E)$, the class of all positive finite measures μ supported in E. If F_μ is the distribution function of μ, write for $\boldsymbol{a}$ in K,

$$\Delta F_\mu(\boldsymbol{a}) = \bigvee \mu(R_a + \boldsymbol{y}) \ , \quad (\boldsymbol{y} \in R^n)$$

where $R_a = \{\boldsymbol{x}\colon 0 \leqq x_i < a_i;\ i = 1, 2, \cdots, n\}$. Finally put

$$K_\tau(\mu) = \liminf\, (\log \Delta F_\mu(\boldsymbol{a}) / \log \tau(R_a)) \ ,$$

as $\tau(R_a) \to 0$, $\boldsymbol{a} \in K$. The number $K_\tau(\mu)$ is called the *Hausdorff dimension of the measure μ with respect to K and τ.* The connection between the Hausdorff dimension of E and the Hausdorff dimension of the measures it supports is given by

THEOREM 1. *For all compact sets E, covering classes K, and convex measures τ,*

$$K_\tau(E) = \sup\{K_\tau(\mu): \mu \in \mathscr{M}(E)\} .$$

Proof. The inequality

$$K_\tau(E) \geqq \sup\{K_\tau(\mu): \mu \in \mathscr{M}(E)\}$$

is immediate. Indeed, if $\mu \in \mathscr{M}(E)$ and $0 < \beta < K_k(\mu)$, then there is $\delta > 0$ for which $\tau(R_a) < \delta$ implies $\tau(R_a)^\beta > \Delta F_\mu(\boldsymbol{a})$, for all $\boldsymbol{a} \in K$. Hence if $\mathscr{A}$ is a countable covering of E by rectangles of K, then

$$\sum_{A \in \mathscr{A}} (\tau(A))^\beta \geqq \mu(E) \wedge \delta^\beta .$$

Since the right-hand side is positive and independent of the covering, it follows that $\beta \leqq K_\tau(E)$. To establish the reverse inequality, assume that the points of K have all coordinates of the form 2^{-m}, m integral. It will be shown later that this assumption is not restrictive. Let $\{\boldsymbol{a}(m)\}$ be a sequence in K tending to the origin. Fixing m, let $K(m)$ denote the points of K in $\{\boldsymbol{x}: x_i \geqq a(m)_i\}$. If β is a positive number and $\beta < K_\tau(E)$, then a measure ν_m can be associated with E as follows. Each $K(m)$ contains a finite number of points which are taken to be lexicographicallv ordered, say

$$\boldsymbol{b}(1) > \boldsymbol{b}(2) > \cdots > \boldsymbol{b}(p) ,$$

with $\boldsymbol{b}(p) = \boldsymbol{a}(m)$. For each $j = 1, 2, \cdots, p$, let A_j denote the partition of R^n induced by the rectangle $R_{b(j)}$. If Q is a subrectangle of A_j, write $\delta(E, Q) = \sup\{\chi_{E\cap Q}(\boldsymbol{x}): \boldsymbol{x} \in R^n\}$,

$$f_0(\boldsymbol{x}) = \sum_{Q \in A_p} \tau(Q)^\beta \delta(E, Q)\chi_Q(\boldsymbol{x})$$

For each index $j = 0, 1, \cdots, p-2$ write

$$f_{j+1}(\boldsymbol{x}) = \sum_{Q \in A_{p-(j+1)}} \left(1/\bigwedge\, (\tau(Q)^\beta \Big/ \int_Q f_j(\boldsymbol{x})d\boldsymbol{x}\right) \cdot \chi_Q(\boldsymbol{x}) \cdot f_j(\boldsymbol{x}) ,$$

allowing $\tau(Q)^\beta \Big/ \int_Q f_j(\boldsymbol{x})d\boldsymbol{x}$ to take the value $+\infty$, when $\int_Q f_j(\boldsymbol{x})d\boldsymbol{x}$ is zero. Finally, the measure ν_m is defined to be

$$\nu_m(A) = \int_A f_{p-1}(\boldsymbol{x})d\boldsymbol{x} .$$

LEMMA 1. *For all R in $\bigcup_{j=1}^p A_j$, $\nu_m(R) \leqq (R)^\beta$. Moreover, for each $\boldsymbol{x}$ in E, there is Q in $\bigcup_{j=1}^p A_j$, containing $\boldsymbol{x}$ and such that $\nu_m(Q) = \tau(Q)^\beta$. This rectangle Q can be selected so that $Q \in A_{p-j}$ implies*

$\int_Q f_{j-1}(x)dx > \tau(Q)^\beta$, while $\int_Q f_j(x)dx = \tau(Q)^\beta$ $(j > 0)$.

Proof. The first assertion follows immediately from the fact that for all x, $f_{j+1}(x) \leqq f_j(x)$. For the second part, let $x \in E$ and $x \in P \in A_p$. If $\nu_m(P) = \tau(P)^\beta$ there would be nothing to prove. Otherwise, let j be such that

$$\int_P f_{j-1}(x)dx > \nu_m(P)$$

and

$$\int_P f_j(x)dx = \nu_m(P) .$$

It would then follow that

$$\int_P f_{j-1}(x)dx > \int_P f_j(x)dx = \left[1 \wedge \tau(Q)^\beta \Big/ \int_Q f_{j-1}(x)dx\right] \cdot \int_P f_{j-1}(x)dx ,$$

where Q is that unique element of A_{p-j} containing P. Hence, $\tau(Q)^\beta < \int_Q f_{j-1}(x)dx$, and so $\int_Q f_j(x)dx = \tau(Q)^\beta$. If there were an index l with $j \leqq l$ and

$$\int_Q f_l(x)dx > \int_Q f_{l+1}(x)dx ,$$

it would follow that $f_l(x) > f_{l+1}(x)$ for all x in P, which in turn would imply that

$$\int_P f_l(x)dx > \int_P f_{l+1}(x)dx ,$$

contradicting the choice of j. Hence $\nu_m(Q) = \tau(Q)^\beta$ and the lemma is proved.

Returning to the proof of the theorem, it follows trivially from the first assertion of Lemma 1, that there is a positive constant A, independent of m, such that $\nu_m(R^n) \leqq A$. Less trivial is the fact that there is another positive constant B, for which $\nu_m(R^n) \geqq B$ for all m. Indeed, let $\mathscr{A}$ be a covering of E by the rectangles of $\bigcup_1^p A_j$, distinguished by Lemma 1, and with the property that no element of $\mathscr{A}$ contains another element of $\mathscr{A}$. Let $P, Q \in \mathscr{A}$, $P \cap Q \neq \varnothing$, $P \in A_{p-j}$, $Q \in A_{p-k}$, $1 \leqq j < k$, and R an arbitrary element of A_p contained in $P \cap Q$. The rectangles P and Q satisfy

$$\nu_m(P) = \tau(P)^\beta = \int_P f_j(x)dx < \int_P f_{j-1}(x)dx$$

and

$$\nu_m(Q) = \tau(Q)^\beta = \int_Q f_k(\boldsymbol{x}) < \int_Q f_{k-1}(\boldsymbol{x})d\boldsymbol{x} .$$

If $\boldsymbol{x}$ belongs to R and $f_j(\boldsymbol{x}) > 0$, then

$$f_k(\boldsymbol{x}) = f_{k-1}(\boldsymbol{x})\tau(Q)^\beta \Big/ \int_Q f_{k-1}(\boldsymbol{x})d\boldsymbol{x} < f_j(\boldsymbol{x}) .$$

Hence

$$\int_P f_k(\boldsymbol{x})d\boldsymbol{x} < \int_P f_j(\boldsymbol{x})d\boldsymbol{x} = \tau(P)^\beta ,$$

contradicting the choice of P and k. Thus $f_j(\boldsymbol{x}) \equiv 0$ on R, which shows $P \cap Q \cap S = \varnothing$, for every $S \in A_p$ intersecting E. If D denotes the union of all S in A_p, intersecting E, then $\mathscr{A}' = \{P \cap D\colon P \in \mathscr{A}\}$ is a disjoint covering of E. Since $\beta < K_\tau(E)$, there is a positive constant B, such that

$$\begin{aligned}\nu_m(R^n) &= \sum_{A \ni \mathscr{A}'} \nu_m(A) = \sum_{P \in \mathscr{A}} \nu_m(P) \\ &= \sum_{P \in \mathscr{A}} \tau(P)^\beta \geqq B .\end{aligned}$$

It follows that the sequence of measures $\{\nu_m\}$, has a subsequence which is weakly convergent to a measure ν for which $A \geqq \nu(R^n) \geqq B$. Since E is compact and $a(m) \to 0$, it follows that $\nu \in \mathscr{M}(E)$. If Q is a rectangle in K, then Q can be covered by 2^n of its translates, Q', for which $\nu_m(Q') \leqq \tau(Q')^\beta$, provided m is sufficiently large. Since τ is translation invariant and sub-additive on convex sets, it follows that

$$\nu(Q) \leqq 2^n\tau(Q)^\beta .$$

Hence, for each $\boldsymbol{a} \in K$, $\Delta F_\nu(\boldsymbol{a}) \leqq 2^n\tau(R_a)^\beta$, which shows $K_\tau(\nu) \geqq \beta$, and thus

$$K_\tau(E) \leqq \sup\{K_\tau(\mu)\colon \mu \in \mathscr{M}(E)\} .$$

Finally, it must be shown that the assumption that K consists only of points having all coordinates of the form 2^{-m}, can be eliminated. Let K be an arbitrary covering class, and let K' be obtained from K by stipulating that $\boldsymbol{a}' \in K'$, if and only if, there is $\boldsymbol{a}$ in K such that for each j,

$$a'_j = 2^{-m} \geqq a_j > 2^{-m-1}$$

for some integer m. By what has already been shown, it is sufficient to prove that for each compact set E, and each finite measure μ of compact support

$$K_\tau(E) \leqq K'_\tau(E)$$

and

$$K'_\tau(\mu) \leqq K_\tau(\mu) \ .$$

To establish the first of these relations, let $\beta < K_\tau(E)$ and let $\mathscr{A}$ be a covering of E by rectangles of K'. If $P' \in \mathscr{A}$, then P' can be covered by 2^n rectangles of K, from which P' was formed. The collection of such rectangles $\mathscr{B}$ is again a covering of E. It follows that there is a positive number B such that

$$B \leqq \sum_{P \in \mathscr{B}} \tau(P)^\beta \leqq 2^n \sum_{P' \in \mathscr{A}} \tau(P')^\beta \ ,$$

and so $\beta \leqq K'_\tau(E)$, which entails $K_\tau(E) \leqq K'_\tau(E)$. By the sub-additivity of τ on convex sets,

$$\tau(R_{a'}) \leqq 2^n \tau(R_a)$$

for each $\boldsymbol{a} \in K$, and so

$$\frac{\log \varDelta F_\mu(\boldsymbol{a}')}{\log \tau(R_{a'})} \leqq \frac{\log \tau(R_a)}{n \log 2 + \log \tau(R_a)} \cdot \frac{\log \varDelta F_\mu(\boldsymbol{a})}{\log \tau(R_a)} \ ,$$

which implies that $K'_\tau(\mu) \leqq K_\tau(\mu)$. The proof of Theorem 1 is now complete.

The following illustrates the usefulness of Theorem 1 in questions dealing with the dimension of compact sets. Since dimension is monotone with respect to covering classes, i.e., $K_1 \subseteqq K_2$ implies $K_1(E) \geqq K_2(E)$ for all E, it is natural to consider the following question. Suppose two covering classes, K and M, are given and are related by a map $\varphi: K \to M$. What conditions on φ will guarantee $K_\tau(E) \geqq M_\tau(E)$ for all compact E? It would be difficult to guess such conditions using only the definitions of § 1. By Theorem 1, however, it is sufficient to obtain conditions on φ implying $K_\tau(\mu) \geqq M_\tau(\mu)$ for all finite measures of compact support in R^n. Since $\varDelta F_\mu$ is sub-additive in each component, it follows that

$$\varDelta F_\mu(\boldsymbol{x}) \leqq 2^n \cdot \left(\sum_{i=1}^n [1 \vee (x_i/s_i)]\right) \cdot \varDelta F_\mu(\boldsymbol{s}) \ ,$$

and so

$$\frac{\log \varDelta F_\mu(\boldsymbol{x})}{\log \tau(R_x)}$$
$$\geqq \frac{\log \tau(R_{\varphi(x)})}{\log \tau(R_x)} \cdot \frac{\log \varDelta F_\mu(\varphi(\boldsymbol{x}))}{\log \tau(R_{\varphi(x)})} + \sum_{i=1}^n \frac{\log (1 \vee (x_i/\varphi(x)_i))}{\log \tau(R_x)} + \frac{\log 2^n}{\log \tau(R_x)} \ .$$

Hence the following

COROLLARY. *Given covering classes K and M, and the convex-measure τ, $M_\tau(E) \leqq K_\tau(E)$ for all compact E, provided there is a map $\varphi\colon K \to M$ with the properties*:

(i) $$\lim_{\tau(R_x)\to 0} (\log \tau(R_{\varphi(x)})/\log \tau(R_x)) = 1 \;, \quad x \in K$$

and

(ii) *For* $j = 1, 2, \cdots, n$,
$$\lim_{\tau(R_x)\to 0} (\log (x_j/\varphi(x)_j)/\log \tau(R_x)) = 0 \;, \quad x \in K \,.$$

REMARK 1. The preceding corollary shows that $\varphi(K)_\tau(E) \leqq K_\tau(E)$ for all compact E and for φ satisfying (i) and (ii). If, in addition, φ has the property that $\tau(R_x) \to 0$ as $\tau(R_{\varphi(x)}) \to 0$, then it is clear that $\varphi(K)_\tau(E) = K_\tau(E)$ for all compact E. This fact will be used without explicit mention in § 3 below.

REMARK 2. The function φ defined by $\varphi(x)_i = x_i \wedge a_i$ for $\boldsymbol{a}$ with $a_i > 0$, $i = 1, 2, \cdots, n$, maps any covering class K into the rectangle R_a. Since K is bounded and

$$\tau(R_x) \cdot \prod_1^n \left(\frac{x_i}{a_i} + 1\right)^{-1} \leqq \tau(R_{\varphi(x)}) \leqq \tau(R_x) \;,$$

it follows that φ satisfies conditions (i) and (ii), and the property mentioned in Remark 1. Thus in the following it will be assumed that covering classes are contained in a rectangle R_a for convenient choice of a.

REMARK 3. In § 3, the conditions (i) and (ii) are shown to be necessary for $M_m(E) \leqq K_m(E)$, in the special case that M consists entirely of cubes. For $n = 1$, the conditions are known to be necessary [2], but complete results are not known at present for $n \geqq 2$.

REMARK 4. The idea for the construction of the measure ν in the proof of Theorem 1 is due to O. Frostman [4], although his construction is carried out in R^1, and for the covering class consisting of all intervals. It seems to be difficult to prove a version of Theorem 1 when covering classes are presumed closed under all rigid transformations.

3. Dimension as a function of the convex-measure τ. If E is a compact subset of R^n, let $K_m(E)$ and $K_\delta(E)$ denote, respectively, the dimension of E relative to K and Lebesgue measure m, and the dimension of E relative to K and diameter δ. In general,

$$nK_m(E) \leqq K_\delta(E) ,$$

since $m(R) \leqq \delta(R)^n$ for rectangles in R^n. The results of this section establish a necessary and sufficient condition for equality to hold in the above relation.

THEOREM 2. *Given a covering class K,*

$$nK_m(E) = K_\delta(E)$$

for all compact subsets E of R^n, if and only if, there is a covering class S, consisting of cubes, for which $S_m(E) = K_m(E)$ for all compact E.

Proof. Suppose $nK_m(E) = K_\delta(E)$ holds for all compact E. Let K^* be the covering class of cubes obtained from K by writing $\boldsymbol{a}^* \in K^*$ if and only if there is $\boldsymbol{a}$ in K such that

$$a_j^* = \max_i a_i \qquad \text{for } j = 1, 2, \cdots, n .$$

If $\beta > K_\delta(E)$ and $\varepsilon > 0$, then there is a covering, $\{R_i\}$, of E in K such that

$$\varepsilon > \Sigma\, \delta(R_i)^\beta .$$

If R_i^* denotes the cube of K^* corresponding to, and concentric with R_i, then $\bigcup R_i^* \supseteqq E$ and

$$n^{\beta/2}\varepsilon \geqq \Sigma\, \delta(R_i^*)^\beta .$$

It follows that $\beta \geqq K_\delta^*(E)$ and so $K_\delta(E) \geqq K_\delta^*(E)$. Consequently,

$$nK_m(E) = K_\delta(E) \geqq K_\delta^*(E) = nK_m^*(E) ,$$

the last equality arising from the fact that

$$m(R) = n^{-n/2}\,\delta(R)^n$$

for cubes R. Hence $K_m(E) \geqq K_m^*(E)$ for all compact sets E. Before proceeding, it will be convenient to introduce some notation and new concepts. Let $\mathscr{F}$ denote the collection of all real-valued function f defined on R^1 and unbounded on the positive portion of R^1 with the properties that $f(0) \leqq 0$ and that $x \leqq y$ implies

$$0 \leqq f(y) - f(x) \leqq y - x .$$

With each such function f associate a compact set $E = E(f)$ in R^1 as follows. Let $\{\xi(j)\}$ be a positive, decreasing sequence for which $f(-\log \xi(j)) = j \log 2$. Since $f(x) - x$ is nonincreasing, it follows that $\Sigma\xi(j) \leqq 1$, and so the set

$$E = \{\xi: \xi = \Sigma\varepsilon_j\xi(j), \varepsilon_j = 0 \text{ or } 1\}$$

is compact. Moreover, the function

$$F_\mu(x) = \sup\{\Sigma\varepsilon_j 2^{-j}: x \geqq \Sigma\varepsilon_j\xi(j)\}$$

is sub-additive and is the distribution function of a finite measure μ, supported on E. Now let $\mathscr{G}$ be the collection of all functions g on R^n which are of the form $g(\boldsymbol{x}) = \sum_1^n f_i(x_i)$ for $f_i \in \mathscr{F}$. With each such g, associate the compact set

$$E_g = E(f_1) \times \cdots \times E(f_n) .$$

If F_i and μ_i denote, respectively, the distribution function and finite measure associated with $E(f_i)$, then

$$F_\mu(\boldsymbol{x}) = \prod_{i=1}^n F_i(x_i)$$

is the distribution function of the product measure $\mu = \mu_1 \times \cdots \times \mu_n$ supported on E_g. Since each F_i is sub-additive, it follows that $\Delta F_\mu \equiv F_\mu$. Finally, if $g \in \mathscr{G}$ and K is a covering class, define

$$K_m(g) = \liminf (g(\boldsymbol{x})/\Sigma x_i) ,$$

taken as $\Sigma x_i \to \infty$ over points $\boldsymbol{x}$ for which there is $\boldsymbol{a}$ in K with $x_i = -\log a_i$, $i = 1, 2, \cdots, n$. The relationship between g and E_g is given by

LEMMA 2. *For all $g \in \mathscr{G}$ and all covering classes K,*

$$K_m(g) = K_m(E_g) .$$

Proof. Assuming that $g(\boldsymbol{x}) = \Sigma f_i(x_i)$, let $\{\xi_i(j)\}$, satisfy

$$f_i(-\log \xi_i(j)) = j \log 2 \qquad (i = 1, \cdots, n; j = 1, 2, \cdots) .$$

Given the point $\boldsymbol{x}$, there are indices $k_1, \cdots, k_1$, for which

$$-\log \xi_i(k_i) \leqq x_i \leqq -\log \xi_i(k_i+1) , \qquad (i = 1, 2, \cdots, n) .$$

It follows that

$$-\log 2 - \log F_i(\exp(-x_i)) \leqq f_i(x_i) \leqq \log 2 - \log F_i(\exp(-x_i)) .$$

Thus, if a satisfies $x_i = -\log a_i$ $(i = 1, 2, \cdots, n)$, then

$$\frac{n \log 2}{\log m(R_a)} + \frac{\log \Delta F_\mu(\boldsymbol{a})}{\log m(R_a)} \leqq \frac{g(\boldsymbol{x})}{\Sigma x_i} + \frac{-n \log 2}{\log m(R_a)} + \frac{\log \Delta F_\mu(\boldsymbol{a})}{\log m\,(R_a)}$$

which implies $K_m(g) = K_m(\mu)$. If $\lambda \in \mathscr{M}(E_g)$ and $\boldsymbol{a} \in K$, with, say, $\xi_i(k_i + 1) \leqq a_i \leqq \xi_i(k_i)$ $(i = 1, 2, \cdots, n)$, then clearly,

$$\log \varDelta F_\lambda(\boldsymbol{a}) \geqq -\sum_i (k_i + 1) \log 2 \geqq -n \log 2 + \log \varDelta F_\mu(\boldsymbol{a}) .$$

It follows that $K_m(\lambda) \leqq K_m(\mu)$ so that by Theorem 1,

$$K_m(g) = K_m(\mu) = K_m(E_g)$$

and the lemma is proved.

At this point it is necessary to establish the fact that, in so far as compact sets are concerned, covering classes K can be assumed to have the property that if $\{R_n\}$ is a sequence of rectangles for which $m(R_n) \to 0$, then $\delta(R_n) \to 0$ as $n \to \infty$.

LEMMA 3. *Given a covering class K, there is a covering class K' such that*

(i) $K_m(E) = K'_m(E)$ *for all compact sets* E,

and

(ii) *If* $\{R_n\} \subseteqq K'$ *with* $m(R_n) \to 0$, *then* $\delta(R_n) \to 0$ $(n \to \infty)$.

Proof. Let p be a permutation of the first n ı
and write

$$K(p) = \{\boldsymbol{a} \in K\colon a_{p(1)} \geqq \cdots \geqq a_{p(n)}\} .$$

Define

$$\varphi(t) = \begin{cases} -1/\log t; & 0 < t \leqq 1/e \\ \quad 1 & , 1/e < t \end{cases}$$

Then φ is nondecreasing and $\varphi(t) \geqq t$ for $t \leqq 1/e$. If $\boldsymbol{x}$ belongs to $K(p)$, define

$$\psi(\boldsymbol{x}) = \boldsymbol{x} ,$$

in the case that $x_{p(i)} < \varphi(x_{p(i+1)})$ for $i = 1, 2, \cdots, n-1$. Otherwise define $\psi(\boldsymbol{x})$ by

$$\psi(x)_{p(i)} = \begin{cases} \varphi(x_{p(j+1)}) \ , & 1 \leqq i \leqq j \\ x_p(i) & , j + 1 \leqq i \leqq n , \end{cases}$$

where j is the largest integer $k \leqq n - 1$ for which

$$x_{p(k)} \geqq \varphi(x_{p(k+1)}) .$$

Consider the set $K(1)$, 1 denoting the identity permutation. The following remarks will apply to $K(p)$ for arbitrary p, by replacing every index j by its image $p(j)$. If $\boldsymbol{x} \in K(1)$, then

$$\frac{\log m(R_{\psi(x)})}{\log m(R_x)} = \left(1 + j\frac{\log \varphi(x_{j+1})}{\log x_{j+1} \cdots x_n}\right)\Big/\left(1 + \frac{\log x_1 \cdots x_j}{\log x_{j+1} \cdots x_n}\right)$$

and, by Remark 2 of § 2 with $R_a = \prod_1^n(0, 1/e)$,

$$0 \leqq \frac{\log x_1 \cdots x_j}{\log x_{j+1} \cdots x_n} \leqq j\frac{\log \varphi(x_{j+1})}{\log x_{j+1} \cdots x_n} \leqq \frac{j}{(n-j)}\frac{\log \varphi(x_{j+1})}{\log x_{j+1}} .$$

Suppose that $K(1)$ contains rectangles of arbitrarily small measure. Let $\varepsilon > 0$ and $\delta > 0$ be such that $0 < t < \delta$ implies

$$0 < \log \varphi(t)/\log t < \varepsilon/(n-1) .$$

Select $\gamma > 0$ so that $0 < t < \gamma$ implies $0 < \varphi^n(t) < \delta$. Now if $x_1 \cdots x_n < \gamma^n$, then $x_n < \gamma$ and so $\varphi^n(x_n) < \delta$. Now $\varphi^i(t) \geqq \varphi^k(t)$ if $i \geqq k$, and so

$$x_{j+1} < \varphi(x_{j+2}) \leqq \cdots \leqq \varphi^{n-j-1}(x_n) < \delta .$$

It follows that $\log \varphi(x_{j+1})/\log x_{j+1} < \varepsilon/(n-1)$, and so

$$\frac{1}{1+\varepsilon} \leqq \frac{\log m(R_{\psi(x)})}{\log m(R_x)} \leqq 1 + \varepsilon .$$

Since $K = \bigcup_p K(p)$, it now follows that

$$\lim_{m(R_x)\to 0} \frac{\log m(R_{\psi(x)})}{\log m(R_x)} = 1 \qquad (x \in K) .$$

A similar analysis shows that condition (ii) of the corollary to Theorem 1 also holds, with $\tau = m$. If $K' = \psi(K)$, then $K_m(E) = K'_m(E)$ for all compact sets E. For the second assertion of the lemma, consider again $K'(1)$, this set being typical of the general case. Let $\varepsilon > 0$, and $\delta > 0$ such that $0 < t < \delta$ implies $\varphi^n(t) < \varepsilon$. As before, if $x_1 \cdots x_n < \delta^n$, then $\varphi^n(x_n) < \varepsilon$ and so,

$$x_{j+1} \leqq \varphi(x_{j+2}) \leqq \cdots \leqq \varphi^{n-j-j}(x_n) < \varepsilon .$$

Hence

$$(\psi(x)_1^2 + \cdots + \psi(x)_n^2)^{1/2} \leqq (j\varphi(\varepsilon)^2 + (n-j)\varepsilon^2)^{1/2} .$$

Since $\psi(x)_1 \cdots \psi(x)_n \to 0$ implies $x_1 \cdots x_n \to 0$, the second assertion is proved.

Since $K_m(E) \geqq K_m^*(E)$ for all compact E, by Lemma 3, the same relation holds for K', i.e., $K'_m(E) \geqq K_m^*(E)$. The proof of the first part of Theorem 2 will be concluded with

LEMMA 4. *Let K be a covering class with the property that if $\{R_n\} \subseteqq K$ and $m(R_n) \to 0$, then $\delta(R_n) \to 0$ $(n \to \infty)$. If S is a covering* *class consisting of cubes and* $K_m(E) \geqq S_m(E)$ *for all compact* E, *then the map* ψ *defined on* K *by* $\psi(a)_j = \max_{1 \leqq i \leqq n} a_i$, $(j = 1, 2, \cdots, n)$ *has the properties* (i) *and* (ii) *listed in the corollary to Theorem* 1 *for* $\tau = m$.

Proof. Let $\varepsilon > 0$ and let p be a permutation of $\{1, 2, \cdots, n\}$. Write

$$K(p, \varepsilon) = \left\{ \boldsymbol{a} \in K \colon a_{p(1)} \leqq \cdots \leqq a_{p(n)} \text{ and } \frac{\log a_{p(1)}}{\log m(R_a)} - \frac{1}{n} \geqq \varepsilon \right\} .$$

Suppose that $K(p, \varepsilon)$ contains rectangles of arbitrarily small measure. Let $\gamma_1, \cdots, \gamma_n$ be selected so that $0 < \gamma_i < 1$ and for all $\boldsymbol{a}$ in $K(p, \varepsilon)$,

$$\left(\frac{1}{n} - \frac{\log a_{p(1)}}{\log m(R_a)} \right) (\gamma_{p(1)} - \gamma_{p(n)}) \geqq \varepsilon/2$$

and

$$\sum_{j=2}^{n-1} | \gamma_{p(j)} - \gamma_{p(n)} | \leqq \varepsilon/4 .$$

For each i, $(i = 1, 2, \cdots, n)$, define

$$f_i(t) = \bigvee_{s \in S} (-\gamma_i \log s_i \wedge (t + (1 - \gamma_i) \log s_i)) .$$

Then $f_i \in \mathscr{F}$ $(i = 1, 2, \cdots, n)$ and hence consider $g(\boldsymbol{x}) = \Sigma f_i(x_i)$ in $\mathscr{G}$. Clearly $S_m(g) = 1/n \sum_1^n \gamma_i$. On the other hand, for $\boldsymbol{a}$ in $K(p, \varepsilon)$ and $\boldsymbol{x}$ with $x_i = -\log a_i$ $(i = 1, \cdots, n)$,

$$\frac{g(\boldsymbol{x})}{\Sigma x_i} = \Sigma \frac{f_i(x_i)}{x_i} \cdot \frac{x_i}{\Sigma x_i} \leqq \Sigma \gamma_i \frac{\log a_i}{\log m(R_a)} .$$

It follows that

$$\begin{aligned} S_m(g) - K_m(p, \varepsilon)(g) &\geqq \frac{1}{n} \sum_1^n \gamma_i - \sum_1^n \gamma_i \frac{\log a_i}{\log m(R_a)} \\ &= \sum^{n-1} \left(\frac{1}{n} - \frac{\log a_{p(i)}}{\log m(R_a)} \right) (\gamma_{p(i)} - \gamma_{p(n)}) \geqq \varepsilon/4 . \end{aligned}$$

If $K_m(p, \varepsilon)$ is a covering class, Lemma 2 implies that there is a compact set E for which

$$S_m(E) - K_m(p, \varepsilon)(E) \geqq \varepsilon/4 .$$

Since $K(p, \varepsilon) \subseteqq K$, $K(p, \varepsilon)$ cannot contain rectangles of arbitrarily small measure, and thus

$$\lim \left(\max_{1 \leqq i \leqq n} \frac{\log a_i}{\log m(R_a)} \right) = 1/n \quad (m(R_a) \longrightarrow 0, \boldsymbol{a} \in K) .$$

Since $\sum_1^n \log a_i/\log m(R_a) = 1$, it also follows that

$$\lim\Big(\min_{1\le i\le n}\frac{\log a_i}{\log m(R_a)}\Big) = 1/n \quad (m(R_a) \longrightarrow 0,\ \boldsymbol{a}\in K)\ ,$$

and so,

$$\lim\Big(\frac{\log m(R_{\psi(a)})}{\log m(R_a)}\Big) = 1 \quad (m(R_a) \longrightarrow 0\ ,\ \boldsymbol{a}\in K)\ .$$

Moreover, for each j,

$$\lim\Big(\frac{\log a_j - \log \psi(a)_j}{\log m(R_a)}\Big) = \lim\Big(\frac{\log a_j}{\log m(R_a)} - \frac{1}{n}\Big) = 0$$
$$(m(R_a) \longrightarrow 0,\ \boldsymbol{a}\in K)$$

and the lemma is proved.

It now follows that $K'_m(E) = \psi(K')_m(E)$ for all compact sets E, and so $K_m(E) = \psi(K')_m(E)$, which concludes the proof of the first part of Theorem 2. For the second part, assume that $K_m(E) = S_m(E)$ for all compact E and some covering class of cubes S. Observe that this condition implies that $\delta(R_j)\to 0$ whenever $m(R_j)\to 0$, $\{R_j\}\subseteqq K$. Indeed, if $\limsup \delta(R_k) = b > 0$, while $m(R_k)\to 0$, then extract a subsequence, say $\{P_j\}$, from $\{R_k\}$ for which $\delta(P_j)\geqq b/2$, and such that there is l, $(1\leqq l\leqq n)$ for which the edges of the rectangles P_k, parallel to the lth coordinate axis have length at least $b/2\sqrt{n}$. Then the set

$$E = \{\boldsymbol{x}\colon x_i = 1/2,\ i\ne l, \text{ and } 0\leqq x_l\leqq b^2/2n\}$$

is such that $S_m(E) = 1/n$, while $K_m(E) = 0$, which contradicts the assumption. Now, by Lemma 4 and the corollary to Theorem 1, $K_m(E)\geqq \psi(K)_m(E)$ for all compact E. Since

$$\psi(K)_m(E) = 1/n\ \psi(K)_\delta(E)\ ,$$

it is sufficient to establish the relation $\psi(K)_\delta(E)\geqq K_\delta(E)$ for all compact E. For this purpose, let φ be defined on $\psi(K)$ by writing $\varphi(\boldsymbol{x}) = z$, for some z in K for which $\psi(z) = \boldsymbol{x}$. For $\boldsymbol{x}$ in $\psi(K)$,

$$\frac{\log x_j - \log\varphi(x)_j}{\log\delta(R_x)}$$
$$= \Big(\frac{\log\psi(z)_j - \log z_j}{\log m(R_z)}\Big)\Big(\frac{\log m(R_z)}{\log m(R_{\psi(z)})}\Big)\Big(\frac{\log m(R_{\psi(z)})}{\log\delta(R_{\psi(z)})}\Big)\ .$$

Now $\log m(R_{\psi(z)})/\log\delta(R_{\psi(z)})$ is bounded for all z, since $R_{\psi(z)}$ is a cube. Moreover, since $m(R_z)\to 0$ as $\delta(R_x)\to 0$, the expressions

$$\frac{\log\psi(z)_j - \log z_j}{\log m(R_z)} \quad\text{and}\quad \frac{\log m(R_z)}{\log m(R_{\psi(z)})}\ ,$$

approach 0 and 1 respectively as $\delta(R_x)$ approaches 0. It follows that

$$\lim \frac{\log x_j - \log \varphi(x)_j}{\log \delta(R_x)} = 0 \qquad (\delta(R_x) \longrightarrow 0\,,\ x \in \psi(K))\,.$$

Also, since $R_x \supseteqq R_{\varphi(x)}$,

$$1 \leqq \frac{\log \delta(R_{\varphi(x)})}{\log \delta(R_x)} \leqq \frac{\log(\max \varphi(x)_i)}{\log \delta(R_x)} = \frac{\log \delta(R_x) - 1/2 \log n}{\log \delta(R_x)}\,,$$

and thus

$$\lim \frac{\log \delta(R_{\varphi(x)})}{\log \delta(R_x)} = 1\,, \qquad (\delta(R_x) \longrightarrow 0\,,\ x \in \psi(K))\,.$$

The map $\varphi\colon \psi(K) \rightarrow K$ thus satisfies the conditions listed in the corollary to Theorem 1 for $\tau = \delta$, and the desired inequality, $\psi(K)_\delta(E) \geqq K_\delta(E)$ is established; and the proof of Theorem 2 is complete.

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