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# CLOSED RANGE THEOREMS FOR CONVEX SETS AND LINEAR LIFTINGS

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**Let $M$ be a closed subspace of a Banach space $E$ such that its annihilator $M^{\perp}$ is the range of a projection $P$. Given a closed convex subset $S$ containing 0, the first problem of this paper is to find a condition for $\tau(S)$ to be closed where $\tau$ is the canonical map from $E$ to $E/M$. Closure is guaranteed if $S$ is splittable in the sense that the polar $S^0$ coincides with the norm-closed convex hull of $P(S^0) \cup Q(S^0)$, where $Q = 1 - P$. The second problem is to give a condition for existence of a linear map $\varphi$, called a linear lifting, from $E/M$ to $E$ such that $\tau \circ \varphi = 1$ and $\varphi \circ \tau(S) \subseteqq S$. A linear lifting exists if and only if $M$ is the kernel of a projection making $S$ invariant. Of special interest is the case where $S$ is a ball or a cone. When the unit ball is splittable, existence of a linear lifting of norm one is guaranteed under suitable conditions on $E/M$, which are satisfied by separable $L_p$ and $C(X)$ on compact metrizable $X$. If further $E$ is an ordered Banach space, and if both $P$ and $Q$ are positive, $M$ is shown to be the kernel of a positive projection of norm one.**

Though the closed range theorem (Theorem 1) yields immediately an abstract version of the Rudin-Carleson-Bishop theorem on norm-preserving extensions of functions defined on a peak set, in §2 further modification (Theorem 2) is shown to include Gamelin's extension [5] of the Rudin-Carleson-Bishop theorem in abstract form. Recently a different approach to generalization of the Gamelin theorem was made by Alfsen and Hirsberg [1]. In §3 it is indicated how the closed range theorem is applied to give unified proofs for results of Davies [4] and Perdrizet [9] on closedness of a cone in a quotient space and on order-preserving extensions. In §4 an idea of Pełczynski-Michael [8] is further developed for the closed range theorem to produce existence of linear liftings under suitable conditions. The Pełczynski-Michael theorems are generalized in abstract form (Theorems 5 and 6).

1. Preliminary. Let $E$ be a real or complex Banach space with unit ball $U$. $E^*$ and $E^{**}$ are its dual and second dual respectively, and $E$ is always imbedded canonically into $E^{**}$. $x, y, z, \cdots$ are vectors in $E$ or $E^{**}$ while $f, g, h, \cdots$ are functionals in $E^*$. For $x \in E^{**}$ and $f \in E^*$ $f(x)$ is used instead of $x(f)$. The weak topology $\sigma(E^*, E)$ on $E^*$ is called the *weak** topology while $\sigma(E^{**}, E^*)$ on $E^{**}$ is the *weak*** topology. For a subset $S$ of $E$ its norm-closure and its weak**

closure (in $E^{**}$) are denoted by $\bar{S}$ and $S^{\sim}$ respectively.

The *polar* $S^0$ is defined as the set of all $f$ such that $\mathrm{Re}\, f(x) \leqq 1$ or $f(x) \leqq 1$ on $S$ according as the scalar field is complex or real. When $S$ is a subspace its polar coincides with its *annihilator* $S^{\perp}$ consisting of all $f$ vanishing on it. The following basic facts are used frequently in this paper. Proofs are found, for instance, in [10]. Let $S_1$ and $S_2$ be closed convex subsets of $E$ containing 0. $(S_1 \cap S_2)^0$ coincides with the weak* closure of conv $(S_1^0 \cup S_2^0)$ where conv $(\cdot)$ denotes the convex hull. The weak** closure $S_1^{\sim}$ coincides with the polar of $S_1^0$ in duality $\langle E^{**}, E^* \rangle$ and $S_1 = E \cap S_1^{\sim}$. Thus coincidence $(S_1 \cap S_2)^{\sim} = S_1^{\sim} \cap S_2^{\sim}$ occurs if and only if the weak* and the norm closure of conv $(S_1^0 \cup S_2^0)$ coincide. In some case conv $(S_1^0 \cup S_2^0)$ becomes itself weak* closed. Here the Krein-Smulian theorem is quite useful: conv $(S_1^0 \cup S_2^0)$ is weak* closed if (and only if) $\gamma U^0 \cap \mathrm{conv}\,(S_1^0 \cup S_2^0)$ is weak* closed for every $0 \leqq \gamma < \infty$. If $S_1$ contains 0 in its interior then $S_1^0$ is weak* compact and the norm closure of conv $(S_1^0 \cup S_2^0)$ is weak* closed. If $S_1$ is a subspace or a cone, the weak* closure of conv $(S_1^0 \cup S_2^0)$ is just that of $S_1^{\perp} + S_2^0$. In case both $S_1$ and $S_2$ are subspaces, $S_1^{\perp} + S_2^{\perp}$ is weak* closed if and only if $S_1 + S_2$ is norm-closed.

Suppose now that $E$ is a real Banach space provided with a closed proper cone $E_+$. $E_+$ gives rise to natural ordering in $E$ under which it becomes the set of all *positive* vectors: $x \leqq y$ means $y - x \in E_+$. In this respect $E_+$ is called the *positive cone*. The dual positive cone $E_+^*$ is defined as the set of $f$ nonnegative on $E_+$, or equivalently $E_+^* = -E_+^0$. $E$ is called an *ordered Banach space* if $E = E_+ - E_+$ and if there is $\gamma < \infty$ with $(U - E_+) \cap (U + E_+) \subseteqq \gamma U$. The latter condition is equivalent to that every subset of the form $\{x; y_1 \leqq x \leqq y_2\}$ is norm-bounded. For notational convenience the relation $x \leqq y + \varepsilon$ in an ordered Banach space means that there is $z \geqq 0$ such that $\|z\| < \varepsilon$ and $x \leqq y + z$. An ordered Banach space or its norm is called *regular* if $\|x\| = \inf\{\|y\|; -y \leqq x \leqq y\}$ for every $x$. A regular norm is monotone on the positive cone in the sense that $0 \leqq x \leqq y$ implies $\|x\| \leqq \|y\|$. An ordered Banach space admits an equivalent regular norm. In fact, the functional $\|x\|_0 = \inf\{\|y\|; -y \leqq x \leqq y\}$ gives a regular norm.

It is known (cf. [2] and [4]) that $E$ is regular if and only if $E^*$ is regular. An ordered Banach space is said to have the *Riesz interpolation property* if for $y_i \geqq x_j$ $(i, j = 1, 2, \cdots, n)$ there is $z$ such that $x_i \leqq z \leqq y_i (i = 1, 2, \cdots, n)$. A regular ordered Banach space is called a *Banach lattice* if it is lattice under the ordering. A Banach lattice has the Riesz interpolation property. It is known (cf. [2] and [4]) that $E$ has the Riesz interpolation property if and only if $E^*$ is a lattice. A continuous linear operator between ordered Banach spaces is called *positive* if it transforms a positive cone into another.

**2. Closed range theorems.** $E$ is a real or complex Banach space with unit ball $U$ and $M$ is a closed subspace. The canonical map from $E$ to the quotient space $E/M$ is denoted by $\tau$.

Throughout this section it is assumed:

*There is a continuous projection $P$ from $E^*$ to $M^\perp$, and $Q$ stands for $1 - P$.*

Remark that the adjoint $Q^*$ projects $E^{**}$ onto $M^\sim$ but $M$ is not necessarily range of any projection. $S$, $S_1$ and $S_2$ will denote closed convex subsets of $E$ containing 0. $S$ is said to be *splittable*, or more precisely, $P$-splittable if its polar $S^0$ coincides with the norm-closure of conv $(P(S^0) \cup Q(S^0))$.

LEMMA 1. *The following conditions are equivalent.*
(a) *$S$ is splittable.*
(b) *$S^\sim = \{x \in E^{**}; P^*x \in S^\sim$ and $Q^*x \in S^\sim\}$.*
(c) *$\theta(f) = \theta(Pf) + \theta(Qf)$ $(f \in E^*)$*
*where $\theta(f)$ is defined by $\theta(f) = \sup\{\mathrm{Re}\, f(x); x \in S\}$.*

*Proof.* Since the polar of $P(S^0)$(resp. of $Q(S^0)$) in $E^{**}$ coincides with the set $\{x \in E^{**}; P^*x$(resp. $Q^*x) \in S^\sim\}$ equivalence of (a) and (b) is clear (cf. §1).

(b) $\Rightarrow$ (c). Obviously $\theta(f)$ can be defined by

$$\theta(f) = \sup\{\mathrm{Re}\, f(x); x \in S^\sim\} .$$

Take $x$ and $y$ in $S$. Then by (b) $P^*x + Q^*y$ belongs to $S^\sim$ so that

$$\mathrm{Re}\, Pf(x) + \mathrm{Re}\, Qf(y) = \mathrm{Re}\, f(P^*x + Q^*y) \leqq \theta(f) ,$$

leading to $\theta(Pf) + \theta(Qf) \leqq \theta(f)$. The reverse inequality is obvious.

(c) $\Rightarrow$ (b). Since the functional $\theta$ is nonnegative because of $S \ni 0$, (c) implies $P(S^0) \cup Q(S^0) \subseteqq S^0$. Therefore $S^\sim$ is contained in the set $\{x \in E^{**}; P^*x \in S^\sim$ and $Q^*x \in S^\sim\}$. Take $x$ with $P^*x, Q^*x \in S^\sim$. Then by (c)

$$\mathrm{Re}\, f(x) = \mathrm{Re}\, Pf(P^*x) + \mathrm{Re}\, Qf(Q^*x) \leqq \theta(Pf) + \theta(Qf) = \theta(f) .$$

Thus $x$ belongs to the polar of $S^0$ in $E^{**}$.

COROLLARY 1. *The unit ball $U$ is splittable if and only if*
$$\|f\| = \|Pf\| + \|Qf\| \quad (f \in E^*) .$$

COROLLARY 2. *If both $S_1$ and $S_2$ are splittable and $(S_1 \cap S_2)^\sim = S_1^\sim \cap S_2^\sim$, then $S_1 \cap S_2$ is splittable.*

COROLLARY 3. *A closed subspace (resp. cone) is splittable if and only if its polar is invariant under $P$ (resp. under $P$ and $Q$).*

*Proof.* Let $N$ be a closed cone. $P(N^0) \subseteqq N^0$ and $Q(N^0) \subseteqq N^0$ implies $N^0 = P(N^0) + Q(N^0) = \operatorname{conv}(P(N^0) \cup Q(N^0))$. If $N$ is further a subspace, $Q(N^0) \subseteqq N^0$ follows already from $P(N^0) \subseteqq N^0$.

LEMMA 2. *If $S_1$ and $S_2$ are splittable, for any $\varepsilon > 0$ and $\rho > 0$ the following inclusion relation holds:*

$$\{S_1 \cap (S_2 + \varepsilon U) + \rho U\} \cap (\overline{S_1 + M}) \cap (\overline{S_2 + M})$$
$$\subseteqq \overline{S_1 \cap (S_2 + \alpha\varepsilon U) + \alpha\rho U \cap M}$$

*where $\alpha = \|Q\|$.*

*Proof.* Take any $x$ in the set on the left hand side. Then it follows by splittability that

$$Q^*x \in S_1^\sim \cap (S_2^\sim + \alpha\varepsilon U^\sim \cap M^\sim) + \alpha\rho U^\sim \cap M^\sim .$$

There is $y \in \alpha\rho U^\sim \cap M^\sim$ such that

$$Q^*(x - y) = Q^*x - y \in S_1^\sim \cap (S_2^\sim + \alpha\varepsilon U^\sim \cap M^\sim)$$

and

$$P^*(x - y) = P^*x \in P^*(\overline{S_1 + M}) \cap P^*(\overline{S_2 + M}) \subseteqq S_1^\sim \cap S_2^\sim .$$

Then by Lemma 1 $x - y \in S_1^\sim$ and there is $z \in \alpha\varepsilon U^\sim \cap M^\sim$ such that $x - y - z \in S_2^\sim$. Finally in view of arguments of §1 $x$ belongs to

$$E \cap \{S_1^\sim \cap (S_2^\sim + \alpha\varepsilon U^\sim) + \alpha\rho U^\sim \cap M^\sim\}$$
$$\subseteqq E \cap \{S_1 \cap (S_2 + \alpha\varepsilon U) + \alpha\rho U \cap M\}^\sim$$
$$= \overline{S_1 \cap (S_2 + \alpha\varepsilon U) + \alpha\rho U \cap M} .$$

By definition of the quotient topology $\tau(x)$ belongs to the closure $\overline{\tau(S)}$ if and only if $x$ is contained in $\overline{S + M}$. In particular, $\tau(S)$ is closed if and only if $S + M$ is closed.

LEMMA 3. *Suppose that $S_1$ and $S_2$ are splittable. If $\tau(x)$ belongs to $\overline{\tau(S_1)} \cap \overline{\tau(S_2)}$ and $\|x - S_1 \cap S_2\| < \gamma$ there is $y \in S_1$ such that $\tau(x) = \tau(y)$ and $\|x - y\| < \gamma\|Q\|$. In case $\|Q\| = 1$ for any $\varepsilon > 0$ $y$ can be chosen in $S_1 \cap (S_2 + \varepsilon U)$.*

*Proof.* Let $\alpha = \|Q\|$ and take $\varepsilon'$ with $0 < \varepsilon' < \varepsilon$. By hypothesis $x$ is contained in

$$\{S_1 \cap (S_2 + \varepsilon' U) + \gamma' U\} \cap (\overline{S_1 + M}) \cap (\overline{S_2 + M})$$

for some $\gamma > \gamma' > 0$. Choose $\varepsilon_n > 0$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \gamma - \gamma'$. By Lemma 2 there is $x_0 \in M$ such that $\|x_0\| \leqq \alpha\gamma'$ and $\|x + x_0 - S_1 \cap (S_2 + \alpha\varepsilon' U)\| < \varepsilon_1$. Then

$$x + x_0 \in \overline{S_i + M} + M = \overline{S_i + M}$$

and

$$x + x_0 \in S_1 \cap (S_2 + \alpha\varepsilon' U) + \varepsilon_1 U .$$

Now inductive procedure based on Lemma 2 makes it possible to find a sequence $\{x_n\}$ in $M$ such that $\|x_n\| \leqq \alpha\varepsilon_n$ and $\|x + \sum_{i=0}^{n-1} x_i - S_1 \cap (S_2 + \alpha^n \varepsilon' U)\| < \varepsilon_n$. Since $\sum_{n=0}^{\infty} \|x_n\| < \infty$, $y = x + \sum_{n=0}^{\infty} x_n$ is well defined. Obviously $y$ belongs to $S_1$, and in case $\alpha = 1$, to $S_1 \cap (\overline{S_2 + \varepsilon' U}) \subseteqq S_1 \cap (S_2 + \varepsilon U)$. Finally $\tau(x) = \tau(y)$ and $\|x - y\| \leqq \alpha \sum_{n=1}^{\infty} \varepsilon_n + \alpha\gamma' < \alpha\gamma$.

Now the main result of this paper is near at hand.

THEOREM 1. *Suppose that the annihilator of a closed subspace $M$ is the range of a projection $P$ and that $S$, $S_1$ and $S_2$ are closed convex subsets containg* 0. *Then* (a) *the image of $S$ under the canonical map $\tau$ from $E$ to $E/M$ is closed whenever $S$ is splittable.* (b) *If both $S_1$ and $S_2$ are splittable and $(S_1 \cap S_2)^{\sim} = S_1^{\sim} \cap S_2^{\sim}$ then $\tau(S_1) \cap \tau(S_2) = \tau(S_1 \cap S_2)$.* (c) *If both $S_1$ and $S_2$ are splittable and $\|1 - P\| \leqq 1$, then the inclusion $\tau(S_1) \cap \tau(S_2) \subseteqq \tau(S_1 \cap (S_2 + \varepsilon U))$ holds for any $\varepsilon > 0$.*

*Proof.* (a) follows from Lemma 3 with $S_1 = S_2 = S$. Also (c) is a direct consequence of Lemma 3. (b) $S_1 \cap S_2$ is splittable by Corollary 2 and $S_1 \cap S_2 + M$ is closed by (a). Now since by hypothesis

$$P^*\{(S_1 + M) \cap (S_2 + M)\} \subseteqq P^*(S_1) \cap P^*(S_2) \subseteqq S_1^{\sim} \cap S_2^{\sim}$$
$$= (S_1 \cap S_2)^{\sim} ,$$

it follows that

$$(S_1 + M) \cap (S_2 + M) \subseteqq E \cap (S_1 \cap S_2 + M)^{\sim} = S_1 \cap S_2 + M ,$$

showing $\tau(S_1) \cap \tau(S_2) \subseteqq \tau(S_1 \cap S_2)$. The reverse inclusion is obvious. This completes the proof.

It follows immediately from Theorem 1 that if the unit ball $U$ is splittable and if $N$ is a closed splittable subspace then $\tau(N \cap U)$ is closed and coincides with $\tau(N) \cap \tau(U)$. Let us show that the same conclusion holds for a non-splittable subspace under suitable conditions.

Since by Corollary 3 splittability of $N$ is characterized by $P(N^{\perp}) \subseteqq N^{\perp} \cap M^{\perp}$, it follows by Corollary 1 that under the splittability of $U$, $N$ is splittable if and only if

$$\|Qf\| = \|f - M^{\perp} \cap N^{\perp}\| \quad (f \in N^{\perp}) .$$

On the other hand, Corollary 1 implies $\|Qf\| = \|f - M^{\perp}\|$. Thus if the unit ball is splittable, splittability of $N$ is characterized by

$$\|f - M^{\perp} \cap N^{\perp}\| \leqq \|f - M^{\perp}\| \quad (f \in N^{\perp}) .$$

LEMMA 4. *Let $N_1$ and $N_2$ be closed subspace. Then for $\rho > 0$ the following assertions are equivalent.*

(a) $\|x - N_1 \cap N_2\| \leqq \rho\|x - N_1\| \ (x \in N_2)$

(b) $\|f - N_1^{\perp} \cap N_2^{\perp}\| \leqq \rho\|f - N_2^{\perp}\| \ (f \in N_1^{\perp})$

*Proof.* (a) means that

$$N_2 \cap (\overline{N_1 + U}) \subseteqq \overline{N_1 \cap N_2 + \rho U}$$

which implies by polar formation

$$(N_1^{\perp} + N_2^{\perp}) \cap U^0 \subseteqq N_2^{\perp} + N_1^{\perp} \cap \rho U^0 .$$

The last relation can be converted to

$$N_1 \cap (N_2^{\perp} + U^0) \subseteqq N_1^{\perp} \cap N_2^{\perp} + \rho U^0$$

which is nothing but (b). The reverse process can be pursued because (b) implies that $(N_1^{T} + N_2^{\perp}) \cap U^0$ is weak* compact, hence by the Krein-Smulian theorem that $N_1^{\perp} + N_2^{\perp}$ is weak* closed.

COROLLARY 4. *Suppose that the unit ball is splittable. Then the following assertions for a closed subspace $N$ are equivalent.*

(a) *$N$ is splittable.*

(b) $\|x - N \cap M\| \leqq \|x - N\| \ (x \in M)$

(c) $\|f - N^{\perp} \cap M^{\perp}\| \leqq \|f - M^{\perp}\| \ (f \in N^{\perp})$.

THEOREM 2. *Let $M$ and $N$ be closed subspaces, and suppose that the annihilator $M^{\perp}$ is the range of a projection $P$ such that*

$$(1) \qquad \|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*) .$$

*If for some $1 \leqq \rho < 2$*

$$(2) \qquad \|x - N \cap M\| \leqq \rho\|x - N\| \quad (x \in M) ,$$

*then the images of the unit ball $U$, $N$ and $N \cap U$ under the canonical map $\tau$ from $E$ to $E/M$ are closed and*

$$\tau(N \cap U) = \tau(N) \cap \tau(U) .$$

*Proof.* Closedness of $\tau(U)$ follows from (1) by Corollary 1 and

Theorem 1. Let $Q = 1 - P$. Then in view of (1) relation (2) is converted by Lemma 4 to

$$(3) \qquad ||Pf - M^{\perp} \cap N^{\perp}|| \leqq \gamma ||Qf|| \quad (f \in N^{\perp})$$

where $\gamma = \rho - 1 < 1$. Then for any $f \in N^{\perp}$ and $g \in M^{\perp}$.

$$\begin{aligned} ||g - M^{\perp} \cap N^{\perp}|| &\leqq ||Pf - M^{\perp} \cap N^{\perp}|| + ||g - Pf|| \\ &\leqq \gamma ||Qf|| + ||g - Pf|| \leqq ||g - f||, \end{aligned}$$

showing

$$||g - M^{\perp} \cap N^{\perp}|| \leqq ||g - N^{\perp}|| \quad (g \in M^{\perp}),$$

which is converted by Lemma 4 to

$$(4) \qquad ||x - M \cap N|| \leqq ||x - M|| \quad (x \in N).$$

This last relation means that the canonical map from the Banach space $N/M \cap N$ onto $\tau(N)$ has bounded inverse. Therefore $\tau(N)$ is closed. Further (4) implies

$$N \cap (U + M) \subseteqq \overline{U + N \cap M}.$$

Let us prove that really

$$N \cap (U + M) \subseteqq U + N \cap M$$

holds, which is equivalent to the required relation:

$$\tau(N) \cap \tau(U) \subseteqq \tau(N \cap U).$$

Suppose for contradiction that there exists $x$ in $N \cap (U + M)$ with $(U - x) \cap N \cap M = \varnothing$. Since $N \cap M$ is a subspace, it follows that

$$\operatorname{conv}((U - x) \cup \{0\}) \cap N \cap M = \{0\}.$$

Since $\operatorname{conv}((U - x) \cup \{0\})$ is closed, the last relation implies by polar formation that $(U - x)^0 + N^{\perp} + M^{\perp}$ is weak* dense in $E^*$. Weak* closedness of $(U - x)^0 + N^{\perp} + M^{\perp}$, if proved, leads to

$$\operatorname{conv}((U^{\sim} - x) \cup \{0\}) \cap N^{\sim} \cap M^{\sim} = \{0\}$$

hence to a contradiction:

$$x \notin E \cap (U^{\sim} + N^{\sim} \cap M^{\sim}) = \overline{U + N \cap M}.$$

Now let us prove the above weak* closedness. To this end, in view of the Krein-Smulian theorem, it suffices to prove that for any $n > 0$

$$\{(U - x)^0 + N^{\perp} + M^{\perp}\} \cap nU^0 \subseteqq \delta U^0 \cap (U - x)^0 + \delta U^0 \cap N^{\perp} + M^{\perp}$$

where $\delta$ is a constant depending on $n$. Remark that $(U-x)^0$ consists of all $f$ with $\|f\| \leqq \operatorname{Re} f(x)+1$. Since $x \in U+M$ implies $\|P^*x\| \leqq 1$, it follows from (1) that

$$\begin{aligned} &1+\operatorname{Re} f(x)-\|f\| \\ \leqq\ &1+\operatorname{Re} Qf(x)-\|Qf\| \\ &-\{\|Pf\|\,\|P^*x\|-|Pf(P^*x)|\} \\ \leqq\ &1+\operatorname{Re} Qf(x)-\|Qf\| . \end{aligned}$$

This indicates that $Q$ makes $(U-x)^0$ invariant. Now take $f \in (U-x)^0$, $g \in N^\perp$ and $h \in M^\perp$ with $\|f+g+h\| \leqq n$. Then by (1) $\|Qf+Qg\| \leqq n$. Since $x \in N \cap (U+M)$ and $g \in N^\perp$,

$$\begin{aligned} \operatorname{Re} Qf(x) &\leqq n\|x\| - \operatorname{Re} Qg(x) = n\|x\| + \operatorname{Re} Pg(x) \\ &\leqq n\|x\| + \|Pg - N^\perp \cap M^\perp\|\ \|x - M\| \\ &\leqq n\|x\| + \|Pg - N^\perp \cap M^\perp\| . \end{aligned}$$

Since $Qf$ belongs to $(U-x)^0$ as $f$, it follows that

$$\begin{aligned} \|Qg\| &\leqq n + \|Qf\| \leqq n + \operatorname{Re} Qf(x) + 1 \\ &\leqq n(\|x\|+2) + \|Pg - N^\perp \cap M^\perp\| . \end{aligned}$$

Then (3) applied to $g$ yields

$$\|Pg - N^\perp \cap M^\perp\| \leqq \frac{\gamma(n\|x\|+2)}{1-\gamma} \equiv \delta_1$$

and consequently

$$\|Qf + g - N^\perp \cap M^\perp\| \leqq n + \delta_1 \equiv \delta_2 .$$

Since $x \in N$, $g \in N^\perp$ and $Qf \in (U-x)^0$,

$$\|Qf\| \leqq \operatorname{Re} Qf(x) + 1 \leqq \delta_2\|x\| + 1 \equiv \delta_3$$

and

$$\|g - N^\perp \cap M^\perp\| \leqq \delta_2 + \delta_3 \equiv \delta .$$

This implies that

$$\begin{aligned} &f+g+h = Qf + g + (Pf + h) \\ &\subseteqq \delta U^0 \cap (U-x)^0 + \delta U^0 \cap N^\perp + M^\perp . \end{aligned}$$

This completes the proof.

Consider the sup-norm Banach space $C(X)$ of continuous functions on a compact Hausdorff space $X$. By the Riesz theorem its dual is realized by the space of regular Borel measures on $X$ with total-variation norm. Given a closed subset $Y$ of $X$, let $M$ be the subspace of functions in $C(X)$ vanishing on $Y$. Then $M^\perp$ is the set of measures with support in $Y$ and becomes the range of a projection $P$: $Pm = \chi m$ for each measure $m$ where $\chi$ is the characteristic function of $Y$. Obviously (1) is satisfied. Now let $N$ be a closed subspace of $C(X)$. As shown in [5] (3) is equivalent to the property that for any $x \in N$ with $|x(t)| < 1 (t \in Y)$ and any closed subset $Z \subset X$ with $Y \cap Z = \emptyset$ there is $y \in N$ such that $x(t) = y(t)$ $(t \in Y)$, $|y(s)| < \gamma$ $(s \in Z)$ and $\|y\| < \max(1, \gamma)$. Remark that $\|x - M\|$ coincides with the norm of the restriction $x \mid Y$ of $x$ to $Y$ and that $x(t) = y(t) (t \in Y)$ is equivalent to $x - y \in M$. Thus Theorem 2 shows that if (3) with $\gamma < 1$, or equivalently (2) with $\rho < 2$, is satisfied then for any $x \in N$ there is $y \in N$ such that $x \mid Y = y \mid Y$ and $\|y\| = \|x \mid Y\|$. The case $\gamma = 0$ is the generalized Carleson-Rudin theorem (cf. [6] Chap. II). As Gamelin [5] shows, Theorem 2 can further yield the following: suppose that (3) with $\gamma < 1$, or equivalently (2) with $\rho < 2$, is satisfied and that $p \in C(X)$ satisfies $p(t) = 1$ $(t \in Y)$ and $p(s) > \gamma$ $(s \varepsilon X)$. Then if $x \in N$ satisfies $|x(t)| \leqq p(t)$ $(t \in Y)$ there is $y \in N$ such that $x(t) = y(t)$ $(t \in Y)$ and $|y(s)| \leqq p(s)$ $(s \in X)$. The case $\gamma = 0$ is the Bishop theorem (cf. [6] Chap. II). Generalization of the Gamelin theorem to other direction is treated by Alfsen and Hirsberg [1].

**3. Ordered Banach spaces.** Let $E$ be an ordered Banach space with positive cone $E_+$. A closed subspace $M$ is called an *ideal* if $(M - E_+) \cap (M + E_+) \subseteqq M$. An ideal $M$ is *hypostrict* if its annihilator $M^\perp$ is the range of a projection $P$ such that $f \geqq Pf \geqq 0$ for every $f \geqq 0$ The requirement means that both $P$ and $Q = 1 - P$ are positive. Perdrizet [9] shows that a closed subspace $M$ is a hypostrict ideal if and only if the following two conditions are satisfied: (1) Given $x_1$, $x_2 \in M$ and $y \in E$ with $x_1, x_2 \leqq y$, for any $\varepsilon > 0$ there is $z \in M$ such that $x_1, x_2 \leqq z \leqq y + \varepsilon$, and (2) given $x \in M$ and $y_1, y_2 \in E_+$ with $x \leqq y_1 + y_2$ there are $x_1, x_2 \in M$ such that $x = x_1 + x_2$ and $x_i \leqq y_i + \varepsilon$ $i = 1, 2$. Under the Riesz interpolation property an ideal $M$ is hypostrict if and only if it is positively generated in the sense: $M = M \cap E_+$ $M \cap E_+$.

When $M$ is an ideal, the Banach space $E/M$ is preordered by the cone $\tau(E_+)$ where $\tau$ is the canonical map from $E$ to $E/M$. The following theorem was first proved by Davies [4] under the Riesz interpolation property and then by Perdrizet [9] in general case. Let us give a proof based on Theorem 1.

THEOREM 3. *Let $E$ be an ordered Banach space with positive cone $E_+$. If $M$ is a hypostrict ideal then $E/M$ is an ordered Banach space with $\tau(E_+)$ as its positive cone. If $E$ is regular in addition, so is $E/M$.*

*Proof.* Since hypostrictness means that $E_+^* = -E_+^0$ is invariant under both $P$ and $Q$, $E_+$ is splittable by Corollary 3. Then $\tau(E_+)$ is closed by Theorem 1. $M^\perp$ is isometric to the dual of $E/M$, and the dual positive cone is identified with $M^\perp \cap E_+^*$. Suppose that $E$ is regular. Then $E^*$ is regular. Since $P$ is positive and is of norm one in this case, $M^\perp$ is a regular ordered Banach space with $M^\perp \cap E_+^*$ as its positive cone. Therefore $E/M$ is regular as stated in §1. This completes the proof because every ordered Banach space admits an equivalent regular norm.

COROLLARY 5. *Suppose that the positive cone $E_+$ has nonempty interior and that $M$ is a hypostrict ideal. If a closed subspace $N$ is splittable and if it contains an interior point of $E_+$ then $\tau(N \cap E_+)$ is closed and $\tau(N \cap E_+) = \tau(N) \cap \tau(E_+)$.*

*Proof.* Since $E_+$ is splittable, in view of Corollary 2 and Theorem 1 it suffices to prove that $(N \cap E_+)^0 = N^\perp + E_+^0$. Remark that $f$ belongs to $(N \cap E_+)^0$ if and only if the restriction of $-f$ to $N$ is positive. However it is known (cf. [10] Chap. V §5) that when $N$ contains an interior point of $E_+$ every continuous positive linear functional on $N$ admits a continuous positive linear extension to $E$, in other words, $-(N \cap E_+)^0 = -(N^\perp + E_+^0)$.

Since $E/M$ is ordered by the cone $\tau(E_+)$, for any $y, z$ with $\tau(z) \leqq \tau(y)$ there is $y'$ such that $z \leqq y'$ and $\tau(y) = \tau(y')$. The next task is to treat the case $\tau(z) \leqq \tau(y) \leqq \tau(x)$ and $z \leqq x$ and to find a condition of existence $y''$ such that $z \leqq y'' \leqq x$ and $\tau(y) = \tau(y'')$.

LEMMA 5. *Let $S_1$ and $S_2$ be closed convex subsets containing 0. If for any $0 < \lambda < 1$, $f \in S_1^0$ and $g \in S_2^0$ there are $f' \in S_1^0$ and $g' \in S_2^0$ such that*

$$\lambda f + (1-\lambda) g = \lambda f' + (1-\lambda) g'$$

*and*

$$\lambda \| f' \|, (1-\lambda) \| g' \| \leqq \| \lambda f + (1-\lambda) g \|$$

*then $(S_1 \cap S_2)^\sim$ coincides with $S_1^\sim \cap S_2^\sim$ where $(\cdot)^\sim$ denotes the weak** closure.*

*Proof.* In view of the Krein-Smulian theorem it suffices to prove that for any $\gamma > 0$ the weak* closure of $\operatorname{conv}(S_1^0 \cup S_2^0) \cap \gamma U^0$ is contained in the norm closure of $\operatorname{conv}(S_1^0 \cup S_2^0)$. Suppose that $0 < \lambda_\alpha < 1$, $f_\alpha \in S_1^0$, $g_\alpha \in S_2^0$ and $\| \lambda_\alpha f_\alpha + (1-\lambda_\alpha) g_\alpha \| \leqq \gamma$ and that the net $\{\lambda_\alpha f_\alpha +$

$(1-\lambda_\alpha)g_\alpha\}$ weak* converges to $h$ and the net $\{\lambda_\alpha\}$ converges to $\lambda$. By hypothesis $\{\lambda_\alpha f_\alpha\}$ and $\{(1-\lambda_\alpha)g_\alpha\}$ can be assumed to be bounded, hence to weak* converge to $f'$ and $g'$ respectively. If $0<\lambda<1$, $\{f_\alpha\}$ and $\{g_\alpha\}$ can be assumed to weak* converge to $f''\in S_1^0$ and $g''\in S_2^0$ respectively. Then $h=\lambda f''+(1-\lambda)g''$ belongs to conv $(S_1^0\cup S_2^0)$. In case $\lambda=0$, $h=f'+g''$ and $nf'$ belongs to $S_1^0$ for any $n>0$. Therefore $h$, as the norm limit of $1/n(nf')+(1-1/n)g''$, belongs to the norm closure of conv $(S_1^0\cup S_2^0)$. The case $\lambda=1$ is treated similarly.

COROLLARY 6. *$(\bigcap_{i=1}^n(x_i-E_+))^\sim=\bigcap_{i=1}^n(x_i-E_+)^\sim$ whenever $x_i\geqq 0$ $i=1,2,\cdots,n$.*

*Proof.* $E$, hence $E^*$, can be assumed to be regular. $(x_i-E_+)^0$ consists of all $0\leqq f$ with $f(x_i)\leqq 1$. Suppose that $\lambda_i\geqq 0$, $\sum_{i=1}^n\lambda_i=1$ and $f_i\in(x_i-E_+)^0$. Since the norm is monotone on the dual positive cone by regularity, it follows that $\lambda_j\|f_j\|\leqq\|\sum_{i=1}^n\lambda_i f_i\|$ $j=1,2,\cdots,n$. Now inductive application of Lemma 5 yields the assertion.

The following theorem was proved by Perdrizet [9]. Let us give a proof based on Theorem 1.

THEOREM 4. *Let $E$ be an ordered Banach space with positive cone $E_+$. Suppose that $M$ is a hypostrict ideal and $E/M$ is ordered by the cone $\tau(E_+)$ where $\tau$ is the canonical map from $E$ to $E/M$. If $z_i\leqq 0\leqq x_i$ and $\tau(z_i)\leqq\tau(y)\leqq\tau(x_i)$ $i=1,2,\cdots,n$, then for any $\varepsilon>0$ there is $y'$ such that $z_i\leqq y'\leqq x_i+\varepsilon$ $i=1,2,\cdots,n$ and $\tau(y)=\tau(y')$. Further $\varepsilon$ can be made $0$ if every $x_i$ is an interior point of $E_+$ or if $E$ has the Riesz interpolation property.*

*Proof.* $E$ is assumed to be regular, hence $Q$ is of norm one. $z_i+E_+$ is a closed convex set containing 0. It is splittable because both $P^*$ and $Q^*$ are positive and $z_i$ is negative. Similary $x_i-E_+$ is splittable. Let $S_1=\bigcap_{i=1}^n(z_i+E_+)$ and $S_2=\bigcap_{i=1}^n(x_i-E_+)$. Then by Corollaries 2, 6, and Theorem 1 both $S_1$ and $S_2$ are splittable and

$$\begin{aligned}&\bigcap_{i=1}^n\tau(z_i+E_+)\cap\bigcap_{i=1}^n\tau(x_i-E_+)\\&=\tau(S_1)\cap\tau(S_2)\subseteqq\tau(S_1\cap(S_2+\varepsilon U))\end{aligned}$$

which is just the first assertion.

If every $x_i$ is an interior point of $E_+$, $S_2$ contains 0 in its interior and by Corollary 2 and Theorem 1

$$\tau(S_1)\cap\tau(S_2)=\tau(S_1\cap S_2)\ .$$

Suppose finally that $E$ has the Riesz interpolation property. Since $E^{**}$ becomes a lattice as stated in §1, $S_1^{\sim}$ consists of all $w \in E^{**}$ with $\bigvee_{i=1}^n z_i \leqq w$, where $\bigvee_{i=1}^n z_i$ denotes the supremum of $z_1, \cdots, z_n$ in $E^{**}$. Then $S_1^0$ consists of all $0 \geqq f$ with $f(\bigvee_{i=1}^n z_i) \leqq 1$. Similarly $S_2^0$ consists of all $0 \leqq g$ with $g(\bigwedge_{i=1}^n x_i) \leqq 1$ where $\bigwedge_{i=1}^n x_i$ denotes the infimum of $x_1, \cdots x_n$, in $E^{**}$. Take $0 < \lambda < 1$, $f \in S_1^0$ and $g \in S_2^0$ and let $h = \lambda f + (1-\lambda)g$. Since $E^*$ is a Banach lattice as stated in §1, and since both $-f$ and $g$ are positive, it follows that $0 \geqq h \wedge 0 \geqq \lambda f$ and $0 \leqq h \vee 0 \leqq (1-\lambda)g$. Let $f' = (1/\lambda)(h \wedge 0)$ and $g' = (1)/(1-\lambda)h \vee 0$. Then it follows from the above characterization of $S_i^0$ that $f' \in S_1^0$, $g' \in S_2^0$ and $h = \lambda f' + (1-\lambda)g'$. Now since $\|h \wedge 0\|, \|h \vee 0\| \leqq \|h\|$, Lemma 5 yields $(S_1 \cap S_2)^{\sim} = S_1^{\sim} \cap S_2^{\sim}$ and the assertion follows from Theorem 1.

**4. Linear lifting.** Let $E$ be a Banach space with unit ball $U$ and $M$ a closed subspace. The canonical map from $E$ to $E/M$ is denoted by $\tau$. A continuous linear map $\varphi$ from $E/M$ to $E$ is called a *linear lifting* if $\tau \circ \varphi = 1$. If $\varphi$ is a linear lifting, $\varphi \circ \tau$ is a projection with $M$ as its kernel. Conversely, a linear lifting exists if $M$ is the kernel of a continuous projection.

In this section it is assumed:

*There is a projection $P$ from $E^*$ to $M^{\perp}$ such that*

$$\|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*)$$

*and $Q$ stands for $1 - P$.*

Let $F$ be a finite dimensional Banach space with unit ball $V$. Consider the dual system $\langle F^* \otimes E, F \otimes E^* \rangle$ of tensor products. When $F^* \otimes E$ is provided with the Minkowski functional of $(V \otimes U^0)^0$ as norm, it is called the *inductive* tensor product of $F^*$ and $E$ and is denoted by $F^* \check{\otimes} E$. When $F \otimes E^*$ is provided with the Minkowski functional of $\operatorname{conv}(V \otimes U^0)$ as norm, it is called the *projective* tensor product of $F$ and $E^*$ and is denoted by $F \hat{\otimes} E^*$. Let $B = B(F, E)$ denote the Banach space of all continuous linear maps from $F$ to $E$, provided with operator-norm. Since $F$ is finite dimensional, $B$ is isometric to the inductive tensor product $F^* \check{\otimes} E$ under the canonical correspondence. The following lemma, whose proof is found in [10] Chap. IV §9, is basic in the subsequent development.

LEMMA 6. *The dual of $B(F, E)$ is isometric to the projective tensor product $F \hat{\otimes} E^*$ while the second dual is isometric to the inductive tensor product $F^* \otimes E^{**}$, hence to $B(F, E^{**})$.*

In view of Lemma 6 $B^{**}$ is always identified with $B(F, E^{**})$.

In this case the imbedding of $B$ to $B^{**}$ is just the natural imbedding of $B(F, E)$ to $B(F, E^{**})$. In accordance with the terminology in §1 the weak** closure of a subset $G$ of $B$ is formed in $B(F, E^{**})$ and is denoted by $G^{\sim}$. When $K$ and $S$ are a subset of $F$ and a closed convex subset of $E$ containing 0 respectively, $G(K, S)$ and $\mathscr{G}(K, S^{\sim})$ denote the set of all $\varphi \in B$ with $\varphi(K) \subseteqq S$ and the set of all $\psi \in B^{**}$ with $\psi(K) \subseteqq S^{\sim}$. Obviously $G(K, S)$ is a closed convex subset of $B$ containing 0 and its weak** closure is contained in $\mathscr{G}(K, S^{\sim})$.

COROLLARY 6. (a) $G(V, U)^{\sim} = \mathscr{G}(V, U^{\sim})$. (b) $\{G(H, 0) \cap G(F, N)\}^{\sim} = \mathscr{G}(H, 0) \cap \mathscr{G}(F, N^{\sim})$ *if $H$ and $N$ are closed subspaces of $F$ and $E$ respectively.* (c) $G(K, S)^{\sim} = \mathscr{G}(K, S^{\sim})$ *if $K$ is a cone generated by a linearly independent basis $\{x_1, \cdots, x_n\}$ of $F$ and $S$ is a cone.*

*Proof.* (a) is an immediate consequence of Lemma 6. (b) $G(F, N)^{\sim} = \mathscr{G}(F, N^{\sim})$ follows from Lemma 6 applied to $N$ instead of $E$. Since $F$ is finite dimensional, $H$ is the kernel of a projection $\sigma$. Then

$$\begin{aligned}\mathscr{G}(H, 0) \cap \mathscr{G}(F, N^{\sim}) &= \mathscr{G}(F, N^{\sim})\circ\sigma = G(F, N)^{\sim}\circ\sigma \\ \subseteqq \{G(F, N)\circ\sigma\}^{\sim} &= \{G(H, 0) \cap G(F, N)\}^{\sim} ,\end{aligned}$$

while the reverse inclusion is obvious. (c) Take any $\varphi$ in $\mathscr{G}(K, S^{\sim})$ and let $y_i = \varphi(x_i)$ $i = 1, 2, \cdots, n$. Since each $y_i$ belongs to $S^{\sim}$, there are nets $\{y_{i,\alpha}\}$ in $S$, weak** converging to $y_i$ $i = 1, 2, \cdots, n$. Consider a net $\{\varphi_\alpha\}$ in $B$ defined by $\varphi_\alpha(x_i) = y_{i,\alpha}$ $i = 1, 2, \cdots, n$. By hypothesis it is contained in $G(K, S)$ and weak** converges to $\varphi$. Thus $\mathscr{G}(K, S^{\sim})$ is contained in $G(K, S)^{\sim}$ with the reverse inclusion is obvious.

Since $B^*$ is identified with the projective tensor product $F \hat{\otimes} E^*$ by Lemma 6, the operators $1 \otimes P$ and $1 \otimes Q$ are considered to define projections in $B^*$. The adjoints of $1 \otimes P$ and $1 \otimes Q$ are realized in $B(F, E^{**})$ according to the following formula:

$$(5) \quad (1 \otimes P)^*\varphi = P^*\circ\varphi \text{ and } (1 \otimes Q)^*\varphi = Q^*\circ\varphi \ (\varphi \in B(F, E^{**})) .$$

LEMMA 7. *The annihilator $G(F, M)^{\perp}$ is the range of $1 \otimes P$.*

*Proof.* Since $Q^*$ is a projection onto $M^{\sim}$, by (5) $(1 \otimes Q)^*$ projects $B^{**}$ onto $\mathscr{G}(F, M^{\sim})$, which coincides with $G(F, M)^{\sim}$ by Corollary 6. Then $1 \otimes P$ is obviously a projection from $B^*$ to $G(F, M)^{\perp}$.

On the basis of Lemma 7, a sentence "$G(K, S)$ is *splittable*" will always mean that $G(K, S)$ is $1 \otimes P$-splittable.

COROLLARY 7. *If $S$ is splittable and $G(K, S)^{\sim} = \mathscr{G}(K, S^{\sim})$ then*

*$G(K, S)$ is splittable.*

*Proof.* This follows from (5) by Lemma 1.

The following lemma can be considered a development of a basic device in Michael and Pełczynski [8], treating linear lifting in a special case. The crucial requirement for $P$ plays a decisive role in the proof.

LEMMA 8. *Suppose that $S$ is splittable and $G(K, S)^{\sim} = \mathscr{G}(K, S^{\sim})$ for a subset $K$ of $F$. If $\psi$ belongs to*

$$G(\pi(K), S) \cap G(\pi(V), U) \cap G(K, S + M) \cap G(V, U + M)$$

*where $\pi$ is a projection of $F$ to a subspace $H$, then for any $\varepsilon > 0$ there is $\varphi$ in $G(K, S) \cap G(V, U)$ such that*

$$\tau\circ\varphi = \tau\circ\psi \quad \textit{and} \quad \|(\varphi - \psi)|H\| < \varepsilon.$$

*Proof.* Remark first of all that the requirement for $P$ means by Corollary 1 that the unit ball $U$ is splittable.

Let $\psi_1 = \psi - Q^*\circ\psi\circ(1 - \pi)$. Since

$$Q^*\circ\psi_1(K) \subseteqq Q^*\circ\psi\circ\pi(K) \subseteqq Q^*(S) \subseteqq S^{\sim}$$

and

$$P^*\circ\psi_1(K) \subseteqq P^*\circ\psi(K) \subseteqq P^*(S + M) \subseteqq S^{\sim}$$

by splittability of $S$, $\psi_1$ belongs to $\mathscr{G}(K, S^{\sim})$ by Lemma 1, hence to $G(K, S)^{\sim}$ by hypothesis. Since $U$ is splittable and $G(V, U)^{\sim} = \mathscr{G}(V, U^{\sim})$ by Corollary 6, the same argument shows that $\psi_1$ belongs also to $G(V, U)^{\sim}$. Moreover it belongs to $\{G(K, S) \cap G(V, U)\}^{\sim}$ because $G(V, U)$ is the unit ball of $B$. On the other hand, $Q^*\circ\psi\circ(1 - \pi)$ belongs to $\mathscr{G}(H, 0) \cap \mathscr{G}(F, M^{\sim})$, hence to $\{G(H, 0) \cap G(F, M)\}^{\sim}$ by Corollary 6. Thus $\psi$ belongs to

$$\{G(K, S) \cap G(V, U) + G(H, 0) \cap G(F, M)\}^{\sim} .$$

It follows that $\psi$ must be contained in the norm closure of

$$G(K, S) \cap G(V, U) + G(H, 0) \cap G(F, M) .$$

Therefore there is $\psi_2 \in B$ such that $\psi - \psi_2 \in G(H, 0) \cap G(F, M)$ and

$$\|\psi_2 - G(K, S) \cap G(V, U)\| < \varepsilon .$$

Since $G(K, S) \cap G(V, U)$ is splittable by hypothesis and Corollary 7, Lemma 3 guarantees that there is $\varphi \in G(K, S) \cap G(V, U)$ such that $\varphi - \psi_2 \in G(F, M)$ and $\|\varphi - \psi_2\| < \varepsilon$. Now $\psi_2 - \psi \in G(H, 0) \cap G(F, M)$

implies that $\tau\circ\varphi = \tau\circ\psi$ and

$$||(\varphi - \psi)\,|\,H|| = ||(\varphi - \psi_2)\,|\,H|| \leqq ||\varphi - \psi_2|| < \varepsilon\ .$$

Let $S$ be a closed splittable subset of $E$ and $L$ a subset of $\tau(S)$. Suppose that there is a sequence of projections $\{\pi_n\}$ in $E/M$ such that (1) the range $F_n$ of $\pi_n$ is of finite dimension, (2) $||\pi_n|| \leqq 1$, (3) $\pi_n\cdot\pi_m = \pi_n$ for $n \leqq m$, (4) $\pi_n(L) \subseteqq L$ and (5) $\pi_n$ converges strongly to the identity as $n \to \infty$ .

Let $\mathscr{G}_n$ denote the set of all $\varphi \in B(F_n, E)$ with $\varphi\circ\pi_n(L) \subseteqq S$ while $G_n$ is the set of all $\psi \in B(F_n, E^{**})$ with $\psi\circ\pi_n(L) \subseteqq S^{\sim}$. As before, the second dual of $B(F_n, E)$ is identified with $B(F_n, E^{**})$.

LEMMA 9. *If the weak** closure of $G_n$ coincides with $\mathscr{G}_n$, $n = 1, 2, \cdots$, then there is a linear lifting $\varphi$ from $E/M$ to $E$ such that $\varphi(L) \subseteqq S$ and $||\varphi|| \leqq 1$.*

*Proof*. Let $\pi_0 = 0$ and $\varphi_0 = 0$. Assume that linear maps $\varphi_j \in B(F_j, E)$ $j = 0, 1, \cdots, n$ have been found in such a way that $\tau\circ\varphi_j = 1$ on $F_j$, $||\varphi_j|| \leqq 1$, $\varphi_j\circ\pi_j(L) \subseteqq S$ and $||(\varphi_{j-1} - \varphi_j)\,|\,F_{j-1}|| < 1/2^{j-1}$ $j = 0, 1, \cdots, n$. Since $F_{n+1}$ is finite dimensional by hypothesis, there is $\psi \in B(F_{n+1}, E)$ such that $\tau\circ\psi = 1$ on $F_{n+1}$. Consider the map $\psi' = \varphi_n\circ\pi_n + \psi\circ(1 - \pi_n)$ from $F_{n+1}$ to $E$. Then by assumption

$$\psi'\circ\pi_n(\pi_{n+1}(L)) = \varphi_n\circ\pi_n(L) \subseteqq S$$

and in view of $||\pi_n|| \leqq 1$

$$\psi'\circ\pi_n(V_{n+1}) = \varphi_n(V_n) \subseteqq U$$

where $V_i$ denotes the unit ball of $F_i$. Since $V_{n+1} \subseteqq \tau(U)$ by Theorem 1 and $\pi_{n+1}(L) \subseteqq \tau(S)$,

$$\psi'(V_{n+1}) \subseteqq U + M \text{ and } \psi'(\pi_{n+1}(L)) \subseteqq S + M\ .$$

Since the weak** closure of $G_n$ coincides with $\mathscr{G}_n$ by hypothesis, Lemma 8, applied to $F_{n+1}, \pi_{n+1}(L)$ and $\pi_n$ instead of $F, K$ and $\pi$, yields that there is $\varphi_{n+1} \in G_{n+1}$ such that $||\varphi_{n+1}|| \leqq 1, \tau\circ\varphi_{n+1} = 1$ on $F_{n+1}$ and $||(\varphi_{n+1} - \varphi_n)\,|\,F_n|| < 1/2^n$, completing induction. Now the sequence $\{\varphi_n\circ\pi_n\}$ is uniformly bounded and

$$\sum_{k=n}^{\infty}\left|\left|(\varphi_{k+1} - \varphi_k)\right|F_n\right|\right| \leqq \sum_{j=n}^{\infty} 1/2^k < \infty$$

guaranteeing convergence of $\varphi_k(x)$ for every $x \in F_n$ as $k \to \infty$. Then $\{\varphi_n\circ\pi_n\}$ converges strongly to some map $\varphi$ from $E/M$ to $E$. Obviously $\varphi$ is a required linear lifting.

It is better to introduce some terminology before stating the main result on linear lifting. A Banach space $E$ is called a *$\pi$-space* if there is a sequence $\{F_n\}$ of finite dimensional subspaces such that $F_1 \subseteqq F_2 \subseteqq \cdots$ with $\overline{\bigcup_{n=1}^{\infty} F_n} = E$ and each $F_n$ is the range of a projection of norm one. Here projections $\pi_n$ can be assumed to have the property that $\pi_n \pi_m = \pi_n$ for $n \leqq m$ and that $\pi_n$ converges strongly to the identity as $n \to \infty$. An ordered Banach space is called a *$\Pi$-space* if, in addition, projections can be chosen positive and if each $F_n$ has the positive cone generated by a linearly independent basis.

THEOREM 5. *Suppose that the annihilator of a closed subspace $M$ is the range of a projection $P$ such that*

$$||f|| = ||Pf|| + ||f - Pf|| \quad (f \in E^*) .$$

*If the quotient space $E/M$ becomes a $\pi$-space then there is a linear lifting of norm one, or equivalently, $M$ is the kernel of a projection of norm one.*

*Proof.* Since the unit ball $U$ is splittable by Corollary 1, all requirements in Lemma 9 are fulfilled with $S = U$ and $L = \tau(U)$ by Corollary 7.

COROLLARY 8. *Let $N$ and $M$ be closed subspaces and suppose that $M^\perp$ is the range of a projection $P$ such that $P(N^\perp) \subseteqq N^\perp$ and*

$$||f|| = ||Pf|| + ||f - Pf|| \quad (f \in E^*) .$$

*If the quotient space $N/N \cap M$ is a $\pi$-space, there is a linear lifting of norm one from $N/N \cap M$ to $N$.*

*Proof.* In view of Theorem 5 it suffices to prove that the annihilator of $N \cap M$ in $N^*$ is the range of a projection $\mathscr{P}$ such that

$$||g|| = ||\mathscr{P}g|| + ||g - \mathscr{P}g|| \quad (g \in N^*) .$$

When $N^*$ is identified with $E^*/N^\perp$, the annihilator of $N \cap M$ becomes the image of $(N \cap M)^\perp$ under the canonical map from $E^*$ to $E^*/N^\perp$. Since hypothesis implies splittability of $N$ by Corollary 3, $N + M$ is closed by Theorem 1 so that $N^\perp + M^\perp$ is weak* closed and coincides with $(N \cap M)^\perp$. Therefore the annihilator of $N \cap M$ in $N^*$ becomes the image of $M^\perp$ in $E^*/N^\perp$. Since $N^\perp$ is invariant under $P$, there arises a natural projection $\mathscr{P}$ from $N^*$ to the annihilator $N \cap M$. Since by hypothesis

$$||f - N^\perp|| \geqq ||Pf - N^\perp|| + ||f - Pf - N^\perp|| ,$$

$\mathscr{P}$ is easily seen to have the required property.

When $E$ is the space of continuous functions on a compact set and $M$ consists of functions vanishing on a fixed closed subset, Corollary 8 was proved by Michael and Pełczynski [8].

THEOREM 6. *Let $M$ be a closed subspace of an ordered Banach space $E$. Suppose that $M$ is the range of a projection $P$ such that* $f \geqq Pf \geqq 0 \quad (f \geqq 0)$ *and*

$$\|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*) .$$

*If $E/M$ is a $\Pi$-space under the canonical ordering, there is a positive linear lifting of norm one, or equivalently, $M$ is the kernel of a positive projection of norm one.*

*Proof.* Since the positive cone $E_+$ is splittable by Corollary 3, all requirements in Lemma 9 are fulfilled with $S = E_+$ and $L = \tau(E_+)$ by definition of a $\Pi$-space and Corollary 6.

To be a $\pi$-space or a $\Pi$-space is not so severe restriction. Let us prove:

*Separable complex (resp. real) $L_p(1 \leqq p < \infty)$ and complex (resp. real) $C(X)$ on compact metrizable $X$ are $\pi$-spaces (resp. $\Pi$-spaces).*

In fact, it suffices for the first part to treat a $L_p$ space on a finite measure space $(\mathscr{B}, \mu)$. Since the Borel field is separable with respect to $\mu$, there is an increasing sequence $\{\mathscr{B}_n\}$ of finite Borel subfields such that $\bigcup_{n=1}^{\infty} L_p(\mathscr{B}_n)$ is dense in $L_p$ where $L_p(\mathscr{B}_n)$ is the subspace of $\mathscr{B}_n$-measurable functions. Each $L_p(\mathscr{B}_n)$ is finite dimensional and the conditional expection relative to $\mathscr{B}_n$ becomes a (positive) projection of norm one from $L_p$ to $L_p(\mathscr{B}_n)$(cf. [3]). The assertion for $C(X)$ is proved in [7] by using peaked partition.

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