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DOUBLE COMMUTANTS OF WEIGHTED SHIFTS

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Let A be an operator (bounded linear transformation) on a (complex) Hilbert space. If the double commutant of A is equal to the weakly closed algebra (with identity) generated by A we shall say that A belongs to the class *(dc)*.

By the von Neumann Double Commutant Theorem any Hermitian operator is in *(dc)*, and it is well known that any operator on a finite dimensional space is in *(dc)*. A generalization of this latter fact was proven in [10]. The object of this paper is to show that any one-sided weighted shift is in *(dc)* and that a two-sided weighted shift is in *(dc)* if and only if it is not invertible.

In their paper "The commutants of certain Hilbert space operators", [9], Shields and Wallen show that any one-sided shift *with nonzero weights* generates a maximal abelian weakly closed algebra and therefore is à fortiori in *(dc)*. Related work on commutants of weighted shifts has also been done by Gellar. (See [2], [3], and [4].)

1. Definitions and notation.

If $\mathfrak{H}$ is a Hilbert space we denote be $B(\mathfrak{H})$ the algebra of all operators on $\mathfrak{H}$. For A belonging to $B(\mathfrak{H})$ we denote by $\mathfrak{A}_A$ the weakly closed algebra with identity generated by A, by $\mathfrak{A}'_A$ the commutant of A, and by $\mathfrak{A}''_A$ the double commutant of A. (The reader is referred to [1, p. 1] for definitions of commutant and double commutant.) The class *(dc)* is the class of all operators A on Hilbert space such that $\mathfrak{A}_A = \mathfrak{A}''_A$.

Let $\mathfrak{H}$ be a separable Hilbert space, let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathfrak{H}$, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a bounded sequence of scalars. Define S on $\mathfrak{H}$ by $Se_n = \alpha_{n+1}e_{n+1}$ for all n, extending linearly and continuously. Then S is called a *one-sided weighted shift* on $\mathfrak{H}$.

Now let $\{e_n\}_{n=-\infty}^{\infty}$ be a two-side orthonormal basis for $\mathfrak{H}$ and let $\{\alpha_n\}_{n=-\infty}^{\infty}$ be a doubly infinite bounded sequence of scalars. Definite S on $\mathfrak{H}$ by $Se_n = \alpha_{n+1}e_{n+1}$ for all n, extending linearly and continuously as before. Then S is called a *two-sided* weighted shift on $\mathfrak{H}$.

Finally let $\mathfrak{H}$ be a finite dimensional Hilbert space and let $\{e_0, \cdots, e_k\}$ be an orthonormal basis for $\mathfrak{H}$. Let $\alpha_1 \cdots \alpha_k$ be scalars. Define S on $\mathfrak{H}$ by $Se_n = \alpha_{n+1}e_{n+1}$ for $0 \leqq n < k$, $Se_k = 0$, extending linearly. Then S is called a *finite* weighted shift on $\mathfrak{H}$.

In the forthcoming we shall drop the term "weighted" and speak simply of *shifts*.

A *backward* (one-sided, two-sided, or finite) shift is the adjoint of a (one-sided, two-sided, or finite) shift respectively. Note that a backward finite or two-sided shift is unitarily equivalent to a (forward) finite or two sided shift respectively. The equivalence is effected simply by renumbering the basis vectors.

Notice that we do not insist that the weights of our shifts (the numbers α_n) be nonzero.

2. Preliminaries.

To facilitate our discussion of shifts we now state and prove the following lemma.

LEMMA 2.1. *Let $A \in B(\mathfrak{H})$ be in the class (dc). Suppose that (a) A has either nontrivial kernel or co-kernel, or (b) A is invertible. Let $\mathfrak{K}$ be a Hilbert space. Then $A \oplus 0 \in B(\mathfrak{H} \oplus \mathfrak{K})$ is in the class (dc).*

Proof. Case (a). It is easily seen that $A \in (dc)$ if and only if $A^* \in (dc)$ so we may assume without loss of generality that A has nontrivial co-kernel. Let f be a unit vector in co-ker (A) and let $g \in \mathfrak{K}$. Define $X: \mathfrak{H} \to \mathfrak{K}$ by $Xh = (h, f)\cdot g$. Then for any $h \in \mathfrak{H}$, $XAh = (Ah, f)\cdot g = 0\cdot g = 0$, so $XA = 0 = 0X$. Now $Xf = ||f||^2\cdot g = g$. Therefore, we conclude that $\bigcup_{XA=0X}$ Range $X = \mathfrak{K}$. This fact, together with the fact that $A \in (dc)$ is easily seen to imply that $A \oplus 0 \in (dc)$.

Case (b). It is clear that $A^{-1} \in \mathfrak{A}_A''$. Since $A \in (dc)$ this implies that $A^{-1} \in \mathfrak{A}_A$, so there is a net of polynomials $\{p_\alpha\}$ such that $p_\alpha(A)$ converges to A^{-1} in the weak operator topology. Therefore, $A\cdot p_\alpha(A)$ tends weakly to $A\cdot A^{-1} = I$. Let $q_\alpha(z) = z\cdot p_\alpha(z)$. Then $q_\alpha(0) = 0$ so $q_\alpha(A \oplus 0) = q_\alpha(A) \oplus 0 = A\cdot p_\alpha(A) \oplus 0$ which converges to $I \oplus 0$. We conclude that $\mathfrak{A}_{A\oplus 0} = \mathfrak{A}_A \oplus \mathfrak{A}_0$, so that $A \oplus 0 \in (dc)$.

Note that $A \in (dc)$ does *not* always imply that $A \oplus 0 \in (dc)$. We shall give an example to illustrate this later on.

REMARK 2.2. As previously stated we make no assumptions about the weights of our shifts being nonzero. However, any weighted shift may be expressed as a direct sum of (one-sided, two-sided, or finite) shifts having nonzero weights. Having noted this fact we proceed to treat weighted shifts by means of a case argument.

As was previously mentioned, a one-sided shift with nonzero weights is known [9, Theorem 1] to be in the class (dc). A few moments reflection therefore shows that the cases we must consider are as follows:

(1) For one-sided shifts

(a) $S = \sum_{i=0}^{\infty} \oplus S_i$

(b) $S = (\sum_{i=0}^{\infty} \oplus S_i) \oplus S_\infty$.

(2) For two-sided shifts

(a) $S = (\sum_{i=0}^{\infty} \oplus S_i) \oplus S_\infty$

(b) $S = (\sum_{i=0}^{n} \oplus S_i) \oplus S_\infty \oplus T$

(c) S has no zero weights,

where each S_i is a finite shift, S_∞ is one-sided shift, and T is a backward one-sided shift, all with nonzero weights.

To eliminate some redundant cases in (2) it is necessary to recall that a backward finite shift is unitarily equivalent to a (forward) finite shift.

REMARK 2.3. Application of Lemma 2.1 allows us to ignore any zero direct summands that may occur in the above.

REMARK 2.4. It is true that a two-sided shift may also assume the forms.

$$\text{(i)} \qquad S = \sum_{i=-\infty}^{\infty} \oplus S_i$$

and

$$\text{(ii)} \qquad S = \left(\sum_{i=0}^{\infty} \oplus S_i\right) \oplus T .$$

However, (i) is really the same as (1) (a) and (taking adjoints) (ii) is really the same as (2) (a).

One more preliminary observation, which we state as Lemma 2.5, is needed. Lemma 2.5 is an easy generalization of a result of Shields and Wallen [9, p. 780]. Its proof depends in turn upon an equally easy generalization of a lemma of Schur [8, Theorem V, p. 11] which Shields and Wallen employ to prove their result. (For a sketch proof of these results see also [7] Lemma 2.1, and Corollary 2.2.) We omit the proof of Lemma 2.5 here.

LEMMA 2.5 *Let $A \in B(\mathfrak{H})$ be a (possibly infinite) direct sum of shifts (finite, one-sided, two-sided, backward, or forward). Let f be a formal power series and suppose that the matrix $f(A)$ defines an operator $D \in B(\mathfrak{H})$ (with respect to the fixed basis relative to which A is realized as a direct sum of shifts). Then D is in $\mathfrak{A}_A$.*

REMARK 2.6. Because of the particularly simple form of the matrix of A, the matrix $f(A)$ always makes sense.

3. Main results. The way is now clear to proceed with our theorems on the double commutants of weighted shifts.

THEOREM 3.1. *Any one-sided shift is in the class (dc).*

Proof. By Remark 2.2 there are two cases to consider:

(a) $$S = \sum_{i=0}^{\infty} \oplus S_i$$

and

(b) $$S = \left(\sum_{i=0}^{n} \oplus S_i\right) \oplus S_\infty \, .$$

Let S_i act on the space $\mathfrak{H}_i$ with orthonormal basis $\{e_{i0}, \cdots, e_{in_i}\}$ and have weights $\alpha_{i1}, \cdots, \alpha_{in_i}$. Let S_∞ act on the space $\mathfrak{H}$ with orthonormal basis $\{e_i\}_{i=0}^{\infty}$ and have weights $\{\alpha_i\}_{i=1}^{\infty}$.

Define $\pi_{i0} = 1$ and $\pi_{ij} = \prod_{k=1}^{j} \alpha_{ik}$ for $1 \leqq j \leqq n_i$, for $0 \leqq i < \infty$. Likewise define $\pi_0 = 1$ and $\pi_j = \prod_{k=1}^{j} \alpha_k$ for $1 \leqq j < \infty$. We now proceed to treat cases (a) and (b) separately.

Case (a). Let $D \in \mathfrak{A}_S''$. It is easily seen that D must have the form $D = \sum_{i=0}^{\infty} \oplus D_i$ where D_i is in $\mathfrak{A}_{S_i}'' = \mathfrak{A}_{S_i}$. (Since S_i acts on a finite dimensional space it is in (dc).) Finite dimensionality also implies that $\mathfrak{A}_{S_i}$ is just the *algebra* generated by S_i, so each $D_i = p_i(S_i)$ where p_i is a polynomial, in fact of degree less than or equal to n_i.

Now for any $n_r \geqq n_s$ define $E_{rs}\colon \mathfrak{H}_r \to \mathfrak{H}_s$ by

$$E_{rs}e_{ri} = \begin{cases} \dfrac{\pi_{si}}{\pi_{ri}} \cdot e_{si} & \text{for } 0 \leqq i \leqq n_s \\ 0 & \text{for } n_s < i \leqq n_r \end{cases}$$

extending linearly. A simple computation shows that $S_sE_{rs} = E_{rs}S_r$, consequently $D_sE_{rs} = E_{rs}D_r$. Now E_{rs} maps $\mathfrak{H}_r$ *onto* $\mathfrak{H}_s$ so for every h in $\mathfrak{H}_r$ there is a g such that $h = E_{rs}g$. Therefore, $D_sh = D_sE_{rs}g = E_{rs}D_rg = E_{rs}p_r(S_r)g = p_r(S_s)E_{rs}g = p_r(S_s)h$. We conclude that $p_s(S_s) = D_s = p_r(S_s)$ for $n_r \geqq n_s$.

If the sequence $\{n_i\}_{i=0}^{\infty}$ is bounded we may choose $n_r = \sup_i n_i$. Then for every i, $D_i = p_i(S_i) = p_r(S_i)$ so that $D = p_r(S) \in \mathfrak{A}_s$.

If the sequence $\{n_i\}_{i=0}^{\infty}$ is unbounded we define a formal power series

$$f(z) = \sum_{k=0}^{\infty} \gamma_k z^k \quad \text{where } \gamma_k$$

is the kth coefficient of any p_i with $n_i \geqq k$. Since $n_r \geqq n_s$ implies $p_r(S_s) = p_s(S_s)$ we see that the kth coefficients of p_r and p_s agree whenever n_r and n_s are both greater than or equal to k, so the numbers γ_k are well-defined.

Now consider $f(S)$, the formal power series in S. For each i, S_i is nilpotent of order $n_i + 1$ and the coefficients of f of index less than or equal to n_i agree with those of p_i. Thus $f(S_i) = p_i(S_i) = D_i$. Therefore, the matrix $f(S)$ defines the operator D, so by Lemma 2.5 D is in $\mathfrak{A}_S$.

Case (b). If D is in $\mathfrak{A}_S''$ then $D = \sum_{i=0}^{\infty} \oplus D_i \oplus D_\infty$ where each D_i is in $\mathfrak{A}_{S_i}''$, $i = 0, \cdots, N, \infty$. Each D_i, $0 \leqq i \leqq N$, is a polynomial p_i in S_i and D_∞ is a formal power series g in S_∞. (See [6, p. 780].) Define $E_r: \mathfrak{H} \rightarrow \mathfrak{H}_r$ by

$$E_r e_i = \begin{cases} \frac{\pi_{ri}}{\pi_i} \cdot e_{ri} & \text{for} \quad 0 \leqq i \leqq n_r \\ 0 & \text{for} \quad i > n_r \end{cases}$$

extending linearly. By arguments similar to those used for case (a) we can show that the coefficients of the polynomials p_i agree (as far as they go) with those of g, whence $D = g(S)$ and so is in $\mathfrak{A}_S$.

We now turn to the discussion of two-sided shifts. These operators need *not* always be in the class (*dc*). The familiar bilateral shift (all weights equal to 1) provides a counter-example. If we denote this shift by S, then $S^* = S^{-1}$ is in $\mathfrak{A}_S''$ but not in $\mathfrak{A}_S$.

As it turns out, this example gives the clue to the general situation. If a two-sided shift S is invertible then $S^{-1} \in \mathfrak{A}_S''$ but $S^{-1} \notin \mathfrak{A}_S$. If S is not invertible then $\mathfrak{A}_S = \mathfrak{A}_S''$.

PROPOSITION 3.2. *If S is a two-sided shift with at least one zero weight then $S \in (dc)$.*

Proof. By Remark 2.2 we have two cases to consider:

(a) $S = (\sum_{i=0}^{\infty} \oplus S_i) \oplus S_\infty$. This case can be treated by exactly the same methods as were used in case (b) of Theorem 3.1.

(b) $S = (\sum_{i=0}^{N} \oplus S_i) \oplus S_\infty \oplus T$. If we can show that an operator in the double commutant of $S_\infty \oplus T$ is a formal power series in $S_\infty \oplus T$, then Lemma 2.5 and the same arguments as were used in Theorem 3.1, part (b), will show that $S \in (dc)$. We therefore turn our attention to operators of the form $R \oplus T$ where R is a forward and T a backward one-sided shift, both with nonzero weights.

Let R act on $\mathfrak{H}$ with orthonormal basis $\{e_i\}_{i=0}^{\infty}$ and have weights $\{\alpha_i\}_{i=1}^{\infty}$. Let T act on $\mathfrak{K}$ with orthonormal basis $\{f_i\}_{i=0}^{\infty}$ and have weights $\{\beta_i\}_{i=1}^{\infty}$.

$$\text{Define } \pi_0 = 1, \pi_j = \prod_{k=1}^{j} \alpha^k, \quad \text{for} \quad 1 \leqq j < \infty .$$

$$\text{Define } \kappa_0 = 1, \kappa_j = \prod_{k=1}^{j} \beta_k, \quad \text{for} \quad 1 \leqq j < \infty .$$

Let D be in the double commutant of $R \oplus T$. Then $D = E \oplus F$ with E in $\mathfrak{A}_R'' = \mathfrak{A}_R$ and F is $\mathfrak{A}_T'' = \mathfrak{A}_T$. By [6, p. 780] E is a formal power series $f(R)$ in R. Likewise F^* is a formal power series in T^*

so F is a formal power series $g(T)$ in T. If we can show that the coefficients of g agree with those of f then we will know that D is a formal power series $f(R \oplus T)$.

To demonstrate this agreement of coefficients define operators $G_r: \mathfrak{H} \to \mathfrak{K}$ by means of the matrix $[\gamma_{ij}]$ where

$$\gamma_{ij} = \begin{cases} \dfrac{1}{\kappa_i \pi_j} & \text{for} \quad i + j \leqq r \\ 0 & \text{for} \quad i + j > r \,. \end{cases}$$

At most finitely many of the numbers γ_{ij} are nonzero so this matrix defines an operator. It is easily seen that $TG_r = G_r R$ and that G_r maps the span of $\{e_0, \cdots, e_r\}$ onto the span of $\{f_0, \cdots, f_r\}$.

Now write $E = f(R) = \sum_{i=0}^{\infty} \mu_i R^i$. (The series converges on finite linear combinations of basis vectors.) Likewise write $F = g(T) = \sum_{i=0}^{\infty} \nu_i T^i$. Then, for any $r \geqq 0$,

$$\begin{aligned} Ff_r &= FG_r h \text{ for some } h \text{ in the span of } \{e_0, \cdots, e_r\} \\ &= G_r E h \\ &= G_r f(R) h \\ &= \sum_{i=0}^{\infty} \mu_i G_r R^i h \\ &= \sum_{i=0}^{\infty} \mu_i T^i G_r h \\ &= \sum_{i=0}^{\infty} \mu_i T^i f_r \\ &= \sum_{i=0}^{r} \mu_i \frac{\kappa_r}{\kappa_i} f_{r-i} \,. \end{aligned}$$

On the other hand

$$\begin{aligned} Ff_r &= g(T) f_r \\ &= \sum_{i=0}^{\infty} \nu_i T^i f_r \\ &= \sum_{i=0}^{r} \nu_i \cdot \frac{\kappa_r}{\kappa_i} f_{r-i} \,. \end{aligned}$$

Comparing coefficients we see that $\nu_r = \mu_r$ for all r.

PROPOSITION 3.3. *Let S be a two-sided shift with nonzero weights $\{\alpha_i\}_{i=-\infty}^{\infty}$. Suppose that* $\inf_i |\alpha_i| = 0$. *Then $S \in (dc)$; in fact S generates a maximal abelian weakly closed algebra.*

Proof. Let $C \in \mathfrak{A}'_S$ with matrix $[\gamma_{ij}]$. Computing $SCe_j = CSe_j$ for arbitrary j and comparing coefficients we obtain the following characterization for the numbers γ_{ij}:

$$\gamma_{ij} = \begin{cases} \dfrac{\alpha_{j+1} \times \cdots \times \alpha_0}{\alpha_{i+1} \times \cdots \times \alpha_{i-j}} \gamma_{i-j0} & \text{for } j \leqq 0 \\ \dfrac{\alpha_{i-j+1} \times \cdots \times \alpha_i}{\alpha_1 \times \cdots \times \alpha_j} \gamma_{i-j0} & \text{for } j > 0 . \end{cases}$$

An easy but tedious case argument shows that this is equivalent to

$$\gamma_{ij} = \begin{cases} \dfrac{\alpha_{j+1} \times \cdots \times \alpha_i}{\alpha_1 \times \cdots \times \alpha_{i-j}} \gamma_{i-j0} & \text{for } i \geqq j \\ \dfrac{\alpha_{i-j+1} \times \cdots \times \alpha_0}{\alpha_{i+1} \times \cdots \times \alpha_j} \gamma_{i-j0} & \text{for } i < j . \end{cases}$$

Now fix a positive integer k and consider the kth super-diagonal $i = j - k$. Along this super-diagonal

$$\gamma_{ij} = \frac{\alpha_0 \times \cdots \times \alpha_{-k+1}}{\alpha_{j-k+1} \times \cdots \times \alpha_j} \gamma_{-k0} .$$

But

$$\inf_j | \alpha_{j-k+1} \times \cdots \times \alpha_j | \leqq \inf_j || S ||^{k-1} | \alpha_j | = || S ||^{k-1} \inf_j | \alpha_j | = 0 .$$

Therefore the numbers γ_{ij} are unbounded along this super-diagonal unless $\gamma_{-k0} = 0$. Since C is an operator we must indeed have $\gamma_{-k0} = 0$, whence the whole kth super-diagonal is zero.

This holds for $k = 1, 2, 3, \cdots$. We conclude that C is the formal power series in S, $\sum_{i=0}^{\infty} \mu_i S^i$ where $\mu_0 = \gamma_{00}$, and $\mu_i = \gamma_{i0}/(\alpha_1 \times \cdots \times \alpha_i)$ for $i > 0$. By Lemma 2.5, $C \in \mathfrak{A}_S$.

THEOREM 3.4. *A two-sided shift is in (dc) if and only if it is not invertible.*

Proof. If S is an invertible two-sided shift with weight sequence $\{\alpha_i\}_{i=-\infty}^{\infty}$, then S^{-1} is the backward shift with weight sequence $\{1/\alpha_i\}_{i=-\infty}^{\infty}$. Since S^{-1} is a backward shift, its matrix is strictly upper triangular and hence it cannot be in $\mathfrak{A}_S$. However, S^{-1} is clearly in $\mathfrak{A}_S''$.

On the other hand, if S is not invertible the infimum of the moduli of its weights is zero. By Propositions 3.2 and 3.3 $S \in (dc)$.

4. Concluding remarks. We may use the above theorem to construct counter-examples to some fairly reasonable-sounding conjectures about the class (*dc*).

First of all we show that the direct sum of operators in (*dc*) need not be in (*dc*): Let S be a two-sided shift with nonzero weights, the infimum of the moduli of which is zero. Since S has zero kernel and dense range, S and the zero operator cannot be intertwined. Thus the commutant of $S \oplus 0$ is $\mathfrak{A}_S' \oplus \mathfrak{A}_0'$, so the double commutant is

$\mathfrak{A}_S'' \oplus \mathfrak{A}_0'' = \mathfrak{A}_S \oplus \mathfrak{A}_0$. Consequently $I \oplus 0$ is in the double commutant of $S \oplus 0$, but it is clearly not in $\mathfrak{A}_{S\oplus 0}$, so $S \oplus 0 \notin (dc)$. However, 0 clearly is in (dc) and $S \in (dc)$ by Theorem 3.4. (This furnishes the example promised after Lemma 2.1.)

Secondly we show that (dc) is not preserved under norm limits. The operator S in the previous example may be taken to be compact. (Let the weights of S tend to 0 in both directions.) Then $S \oplus 0$ is compact also, so $S \oplus 0$ is the limit in norm of finite rank operators. Any finite rank operator belongs to (dc), but $S \oplus 0$ does not.

To see that finite rank operators are all in (dc), let F be finite rank and write F as $E \oplus 0$ where E acts on a finite dimensional space. The operator $E \in (dc)$ and is either invertible or has nonzero kernel. By Lemma 2.1 $F \in (dc)$.

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