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WALLMAN COMPACTIFICATIONS ON E-COMPLETELY REGULAR SPACES

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The Wallman space on E-completely regular spaces is considered. Let $\mathscr{F}$ be the family of all E-closed subsets of an E-completely regular space X. Then the Wallman space $\mathscr{W}(X, \mathscr{F})$ is a compactification of X. In particular, if E is such that $I = [0, 1]$ is E-completely regular, then $\mathscr{W}(X, \mathscr{F})$ is an E-compactification. An example is given to show that I being E-completely regular is necessary.

Recently, the relations between Stone-Čech compactifications and Wallman compactifications, and those between realcompactifications and Wallman compactifications have been studied by Frink [7], Njåstad [11], the Steiners [12], [13], Alo and Shapiro [1], [2], [3], [4], and some others.

Frink [7] introduced the concept of a normal base. (A normal base in a T_1-space X is a base, $\mathscr{F}$, for the closed subsets of X such that (i) $\mathscr{F}$ is disjunctive, i.e., given any closed set F in X and any point x in $X\backslash F$, there is a member A of $\mathscr{F}$ which contains x and is disjoint from F; (ii) $\mathscr{F}$ is a ring, i.e., $\mathscr{F}$ contains all finite unions and intersections of its members; and (iii) any two disjoint members A and B of $\mathscr{F}$ are separated by disjoint complements of two members of $\mathscr{F}$, i.e., there exist elements C and D of $\mathscr{F}$ such that $A \subset X\backslash C$, $B \subset X\backslash D$, and $(X\backslash C) \cap (X\backslash D) = \varnothing$.) Frink showed that if X has a normal base $\mathscr{F}$, equivalently X is Tychonoff, then the Wallman space $\mathscr{W}(X, \mathscr{F})$, consisting of the $\mathscr{F}$-ultrafilters, is a Hausdorff compactification of X. Hence, the Stone-Čech compactification is always such a Wallman compactification. Njåsted [11] came along and gave a condition for a Hausdorff compactification to be of the Wallman type as defined by Frink. The condition is that the corresponding proximity admits a productive base consisting of closed subsets. Alo and Shapiro [2]* used another approach for the results. While Alo and Shapiro in [1]* imposed some conditions on the normal base $\mathscr{F}$ (see Theorem 2, [1]), and gave similar results for a wider class of compactifications, Njåstad showed that Alexandroff, Stone-Čech, Freudenthal [6], Fan-Gottesman [5], and Gould [9] compactifications satisfy the conditions in his theorem.

In [3]*, Also and Shapiro used a delta normal base on a Tychonoff

* The authors wish to express their thanks to the referee for calling these articles to their attention.

space X. (A delta normal base $\mathscr{F}$ is a normal base which is closed under countable intersections, and such that for each $A \in \mathscr{F}$ there exist $B_1, B_2, \cdots \in \mathscr{F}$ with $Z = X \backslash \bigcup_{i=1}^{\infty} B_i$). They show that the subspace $\rho(X, \mathscr{F})$ of $\mathscr{W}(X, \mathscr{F})$ which consists of all $\mathscr{F}$-ultrafilters with the countable intersection property assigned is realcompact. They [4] also used the notion of $\mathscr{F}$-ultrafilters in a countably productive normal base $\mathscr{F}$ to introduce a new space $\eta(X, \mathscr{F})$ consisting of all those $\mathscr{F}$-ultrafilters with the countable intersection property. They showed that if $\mathscr{F}$ is the collection of all zero-sets, then $\eta(X, \mathscr{F})$ is precisely the Hewitt realcompactification. However, the Steiners [13] provided an example to show that not every realcompactification can be obtained as an $\eta(X, \mathscr{F})$. They also gave an example of a space which is an $\eta(X, \mathscr{F})$ but not realcompact.

E. F. Steiner [12] generalized Frink's results and established the necessary and sufficient conditions for a Wallman space to be a compactification. The Steiners [13] used the notion of separating (see Definition 3) nest generated intersection rings (see (1.1), [13]) and studied the Wallman compactification $\mathscr{W}(X, \mathscr{F})$ and the Wallman realcompactification $\nu(X, \mathscr{F})$. Incidentally, the concept of a delta normal base, introduced by Alo and Shapiro [3], is equivalent to that of separating nest generated intersection rings for collections $\mathscr{F}$ of closed sets.**

This note is to consider the Wallman compactification of an E-completely regular space. (See [10].) We have found a class of Hausdorff spaces, E, for which the Wallman compactification arising out of the ring of all E-closed subsets of X is an E-compactification. In light of the examples in [13], we know that not every E-compactification can be obtained as a Wallman compactification.

We first recall come terminologies from [10].

DEFINITION 1. Let E be any Hausdorff space. A T_1-space X is said to be E-completely regular if $\bigcup_{n=1}^{\infty} C(X, E^n)$ separates the closed subsets and points in X. Here, $C(X, E^n)$ is the set of all continuous functions from X into the Cartesian product E^n.

Note that this is equivalent to saying that for each closed subset A of X and for each $p \in (X \backslash A)$, there is a positive integer n and a continuous function $f \in C(X, E^n)$ such that $f(p) \notin \operatorname{cl} f[A]$. This is also equivalent to saying that X is homeomorphic to a subset of E^α for some cardinal α. (See [10].)

We will always assume that E is a Hausdorff space.

DEFINITION 2. A subset A in a space X is called an E-closed

** The authors wish to thank the referee for pointing out this fact.

subset of X if there is a positive integer n and a continuous function $f \in C(X, E^n)$ such that $A = f^{-1}[F]$ for some closed subset F of E^n.

One can easily show that a finite union and a finite intersection of E-closed subsets of X is E-closed. (See 3.18 [10].) That is, the family of all E-closed subsets of X forms a ring.

Combining these two definitions, we have:

LEMMA 1. *A T_1-space X is E-completely regular if and only if each closed subset F of X and each point $x \in X \backslash F$ are separated by disjoint E-closed sets; i.e., there are disjoint E-closed subsets A and B of X such that $x \in A$ and $F \subset B$.*

Proof. Necessity. By definition of E-complete regularity, there is a positive integer n and a continuous function $f \in C(X, E^n)$ such that $f(x) \notin \mathrm{cl}_{E^n} f[F]$. Let $A = f^{-1}[f(x)]$ and $B = f^{-1}[\mathrm{cl}_{E^n} f[F]]$. Then $A \cap B = \varnothing$ and A and B are E-closed subsets of X.

Sufficiency. Let F be a closed subset in X and $x \notin F$. By assumption, there are disjoint E-closed sets A and B such that $x \in A$, $F \subset B$ and $A \cap B = \varnothing$. Since B is E-closed, there exist a positive integer n, and an $f \in C(X, E^n)$ such that $B = f^{-1}[D]$, for some closed subset D in E^n. Now, since $x \notin B = f^{-1}[D]$, $f(x) \notin D$. This implies that $f(x) \notin \mathrm{cl}_{E^n} f[B]$ as $\mathrm{cl}_{E^n} f[B] \subset D$. Hence, X is E-completely regular.

Before stating our next result, we give the following:

DEFINITION 3. A family $\mathscr{F}$ of closed subsets of a space X is called *separating* if for each closed subset F of X and each point $x \in X \backslash F$, there are disjoint elements A and B of $\mathscr{F}$ such that $x \in A$ and $F \subset B$. (See]12].)

E. F. Steiner in [12] proved:

THEOREM 2. *If X is a T_1-space and $\mathscr{F}$ is a separating family, then the Wallman space $\mathscr{W}(X, \mathscr{F})$ is a compactification. If the Wallman space $W(X, \mathscr{F})$ is a compactification, then X is T_1 and the ring generated from $\mathscr{F}$ is separating.*

Now, suppose X is E-completely regular. Then there is a cardinal α, and a homeomorphism, h, from X into E^α. Let $\mathscr{S}$ denote the family of all E-closed subsets of E^α, and $\mathscr{F} = \{F \subset X \colon F = h^{-1}(F'),$ for some $F' \in \mathscr{S}\}$. Then we have:

THEOREM 3. *The Wallman space $\mathscr{W}(X, \mathscr{F})$ is a compactification of X.*

Proof. By Theorem 2, we only have to show that $\mathscr{F}$ is a separating ring. However, by remark of Definition 2, $\mathscr{S}$ is a ring, so that $\mathscr{F}$ is a ring. Now, let F be any closed of X, and $x \in X \backslash F$. Then that $h(x) \notin \mathrm{cl}_{E^\alpha} h[F]$ is clear. Since E^α is E-completely regular, $h(x)$ and $\mathrm{cl}_{E^\alpha} h[F]$ are separated by two disjoint E-closed sets, say A_1 and A_2, where $h(x) \in A_1$ and $\mathrm{cl}_{E^\alpha} h[F] \subset A_2$. Then $B_i = h^{-1}[A_i]$, $i = 1, 2$ are in $\mathscr{F}$ and $x \in B_1$ and $F \subset B_2$.

THEOREM 4. *Let X be a T_1 space and $\mathscr{F}$ be the family of all E-closed subset of X. Then the Wallman space $\mathscr{W}(X, \mathscr{F})$ is a compactification of X if and only if X is E-completely regular.*

Proof. *Sufficiency*. We know that $\mathscr{F}$ is a ring, and by Lemma 1, $\mathscr{F}$ is separating. Hence, $\mathscr{W}(X, \mathscr{F})$ is a compactification.

Necessity. If $\mathscr{W}(X, \mathscr{F})$ is a compactification of X, then the ring $\mathscr{F}$ is separating by Theorem 2, and, and by Lemma 1, X is E-completely regular.

In general, we do not know if $\mathscr{W}(X, \mathscr{F})$ is E-completely regular.

Next, we would like to determine under what conditions the Wallman compactification defined by the ring of all E-closed subsets of an E-completely regular space is an E-compactification.

We recall that an E-completely regular space X is E-compact if and only if X is homeomorphic to a closd subset of E^α for some cardinal α. Hence, each compact E-completely regular space is E-compact. Then we have:

THEOREM 5. *If E, a Hausdorff space, is such that $I = [0, 1]$ with the usual topology is E-completety regular, then if X is an E-completely regular space, the Wallman space $\mathscr{W}(X, \mathscr{F})$ generated by the ring $\mathscr{F}$ of all E-closed subsets of X is an E-compactification of X.*

Proof. By Theorem 2, $\mathscr{W}(X, \mathscr{F})$ is T_2-compact. Since I is E-completely regular and compact, I is E-compact and $\mathscr{W}(X, \mathscr{F})$ is I-compact. Thus, $\mathscr{W}(X, \mathscr{F})$ is E-compact by (4.6) [10]. Hence, it is an E-compactification of X.

REMARK. (1) We know that there exists a space E such that I is E-completely regular. For example, let E_1 be any Hausdorff space. Define E to be the topological sum of I and E_1. Then I is clearly E-completely regular, as I is homeomorphic with a subspace (namely I) of E. Note that as long as E_1 is Hausdorff and not completely regular, E is not completely regular.

(2) Next we point out that the condition that I be E-completely regular cannot be omitted, for consider $E = X$, where X is the space of Knaster and Kuratowski. We still recall it here (see p. 210 of [14]). Let C denote the Cantor middle third set, and Q the end points in C. Let $p = (1/2, 1/2) \in R^2$, and for each $x \in C$, denote by L_x the straight line segment joining p and x.

Define

$$L_x^* = \begin{cases} \{(x_1, x_2) \in L_x \colon x_2 \text{ is rational}\}, & \text{if } x \in Q \\ \{(x_1, x_2) \in L_x \colon x_2 \text{ is irrational}\}, & \text{if } x \in C \backslash Q. \end{cases}$$

Then $E = X = \bigcup_{x \in C} L_x^* \backslash \{p\}$. Here $\bigcup_{x \in C} L_x^*$ is connected, while $E = X$ is T_2, totally disconnected, and $\dim X = \dim E \neq 0$ (see 29.8 [14]). It is then clear that I is not E-completely regular, since E^α is totally disconnected and so is any subset of E^α. (See 29.3 [14].)

Now, X is E-completely regular, a metric space (see 29.8 [14]), and is hence normal. Consider $\mathscr{F}$, the family of all E-closed subsets of X. $\mathscr{F}$, in fact, consists of all closed subsets of X. Thus, the Wallman compactification $\mathscr{W}(X, \mathscr{F})$ is βX, the Stone-Čech compactification (see [8], p. 269).

Finally, βX is T_2 compact space, but βX is not totally disconnected, for otherwise by Theorem 16.17 in [8], we would have $\dim \beta X = 0$. But Theorem 16.11 [8] says that $\dim \beta X = \dim X$, and we know that $\dim X \neq 0$.

Therefore, $X = \mathscr{W}(X, \mathscr{F})$ cannot be E-completely regular, and is thus not an E-compact space.

In view of Remark (2), we have:

COROLLARY 6. *For a Hausdorff space E, if X is a T_1 zero-dimensional normal space having more than one point and such that every closed subset of X is E-closed, then the Wallman space $\mathscr{W}(X, \mathscr{F})$ generated by the ring of all closed subsets of X is an E-compactification of X.*

Proof. Since $\dim X = 0$, $\dim \beta X = 0$. Also, $\beta X = \mathscr{W}(X, \mathscr{F})$ since X is normal. Now, $\mathscr{W}(X, \mathscr{F})$ is T_1 and zero-dimensional. One can easily show that it is E-completely regular. Hence, $\mathscr{W}(X, \mathscr{F})$ is E-compact.

COROLLARY 7. *If X is discrete, then $\mathscr{W}(X, \mathscr{F})$ is an E-compactification of X, where $\mathscr{F}$ is the family of all closed subsets of X.*

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