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SOME ISOLATED SUBSETS OF INFINITE SOLVABLE GROUPS

D. S. PASSMAN

The main theorem of this paper offers necessary and sufficient conditions for a solvable group G to be covered by a finite union of certain types of isolated subsets. This result will have applications to the study of the semisimplicity problem for group rings of solvable groups.

Let H be a subgroup of G. We define

$$\sqrt[G]{H} = \sqrt{H} = \{x \in G \mid x^m \in H \text{ for some } m \geqq 1\} .$$

Observe that $\sqrt{H}$ need not be a subgroup of G even if G is solvable. We say that H has locally finite index in G and write $[G: H] = l.f.$ if for every finitely generated subgroup L of G we have $[L: L \cap H] < \infty$. Suppose $[G: H] = l.f.$ and let $x \in G$. Then $[\langle x\rangle: \langle x\rangle \cap H] < \infty$ so $x^m \in H$ for some $m \geqq 1$ and $x \in \sqrt{H}$. Thus $G = \sqrt{H}$. The main result of this paper is a generalized converse of this fact for solvable groups G.

THEOREM. *Let G be a solvable group and let $H_1, H_2, \cdots, H_n$ be subgroups with*

$$G = \bigcup_1^n \sqrt{H_i} .$$

Then for some $i = 1, 2, \cdots, n$ we have $[G: H_i] = l.f.$

This paper constitutes one third of the solution of the semisimplicity problem for group rings of solvable groups. The remaining two thirds can be found in [1] and [4]. Moreover a description of this latter result as well as an analogue of the above theorem for linear groups will appear in [3].

We first list some basic properties of subgroups of locally finite index.

LEMMA 1. *Let $G \supseteqq W \supseteqq H$, $G \supseteqq W_1 \supseteqq H_1$ and let $N \lhd G$.*
(i) *$[G: H] = l.f.$ implies $[G: W] = l.f.$*
(ii) *$[G/N: HN/N] = l.f.$ implies $[G: HN] = l.f.$*
(iii) *$[W: H] = l.f.$ implies $[WN: HN] = l.f.$*
(iv) *$[W: H] = l.f.$ and $[W_1: H_1] = l.f.$ implies $[W \cap W_1: H \cap H_1] = l.f.$*
(v) *$[G: W] = l.f.$ and $[W: H] = l.f.$ implies $[G: H] = l.f.$*

Proof. (i) If $L \subseteqq G$ then $[L: W \cap L] \leqq [L: H \cap L]$ so this is clear. (ii) Let L be a finitely generated subgroup of G. Then LN/N

is finitely generated so

$$[LN/N: (HN/N) \cap (LN/N)] < \infty .$$

Thus $[LN: HN \cap LN] < \infty$. Since $L \subseteq LN$ this yields $[L: HN \cap L] < \infty$.

(iii) Let L be a finitely generated subgroup of WN. Then there exists a finitely generated subgroup $S \subseteq W$ with $LN = SN$. Now $[S: S \cap H] < \infty$ so $[SN: (S \cap H)N] < \infty$. Observe that $(S \cap H)N \subseteq SN \cap HN$ so $[SN: SN \cap HN] < \infty$. Finally $L \subseteq LN = SN$ yields $[L: L \cap HN] < \infty$ and $[WN: HN] = l.f.$

(iv) Let L be a finitely generated subgroup of $W \cap W_1$. Then $L \subseteq W$ yields $[L: H \cap L] < \infty$ and similarly $[L: H_1 \cap L] < \infty$. Thus $[L: (H \cap H_1) \cap L] < \infty$ and $[W \cap W_1: H \cap H_1] = l.f.$

(v) Finally let L be a finitely generated subgroup of G. Since $[G: W] = l.f.$ we have $[L: L \cap W] < \infty$. Thus by [1, Lemma 6.1] $L \cap W$ is finitely generated and since $[W: H] = l.f.$ we have

$$[L \cap W: L \cap W \cap H] < \infty .$$

This yields $[L: L \cap H] < \infty$ and the lemma is proved.

LEMMA 2. *Let AH be a group with A a normal abelian subgroup. Set*

$$B = \{a \in A \mid [H: H \cap H^a] = l.f.\} .$$

Then we have

(i) $A \cap H \lhd AH$

(ii) *if* $a \in A$ *then* $H \cap H^a = N_H(a(H \cap A))$

(iii) B *is a subgroup of* A *and* $B \lhd AH$.

(iv) *if* $[A: B] < \infty$ *and* $B/(A \cap H)$ *is torsion, then* $[AH: H] = l.f.$

Proof. (i) Since $A \lhd AH$ we have $A \cap H \lhd H$. Since A is abelian we have $A \cap H \lhd A$. Thus $A \cap H \lhd AH$.

(ii) Let $h \in H \cap H^a$. Then $h \in H$ and $h^{a^{-1}} \in H$ so $h^{-1}h^{a^{-1}} \in H \cap A$ since A is normal. Thus h centralizes a modulo $H \cap A$ so h normalizes $a(H \cap A)$ and $H \cap H^a \subseteq N_H(a(H \cap A))$.

Let $h \in N_H(a(H \cap A))$. Then $h \in H$ and $h^a \equiv h$ modulo $H \cap A$. Since $H \cap A \lhd AH$ we have $H^a \supseteq H \cap A$ and $h \in H^a(H \cap A) = H^a$. Thus $h \in H \cap H^a$.

(iii) Clearly $1 \in B$. Since $[a(H \cap A)]^{-1} = a^{-1}(H \cap A)$ we see that $N_H(a(H \cap A)) = N_H(a^{-1}(H \cap A))$. Thus $a \in B$ implies $a^{-1} \in B$. Finally let $a, b \in B$. Then $[H: H \cap H^a] = l.f.$ implies $[H^b: H^b \cap H^{ab}] = l.f.$ so by Lemma 1 (iv), $[H \cap H^b: H \cap H^b \cap H^{ab}] = l.f.$ Now $[H: H \cap H^b] = l.f.$ so Lemma 1 (v) yields $[H: H \cap H^b \cap H^{ab}] = l.f.$ Since $H \cap H^{ab} \supseteq H \cap H^b \cap H^{ab}$ we have $[H: H \cap H^{ab}] = l.f.$ and B is a group. Clearly $B \lhd AH$.

(iv) By Lemma 1 (ii) since $A \cap H \lhd AH$, $A \cap H \subseteqq B$, $A \cap H \subseteqq H$ it clearly suffices to work in $AH/(A \cap H)$ or in other words we may assume that $A \cap H = \langle 1 \rangle$. Thus AH is the semidirect product of A by H. Now $[AH: BH] < \infty$ so by Lemma 1 (v) it suffices to show that $[BH: H] = l.f.$

Let L be a finitely generated subgroup of BH. Then there exists a finitely generated subgroup B_1 of B and a finitely generated subgroup H_1 of H such that $L \subseteqq B_1^{H_1} \cdot H_1$. By definition of B and by (ii) each element of B_1 has only finitely many conjugates under the action of H_1. Thus $B_1^{H_1}$ is a finitely generated abelian group. Since this group is torsion by assumption we have

$$|B_1^{H_1}| < \infty \quad \text{and} \quad [B_1^{H_1} \cdot H_1: H_1] = |B_1^{H_1}| < \infty .$$

Finally $L \subseteqq B_1^{H_1} \cdot H_1$ so $[L: L \cap H_1] < \infty$. Since $L \cap H = L \cap (B_1^{H_1} \cdot H_1) \cap H = L \cap H_1$, the result follows.

We can now obtain the main result.

Proof of the Theorem. By induction on $d(G)$, the derived length of G. If $d(G) = 0$ then $G = \langle 1 \rangle$ so the result is clear. Assume the result for all groups G with $d(G) \leqq d$. For any group G let $DG = G^{(d)}$ be the dth derived subgroup of G.

Suppose $d(G) = d + 1$. Since $G = \bigcup_1^n \sqrt{H_i}$ we have clearly

$$G/(DG) = \bigcup_1^n \sqrt{H_i(DG)/(DG)} .$$

By induction some of these groups have locally finite index in $G/(DG)$. Thus by Lemma 1 (ii) we have for a suitable ordering of the H_i's that $[G: H_i(DG)] = l.f.$ for $i = 1, 2, \cdots, s$ (some $s \geqq 1$) and $[G: H_i(DG)] \neq l.f.$ for $i > s$. We call s the parameter of the situation and we prove the $d(G) = d + 1$ case by induction on the parameter starting with $s = 0$ which does not occur.

Assume the result for all groups G with either $d(G) \leqq d$ or $d(G) = d + 1$ and parameter $< s$. Now fix G and suppose $d(G) = d + 1$, $G = \bigcup_1^n \sqrt{H_i}$ and the parameter of this situation is s. Set $A = DG$ so A is a normal abelian subgroup of G and say $H_1A, H_2A, \cdots, H_sA$ have locally finite index in G. For each $i \leqq s$ set

$$B_i = \{a \in A \mid [H_i: H_i \cap H_i^a] = l.f.\} .$$

By Lemma 2 (iii) B_i is a subgroup of A.

Step 1. For each $i \leqq s$ set

$$A_{1i} = \{a \in A \mid [H_1: H_i^a \cap H_1] = l.f.\} .$$

Then $A = \bigcup_1^s A_{1i}$.

Proof. Fix $a \in A$ and let $x \in H_1$. Then $(axa^{-1})^m \in H_j$ for some j so $x^m \in H_j^a \cap H_1$. Thus

$$H_1 = \bigcup_1^n \sqrt[H_1]{H_i^a \cap H_1} \ .$$

If $d(H_1) \leqq d$ then by induction $[H_1: H_i^a \cap H_1] = l.f.$ for some i and as in the argument below $i \leqq s$ so $a \in A_{1i}$. Assume that $d(H_1) = d + 1$ and consider the parameter of this situation. Observe that $DH_1 \subseteqq A \cap H_1$.

Suppose $[H_1: (H_1^a \cap H_1)DH_1] = l.f.$ Now $H_1 \supseteqq DH_1$ and $H_1^a \supseteqq (DH_1)^a = DH_1$ since A is abelian. Thus $(H_1^a \cap H_1)DH_1 = H_1^a \cap H_1$ so $[H_1: H_1^a \cap H_1] = l.f.$ and $a \in A_{11}$.

Thus we may suppose that $[H_1: (H_1^a \cap H_1)DH_1] \neq l.f.$ Let $[H_1: (H_j^a \cap H_1)DH_1] = l.f.$ Since A is normal in G and $A \supseteqq DH_1$ we have by Lemma 1 (iii)

$$[H_1A: (H_j^a \cap H_1)A] = [H_1A: (H_j^a \cap H_1)(DH_1)A] = l.f.$$

Now $[G: H_1A] = l.f.$ so by Lemma 1 (v) we have $[G: (H_j^a \cap H_1)A] = l.f.$ Now $H_jA \supseteqq (H_1 \cap H_j^a)A$ so $[G: H_jA] = l.f.$ by Lemma 1 (i) and $j \leqq s$. Since $j \neq 1$ the parameter of this situation is $< s$.

By induction $[H_1: H_1 \cap H_i^a] = l.f.$ for some $i \leqq n$. But then by Lemma 1 (i) $[H_1: (H_1 \cap H_i^a)DH_1] = l.f.$ so $i \leqq s$ by the above. Thus $a \in A_{1i}$.

Step 2. If $A_{1i} \neq \varnothing$ and $a_i \in A_{1i}$ then $A_{1i} = B_i a_i$.

Proof. Suppose $A_{1i} \neq \varnothing$ and fix $a_i \in A_{1i}$ and let $a \in A_{1i}$. Then $[H_1: H_i^a \cap H_1] = l.f.$ and $[H_1: H_i^{a_i} \cap H_1] = l.f.$ yield by Lemma 1 (iii) (iv) first $[H_1: H_1 \cap H_i^a \cap H_i^{a_i}] = l.f.$ and then $[H_1A: (H_1 \cap H_i^a \cap H_i^{a_i})A] = l.f.$ Since $[G: H_1A] = l.f.$ we have by Lemma 1 (v) $[G: (H_1 \cap H_i^a \cap H_i^{a_i})A] = l.f.$ Now

$$(H_1 \cap H_i^a \cap H_i^{a_i})A \subseteqq (H_i^a \cap H_i^{a_i})A = (H_i \cap H_i^{aa_i^{-1}})A$$

so we have by Lemma 1 (i) (iv) $[G: (H_i \cap H_i^{aa_i^{-1}})A] = l.f.$ and

$$[H_i: H_i \cap (H_i \cap H_i^{aa_i^{-1}})A] = l.f.$$

Observe that $H_i \cap H_i^{aa_i^{-1}} \supseteqq H_i \cap A$ and thus

$$H_i \cap (H_i \cap H_i^{aa_i^{-1}})A = (H_i \cap H_i^{aa_i^{-1}})(H_i \cap A) = H_i \cap H_i^{aa_i^{-1}} \ .$$

Therefore the above yields $[H_i: H_i \cap H_i^{aa_i^{-1}}] = l.f.$ so $aa_i^{-1} \in B_i$ and $a \in B_i a_i$. Hence $A_{1i} \subseteqq B_i a_i$.

Now let $b \in B_i$. Then $[H_i: H_i^b \cap H_i] = l.f.$ yields $[H_i^{a_i}: H_i^{ba_i} \cap H_i^{a_i}] =$

l.f. so by Lemma 1 (iv) $[H_1 \cap H_i^{a_i}: H_1 \cap H_i^{ba_i} \cap H_i^{a_i}] = l.f.$ Since $[H_1: H_1 \cap H_i^{a_i}] = l.f.$ Lemma 1 (v) yields $[H_1: H_1 \cap H_i^{ba_i} \cap H_i^{a_i}] = l.f.$ Since $H_1 \cap H_i^{ba_i} \supseteqq H_1 \cap H_i^{ba_i} \cap H_i^{a_i}$ we have $[H_1: H_1 \cap H_i^{ba_i}] = l.f.$ and $ba_i \in A_{1i}$. Thus $B_i a_i \subseteqq A_{1i}$ and this fact follows.

Step 3. We may assume that for all $i = 1, 2, \cdots, s$ we have $[A: B_i] < \infty$ and $B_i/(A \cap H_i)$ not torsion.

Proof. By Steps 1 and 2 we have

$$A = \cup B_i a_i \quad \text{over all} \quad A_{1i} \neq \emptyset$$

and hence by Lemma 5.2 of [1]

$$A = \bigcup B_i a_i \quad \text{over all} \quad A_{1i} \neq \emptyset \ , \qquad [A: B_i] < \infty \ .$$

In particular since $1 \in A$ there exists $k \leqq s$ with $[A: B_k] < \infty$ and $1 \in A_{1k}$.

Suppose $k \neq 1$. Then $1 \in A_{1k}$ implies that $[H_1: H_k \cap H_1] = l.f.$ and hence as we observed earlier this yields ${}^{H_1}\sqrt{H_k \cap H_1} = H_1$. Since this clearly yields ${}^{G}\sqrt{H_1} \subseteqq {}^{G}\sqrt{H_k}$ we then have $G = \bigcup_2^n \sqrt{H_i}$. Observe that here $[G: H_i(DG)] = l.f.$ precisely for $i = 2, 3, \cdots, s$ so that parameter of this new situation is $s - 1$. By induction $[G: H_i] = l.f.$ for some i and the result follows. Thus we may assume that $k = 1$. Hence $[A: B_1] < \infty$.

Note that $B_1 \supseteqq A \cap H_1$ since $A \cap H_1 \lhd AH_1$. If $B_1/(A \cap H_1)$ is torsion then Lemma 2 (iv) implies that $[H_1A: H_1] = l.f.$ Since $[G: H_1A] = l.f.$ we conclude by Lemma 1 (v) that $[G: H_1] = l.f.$ and the result follows again. Thus we may assume that $B_1/(A \cap H_1)$ is not torsion.

In a similar manner for each $j \leqq s$ we can define sets A_{ji} for $i = 1, 2, \cdots, s$ and conclude that we may assume $[A: B_j] < \infty$ and $B_j/(A \cap H_j)$ is not torsion.

Step 4. Completion of the proof.

Proof. Now A is abelian so ${}^{A}\sqrt{A \cap H_i}$ is a group. Since $A \neq {}^{A}\sqrt{A \cap H_i}$ for $i \leqq s$ by Step 3 we cannot even have $[A: {}^{A}\sqrt{A \cap H_i}] < \infty$. Thus by Lemma 1.2 of [2], $A \neq \bigcup_1^s {}^{A}\sqrt{A \cap H_i}$ so choose $a \in A, a \notin {}^{A}\sqrt{A \cap H_i}$ for all $i \leqq s$.

Let $B = B_1 \cap B_2 \cap \cdots \cap B_s$. Then $[A: B] < \infty$ and say $a^t = b \in B$ with $t \geqq 1$. Then clearly $b \notin {}^{A}\sqrt{A \cap H_i}$ for all $i \leqq s$. For each $i \leqq s$ let $E_i = H_i \cap H_i^b = N_{H_i}(b(H_i \cap A))$ by Lemma 2 (ii). Then $b \in B_i$ implies that $[H_i: E_i] = l.f.$ so by Lemma 1 (iii) (v) since $[G: H_iA] = l.f.$ we have $[G: E_iA] = l.f.$ Observe that A abelian implies that

$E_iA \subseteqq N_G(b(H_i \cap A))$. If $E = \bigcap_1^s E_1A$ then by Lemma 1 (iv), $[G: E] = l.f.$

Let $e \in E$. Now $G = \bigcup_1^n \sqrt{H_i}$ so for the $n+1$ elements e, be, b^2e, $\cdots$, b^ne there exists integers $m_j, k_j \geqq 1$ with

$$(b^je)^{m_j} \in H_{k_j} \quad \text{for} \quad j = 0, 1, \cdots, n .$$

By the pigeon hole principle there exists $i \neq j$ with $(b^ie)^{m_i}$, $(b^je)^{m_j}$ both in H_k. Thus if $m = m_im_j$ then $(b^ie)^m$, $(b^je)^m$ both belong to H_k.

Suppose that $k \leqq s$. Now $e \in E \subseteqq E_kA \subseteqq H_kA$ so e normalizes the cosets $b(H_k \cap A)$ and $(H_k \cap A)$. Thus

$$(b^ie)^m \in b^{im}e^m(H_k \cap A) , \qquad (b^ie)^m \in H_k$$

so $b^{im}e^m \in H_k$. Similarly $b^{jm}e^m \in H_k$ and hence $b^{(i-j)m} = (b^{im}e^m)(b^{jm}e^m)^{-1} \in H_k$, a contradiction since $(i-j)m \neq 0$ and $b \notin \sqrt{H_k \cap A}$. Thus $k > s$.

Since $(b^ie)^m \in H_k$ for $k > s$ and $b \in A$ we see that $e^m \in H_kA$ and hence $E = \bigcup_{s+1}^n \sqrt[E]{H_kA \cap E}$. Thus $E/A = \bigcup_{s+1}^n \sqrt{(H_kA \cap E)/A}$. Since $DE \subseteqq A$ we have $d(E/A) \leqq d$ so by induction and Lemma 1 (ii), $[E: H_kA \cap E] = l.f.$ for some $k > s$. Since $[G: E] = l.f.$ we then have by Lemma 1 (v) (i) $[G: H_kA] = l.f.$ for some $k > s$. However this contradicts the definition of the parameter s and the theorem is proved.

We close with a few comments about the theorem and proof.

First, some assumption on G is obviously needed in the theorem. For example let G be the finitely generated infinite p-group constructed by E. S. Golod (see Corollary 27.5 of [2]). Then $G = \sqrt{\langle 1 \rangle}$ but $[G: \langle 1 \rangle] \neq l.f.$

Second, one might be tempted to guess that the appropriate definition of locally finite index should be $[G: H] = \widetilde{l.f.}$ if and only if $[\langle H, S \rangle : H] < \infty$ for every finite subset S of G. However this is not the right condition here. For example let $G = Z_p \wr Z_{p^\infty}$ and let $H = Z_{p^\infty}$. Then G is solvable and periodic so $G = \sqrt{H}$ but

$$[\langle H, Z_p \rangle : H] = \infty .$$

Third, it is interesting to observe in the proof that if $G \neq \langle 1 \rangle$ is abelian, then $G = A$ so the results of the first three steps are trivial in this case. The proof for $G = A$ is contained in the first paragraph of the fourth step.

Finally, we remark that the proof of the special case of this result in which G is assumed to equal $\sqrt{H}$ is very much simpler.

REFERENCES

1. C. R. Hampton and D. S. Passman, *On the semisimplicity of group rings of solvable groups*, Trans. Amer. Math. Soc., **173** (1972), 289-301.

2. D. S. Passman, *Infinite Group Rings*, Marcel Dekker, New York, 1971.
3. D. S. Passman, *On the semisimplicity of group rings of linear groups*, Pacific. J. Math., to appear.
4. A. E. Zalesskii, *A semisimplicity criteria for the group ring of a solvable group* (in Russian), Doklady Akad. Nauk CCCP, to appear.

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UNIVERSITY OF WISCONSIN