PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 1, 1973

VARIETIES OF IMPLICATIVE SEMI-LATTICES II

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This paper is concerned with a process of coordinatization of the lattice of varieties of implicative semilattices. Equational descriptions of some elements in each coordinate class, and a complete equational description of one coordinate class are given.

1. Introduction. This paper is a continuation of [8]. Familiarity with [8] and [6] is assumed. After stating some of the consequences of the local finiteness of the variety of implicative semilattices, we describe a system for partitioning the lattice of varieties of implicative semi-lattices into coordinate intervals, and give some results that can be obtained from a study of this coordinatization. Finally, we give equational descriptions for the largest and smallest varieties in each coordinate class, the covers of the smallest variety in each coordinate class and a complete equational description of the coordinate class 4,2.

Recall that an implicative semi-lattice is subdirectly irreducible if and only if it has a single dual atom. In accordance with the usage of [8], this dual atom will be denoted by u. If in a subdirectly irreducible implicative semi-lattice, the dual atom is deleted, the remaining structure is both a subalgebra and a homomorphic image of the original. Thus every subdirectly irreducible implicative semi-lattice may be thought of as obtained by appending a single dual atom to some already given implicative semi-lattice. If L is an implicative semilattice, the subdirectly irreducible implicative semi-lattice obtained in this manner will be denoted by $\hat{L}$.

2. Local finiteness. The following theorem was proven first by A. Diego [2] in a slightly different context. McKay [4] extended the result to implicative semi-lattices. We present a much simpler proof here.

THEOREM 2.1. *The variety of implicative semi-lattices is locally finite.*

Proof. Let F_n denote the free implicative semi-lattice on n generators. The proof proceeds by induction. F_1 has two elements. Assume that F_n is finite. $F_{n+1} \leqq_s \prod \hat{L}_i$, where each $\hat{L}_i$ is $n+1$ generated. Hence each L_i is n generated. It follows from the induction assumption that there are only a finite number of distinct L_i each of which is finite. Therefore the same statement applies to the $\hat{L}_i$, and hence F_{n+1} is finite.

COROLLARY 2.2. *Every variety of implicative semi-lattices is generated by its finite sub-directly irreducible members.*

COROLLARY 2.3. *If f is a homomorphism of an implicative semi-lattice L onto a finite implicative semi-lattice M, then there exists $L' \leqq_s L$ such that $f \mid L'$ is an isomorphism.*

COROLLARY 2.4. *The lattice of all varieties of implicative semi-lattices is itself implicative.*

COROLLARY 2.5. *If L is a finite subdirectly irreducible implicative semi-lattice, then the class of all those implicative semilattices which do not contain a sub-implicative semi-lattice isomorphic to L is a variety.*

3. Coordinates of varieties. In this section, A will denote a subdirectly irreducible implicative semi-lattice. Also the term "algebra" will be used in place of "implicative semi-lattice". Let $\mathscr{C}_n$ denote the variety generated by C_n, the n chain, and $\mathscr{B}_n$ denote the variety generated by $\hat{B}_n$, where B_n is the Boolean algebra with n atoms. Let $\overline{\mathscr{C}}_n$ denote the variety of all algebras which do not have $n+1$ chains as sub-algebras, and similarly let $\overline{\mathscr{B}}_n$ denote the variety of all algebras which do not have sub-algebras isomorphic to $\hat{B}_{n+1}$. (Throughout n and m will denote natural numbers.) Let $W_{n,m} = \mathscr{C}_n \vee \mathscr{B}_m$, and $V_{n,m} = \overline{\mathscr{C}}_n \cap \overline{\mathscr{B}}_m$. We say that a variety has coordinates n, m if it is in the interval $[W_{n,m}, V_{n,m}]$.

LEMMA 3.1. *If $A \in V_{n,m}$, and if A is finite, then $|A| \leqq 2^{m(n-3)}(2^m + 1)$, where $|A|$ denotes the number of elements in A.*

Proof. Since A is subdirectly irreducible and does not contain $\hat{B}_{m+1}$ as a subalgebra, A cannot contain B_{m+1}. Thus the closed algebra of A has at most m atoms. The proof now proceeds by induction. The case $n = 3$ holds since $A \in V_{3,m}$ implies $A = \hat{B}_l$ for some $l \leqq m$. Assume that the proposition holds for some n, and let $A \in V_{n+1,m}$. Then the dense filter D of A is an element of $V_{n,m}$. Thus $|D| \leqq 2^{m(n-3)}(2^m + 1)$. The proposition follows for the $n+1$ case since every element of A is the meet of a closed element and a dense element.

COROLLARY 3.2. *$V_{n,m}$ contains only a finite number of distinct finite subdirectly irreducible algebras.*

THEOREM 3.3. *$V_{n,m}$ contains no infinite subdirectly irreducible algebras.*

Proof. Assume the contrary, and let n be the least integer for which there is an m such that $V_{n,m}$ has an infinite subdirectly irreducible algebra, A. Now A is unbounded, since if A were bounded, the dense filter of A would be an infinite subdirectly irreducible algebra in $V_{n-1,m}$. This reasoning also shows that any principal filter of A is bounded in size by the bound of Lemma 3.1, and this in turn implies that A is bounded, which establishes a contradiction.

COROLLARY 3.4. *If V is a variety of implicative semi-lattices, then the following are equivalent:*

(i) *V has only finitely many subvarieties.*

(ii) *V is generated by a finite algebra.*

(iii) *V has coordinates n, m for some natural numbers n and m.*

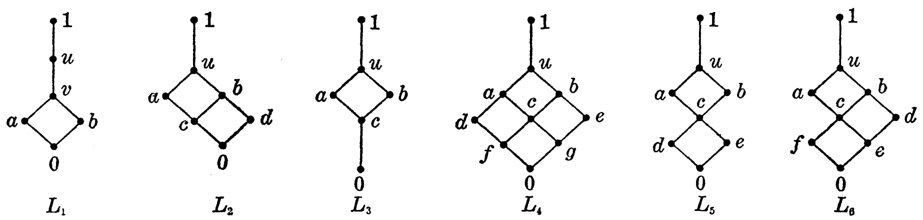

FIGURE 1

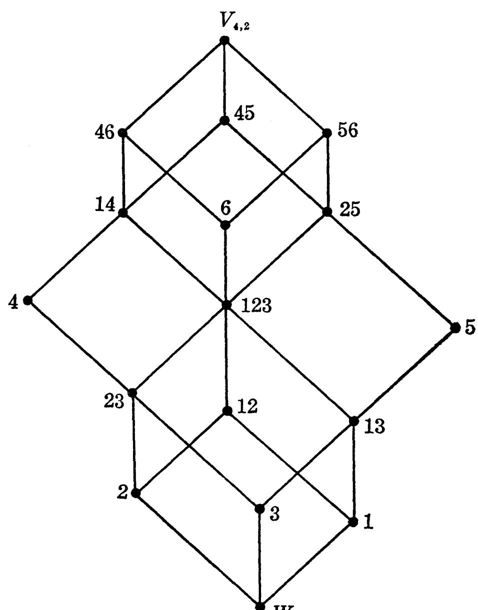

FIGURE 2

In order for A to be in $V_{4,2}$, the closed algebra of A must be B_1 or B_2, and the dense filter of A must be $\hat{B}_2$, C_2 or C_3. In [6] a method is given for constructing all algebras having a given closed algebra and a given dense filter. We omit the details, but using this process one finds that the subdirectly irreducible members of $V_{4,2} - W_{4,2}$ are those shown in Figure 1. We have $L_1 \leqq_s L_5$, $L_2 \leqq_s L_5$; $L_2, L_3 \leqq_s L_4$; $L_1, L_2, L_3 \leqq_s L_6$; and these are the only subalgebra relations holding among these six algebras. Thus the interval $[W_{4,2}, V_{4,2}]$ is as pictured in Figure 2, where the numbers beside a point in the lattice correspond to the indices of the algebras which generate that variety.

For $n \leqq 4$ and $m \leqq 2$, it is clear that the varieties $W_{n+1,m}$, $W_{n,m+1}$, and $W_{n,m} \vee \{L_i\}^e$ for $i = 1, 2, 3$ cover $W_{n,m}$. ($\{L\}^e$ is the variety generated by L.) It is also clear that any other cover of $W_{n,m}$ would have to be a subvariety of $V_{n,m}$. We now show that there are no additional covers of $W_{n,m}$.

DEFINITION 3.5. For $B, D \leqq_s L$, we say B is fixed with respect to D *if* $d^*b = b$ for $b \in B$, and $d \in D$. We say that D is total with respect to B if $b*d \in D$ for $b \in B, d \in D$. Let $B \bigtriangledown D = \{b \wedge d \mid b \in B, d \in D\}$.

It was shown in [5] that $B \bigtriangledown D$ is a subalgebra of L if B is fixed with respect to D and D is total with respect to B.

THEOREM 3.6. *If L is a subdirectly irreducible implicative semilattice, and if $C_4 \leqq_s L$, then either L is a chain or $L_i \leqq_s L$ for some $i = 1, 2, 3$.*

Proof. First, consider the case where L is bounded. If the dense filter of L is not a chain, then it contains $\hat{B}_2$ as a subalgebra, and thus $L_3 \leqq_s L$. Hence, we may assume that the dense filter of L is a chain. If the closed algebra of L is simple, then L is also a chain. Therefore we may assume that the closed algebra of L contains a subalgebra $\{1, b, b', 0\}$, where b' is the complement of b in the closed algebra. Now either $b*d = 1$ for every dense element d, or there is a dense element $d < 1$ such that $b*d = d$. If $b*d = d$, then $b'*d = 1$. Thus in either case, we have a subalgebra $D = \{1, u, d\}$ of the dense filter of L such that B is fixed with respect to D and D is total with respect to B. Hence $B \bigtriangledown D \leqq_s L$. We may assume that $b' \leqq d$. If $b \leqq d$, then $B \bigtriangledown D = L_1$. If $b \nleqq d$, then $B \bigtriangledown D = L_2$. Now suppose that L is not bounded and that $L_i \nleqq_s L$ for any $i = 1, 2$, or 3. Let $a, b \in L$, and let d be the least element of some example of C_4 in L. Then from consideration of the bounded case, it follows that the principal filter generated by $a \wedge b \wedge d$ is a chain. Thus a and b are comparable and so L is a chain.

COROLLARY 3.7. *For $n \geqq 4$ and $m \geqq 2$, $W_{n,m}$ has exactly five covers.*

COROLLARY 3.8. *$\mathscr{C}_n \vee \overline{\mathscr{C}_3}$ and $\mathscr{B}_m \vee \overline{\mathscr{B}_2}$ have exactly three covers.*

4. **Identities.** If $g(x_1, \cdots, x_n)$ is an implicative semi-lattice term and if L is an implicative semi-lattice, then we say that $g(x_1, \cdots, x_n)$ holds in L, or simply that g holds in L, provided the equation $g(x_1, \cdots, x_n) = 1$ holds in L. If this is not the case we say that g fails in L. We let $V(g)$ denote the variety of all implicative semi-lattices in which g holds. We are interested here only in subdirectly irreducible implicative semi-lattices, and we let u denote the dual atom in any such algebra. If there exist elements $a_1, \cdots, a_n \in L$ such that $g(a_1, \cdots, a_n) = u$, then we say that g u-fails in L. If g u-fails in every subdirectly irreducible algebra in which it fails, then we say that g has property U.

We let $a + b$ denote the psuedo-join (see [7]) of the elements a and b(i.e. $a + b = ((a^*b)^*b) \wedge ((b^*a)^*a))$. In general this is not an associative operation, and when not indicated otherwise, we intend for the grouping to be to the left (i.e. $a + b + c = (a + b) + c$). If a and b are comparable elements, then $a + b$ is the larger of the two.

LEMMA 4.1. *If $a_1 \geqq a_i$ for $i = 2, \cdots, n$, then*

$$a_1 + a_2 + \cdots + a_n = a_1 .$$

We should note that this lemma depends on our convention of association.

DEFINITION 4.2. If $g_1(x_1, \cdots, x_n)$ and $g_2(x_1, \cdots, x_m)$ are terms, then we let

$$(g_1 \oplus g_2)(x_1, \cdots, x_{n+m}) = g_1(x_1, \cdots, x_n) + g_2(x_{n+1}, \cdots, x_{n+m})$$

and

$$(g_1 \wedge g_2)(x_1, \cdots, x_{n+m}) = g_1(x_1, \cdots, x_n) \wedge g_2(x_{n+1}, \cdots, x_{n+m}) .$$

LEMMA 4.3. *If g_1 u-fails in L and if g_2 fails in L, then $g_1 \oplus g_2$ u-fails in L. Thus if g_1 has property U, then so does $g_1 \oplus g_2$.*

LEMMA 4.4. *If g_1 has property U, then $V(g_1) \vee V(g_2) = V(g_1 \oplus g_2)$.*

Proof. By [2, Lemma 4.1] any subdirecly irreducible member, L, of $V(g_1) \vee V(g_2)$ is in $V(g_1) \cup V(g_2)$. Thus g_1 holds in L or g_2 holds in L. Hence $g_1 \oplus g_2$ holds in L.

On the other hand, if L is any subdirectly irreducible not in

$V(g_1) \vee V(g_2)$, then g_1 and g_2 both fail in L. Thus g_1 u-fails in L; so $g_1 \oplus g_2$ fails in L.

LEMMA 4.5. *$V(g_1) \wedge V(g_2) = V(g_1 \wedge g_2)$. Furthermore, if g_1 and g_2 both have property U, then so does $g_1 \wedge g_2$.*

The main idea in the following theorem is present in a similar theorem for Heyting algebras due to Alan Day [1].

THEOREM 4.6. *Letting t^* denote $t^*(x_1 \wedge \cdots \wedge x_{n+1})$ and l_{ij} denote $x_i^{**} * x_j^{**}$, we have*

$$\overline{\mathscr{B}_n} = V(P_n)$$

where

$$P_n(x_1, \cdots, x_{n+2}) = x_{n+2} + l_{12} + l_{21} + \cdots + l_{n+1,n}$$

where each l_{ij} with $i \neq j$ and $i, j \leqq n+1$ occurs exactly once. Also, P_n has property U.

Proof. Let $a_1, \cdots, a_{n+1}$ be the atoms of $\hat{B}_{n+1}$. Then $a_i^{**} = a_i$ and $a_i^{**} * a_j^{**} < 1$ if $i \neq j$. Thus $P_n(a_1, \cdots, a_{n+1}, u) = u$. Hence $V(P_n) \subsetneqq \overline{\mathscr{B}_n}$.

Suppose now that L is any subdirectly irreducible member of $\overline{\mathscr{B}_n}$ and that $P_n(a_1, \cdots, a_{n+2}) < 1$ in L. Then $a_1^{**}, \cdots, a_{n+1}^{**}$ are pairwise incomparable closed elements in the principal filter generated by $a_1 \wedge \cdots \wedge a_{n+1}$. Thus $\hat{B}_{n+1} \leqq_s L$, a contradiction. Hence P_n holds in L.

In [8] terms were given which characterize the varieties $\mathscr{C}_n$ and $\overline{\mathscr{C}_n}$. Denote these terms by q_n and r_n, respectively. It is easy to see that q_n and r_n have property U.

COROLLARY 4.7. *$V_{n,m} = V(P_m \wedge r_n)$. In particular, $\mathscr{B}_m = V(P_m \wedge r_3)$.*

COROLLARY 4.8. *$W_{n,m} = V(q_n \oplus (P_m \wedge r_3))$.*

We now turn our attention to the varieties of the interval $[W_{4,2}$ $V_{4,2}]$. First we shall give an indexed list of identities which can be used to describe these varieties. Note that for a term t, t^* is as defined in Theorem 4.6.

$$g_1 = x_4 + ((x_1 \wedge x_2)*(x_1 \wedge x_2 \wedge x_3)) + (x_1*x_2) + (x_2*x_1)$$
$$g_{12} = x_4 + (x_1*x_2) + (x_2*x_1) + (x_1 \wedge x_2)^* + (x_1^{**}*x_1) + (x_2^{**}*x_2)$$
$$g_{23} = x_4 + (x_4*x_3) + (x_1*x_2) + (x_2*x_1) + (x_3 + (x_3*x_1)) + (x_3 + (x_3*x_2))$$
$$g_2 = g_{12} \wedge g_{23}$$
$$g_3 = x_4 + (x_4*x_3) + (x_1*x_2) + (x_2*x_1) + (x_3 + (x_3*(x_1 \wedge x_2)))$$
$$g_{13} = g_1 \oplus g_3$$
$$g_{123} = g_{12} \oplus g_3$$
$$g_4 = x_4 + (x_4*x_3) + ((x_3 \wedge x_1)*(x_3 \wedge x_2)) + ((x_3 \wedge x_2)*(x_3 \wedge x_1)) + ((x_3 + (x_3*(x_3 \wedge x_1))) + (x_3 + (x_3*(x_3 \wedge x_2))))$$
$$g_{14} = g_1 \oplus g_4$$
$$g_5 = x_4 + (x_1*x_2) + (x_2*x_1) + (x_1*x_3) + (x_3*x_1) + (x_2*x_3)$$
$$g_{25} = g_2 \oplus g_5$$
$$g_{45} = g_4 \oplus g_5$$
$$g_{56} = x_4 + (x_1*x_2) + (x_2*x_1) + (x_1*x_3) + (x_3*x_1) + (x_2*x_3) + (x_3*x_2)$$
$$g_{46} = x_5 + (x_1*x_2) + (x_2*x_1) + (x_1 \wedge x_2 \wedge x_3)*(x_1 \wedge x_2 \wedge x_4)) + (x_1 \wedge x_2 \wedge x_4)*(x_1 \wedge x_2 \wedge x_3)) + (x_1 + ((x_1 \wedge x_2) + (x_1 \wedge x_2 \wedge x_3))) + (x_1 + ((x_1 \wedge x_2)*(x_1 \wedge x_2 \wedge x_4))) + (x_2 + ((x_1 \wedge x_2)*(x_1 \wedge x_2 \wedge x_3))) + (x_2 + ((x_1 \wedge x_2)*(x_1 \wedge x_2 \wedge x_4))) .$$

THEOREM 4.9. *For $i, j = 1, \cdots, 6$ let $h_i = g_i \wedge P_4 \wedge r_3$, $h_{ij} = g_{ij} \wedge P_4 \wedge r_3$, $h_{123} = g_{123} \wedge P_4 \wedge r_3$.*
Then

(i) $\{L_i\}^e = V(h_i)$

(ii) $\{L_i, L_j\}^e = V(h_{ij})$ *for* $\{i, j\} = \{1, 3\}, \{1, 2\}, \{2, 3\}, \{1, 4\}, \{2, 5\}, \{4, 5\}, \{4.6\}$, *and* $\{5.6\}$.

(iii) $\{L_1, L_2, L_3\}^e = V(h_{123})$.

COROLLARY 4.10. *For i, j as in the previous theorem and $n > 4$, $m > 2$ we have*

(i) $\{L_i\}^e \vee W_{n,m} = V(h_i \oplus (q_n \oplus (P_m \wedge r_3)))$,

(ii) $\{L_i, L_j\}^e \vee W_{n,m} = V(h_{ij} \oplus (q_n \oplus (P_m \wedge r_3)))$,

(iii) $\{L_1, L_2, L_3\}^e \vee W_{n,m} = V(h_{123} \oplus (q_n \oplus (P_m \wedge r_3)))$.

In some cases the identities given can be simplified somewhat, but these were chosen for convenience in presentation.

Proof. The proof amounts to showing that each of the indexed polynomials g is valid in the corresponding variety of the diagram of figure 2 and its subvarieties, and that it fails elsewhere in the diagram. Note that each of these identities has property U. We shall establish the validity of three of the more complicated identities only.

(1) g_{12} holds in L_1 and L_2, but fails in L_3: If $g_{12}(a_1, \cdots, a_4) < 1$ in L_1, then a_1 and a_2 are incomparable and $(a_1 \wedge a_2)^* = 1$, a contradiction.

If $g_{12}(a_1, \cdots, a_4) < 1$ in L_2, then we must have $\{a_1, a_2\} = \{a, b\}$ and $a_1 \wedge a_2 \wedge a_3 = 0$. However, $a^{**}*a = 1$ then yields a contradiction.

In L_3 we have

$$g_{12}(a, b, 0, u) = u + b + a + (c*0) + (1*a) + (1*b) = u\ .$$

(2) g_4 holds in L_4 but fails in L_1: In L_1 we have $g_4(a, b, v, u) = u + v + b + a + ((u + a) + (v + b)) = u$.

If $g_4(a_1, \cdots, a_4) < 1$ in L_4, then $a_3 < u$. In fact $a_3 = a, b$, or c since there must be a pair of incomparable elements below a_3. If $a_3 = a$ we have $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{d, c\}, \{d, g\}$, or $\{f, g\}$. If $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{d, c\}$, then $a_3 + (a_3*c) = a + b = 1$. If $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{d, g\}$, then $a_3 + (a_3*g) = a + e = 1$. If $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{f, g\}$, then we get the same contradiction as in the preceding case. The case $a_3 = b$ is completely analogous. If $a_3 = c$, then $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{f, g\}$. Then $(a_3 + (a_3*f)) + (a_3 + (a_3*g)) = (c + d) + (c + e) = a + b = 1$, a contradiction.

(3) g_{46} holds in L_4 and L_6, but fails in L_5: If $g_{46}(a_1, \cdots, a_5) < 1$, then a_1 and a_2 are incomparable and there must be a pair of incomparable elements, $a_1 \wedge a_2 \wedge a_3$ and $a_1 \wedge a_2 \wedge a_4$, which are less than $a_1 \wedge a_2$. Thus in L_4 we would have to have $\{a_1, a_2\} = \{a, b\}$ and $\{a_1 \wedge a_2 \wedge a_3, a_1 \wedge a_2 \wedge a_4\} = \{f, g\}$. However, we have $a + ((a \wedge b)^*g) = 1$ which would give a contradiction. In L_6 we would have to have $\{a_1, a_2\} = \{a, b\}$ and $\{a_1 \wedge a_2 \wedge a_3, a_1 \wedge a_2 \wedge a_4\} = \{f, e\}$. This would lead to a contradiction, however, since $a + ((a \wedge b)*e) = 1$.

In L_5 we have

$$\begin{aligned} g_{46}(a, b, d, e, u) &= u + b + a + e + d \\ &\quad + (a + d) + (a + e) + (b + d) + (b + e) = u\ . \end{aligned}$$

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Received October 10, 1971. The second named author was supported in this work by a grant from the Research Corporation.

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