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MULTIPLIERS OF TYPE (p, p) AND MULTIPLIERS OF THE GROUP L_p-ALGEBRAS

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Let G be a locally compact group with left Haar measure λ and suppose $1 \leqq p < \infty$. The purpose of this paper is to exhibit an isometric isomorphism ω of the Banach algebra M_p of all right multipliers on $L_p = L_p(G, \lambda)$ into the normed algebra m_p of all right multipliers on the group L_p-algebra L_p^t. When G is either commutative or compact, ω is surjective.

A function $f \in L_p$ is said to be *p-temperate* if

(1) $h*f(x) = \int_G f(t)h(t^{-1}x)d\lambda(t)$ exists for λ-almost all $x \in G$ whenever h is in L_p;

(2) $h*f$ is in L_p for all $h \in L_p$;

(3) $\sup\{\|h*f\|_p: h \in L_p, \|h\|_p \leqq 1\} < \infty$.

It was shown in [6], Theorem 1, that $f \in L_p$ is p-temperate if

(4) $\sup\{\|h*f\|_p: h \in C_{\infty}, \|h\|_p \leqq 1\} < \infty$

where C_{00} denotes the set of all continuous complex-valued functions on G with compact support. The set of all p-temperate functions will be written as L_p^t. Each function $f \in C_{00}$ is in L_p^t and so L_p^t comprises a dense subspace of L_p. For $f \in L_p^t$, the number given by either (3) or (4) will be written as $\|f\|_p^t$. The function $\| \ \|_p^t$ so defined is a norm under which L_p^t is a normed algebra. This normed algebra will be referred to as the *group L_p-algebra*.

By a right multiplier on L_p^t will be meant a bounded linear operator T on L_p^t such that

(5) $T(f*g) = f*T(g)$ for all f and g in L_p^t.

The set of all such T, which constitutes a normed algebra under the usual operator norm, will be written as $\mathfrak{m}_p$. Write $\mathfrak{B}p$ for the Banach algebra of all bounded linear operators on L_p. An operator $T \in \mathfrak{B}p$ is said to be a right multiplier of type (p, p) (see [3]) if

(6) $T({}_xf) = {}_xT(f)$ for all $f \in L_p$

where ${}_xh(y) = h(xy)$ for each function h on G. The set of all such T will be written as M_p. It is a complete sub-algebra of $\mathfrak{B}_p$.

The group L_p-algebra was utilized in [6] to study a related algebra A_p, of which the Banach algebra of left multipliers was found to be isomorphic to M_p. The situation is reversed here. For $f \in L_p^t$, an operator W_f in $\mathfrak{B}_p$ is defined by

$$W_f(g) = g*f \qquad \text{for all } g \in L_p \tag{7}$$

and, consequently,

$$\| W_f \| = \| f \|_p^t . \tag{8}$$

The closure in $\mathfrak{B}_p$ of the linear span of the set $\{W_{f*g}: f \in L_p^t, g \in C_\infty\}$ will be written as A_p. It is a Banach algebra with a minimal left approximate identity ([6], Theorem 3). Concrete interpretations of both A_p and L_p^t, in the cases where G is either commutative or compact, may be found in [6]. It will be mentioned here only that L_1^t is the group algebra L_1 and that L_2^t is the group Hilbert algebra (see [1] and [2] for example).

PROPOSITION 1. *Let T be in M_p and f and g be in L_p. Then*
(i) $T(f*g) = f*T(g)$ *if* $f \in L_1$;
(ii) $T(g)$ *is in* L_p^t *if* g *is in* L_p^t;
(iii) $T(f*g) = f*T(g)$ *if* g *is in* L_p^t.

Proof. Part (i) was proved in the corollary to Theorem 4 in [6]. Let g be in L_p^t. By (i),

$$\begin{aligned} &\sup\{\| h*T(g) \|_p: h \in C_{00}, \| h \|_p \leqq 1\} \\ &= \sup\{\| T(h*g) \|_p: h \in C_{00}, \| h \|_p \leqq 1\} \leqq \| T \| \cdot \| g \|_p^t . \end{aligned}$$

By (4), this implies that $T(g)$ is in L_p^t.

Let again g be in L_p^t and choose a sequence $\{f_n\}$ in C_∞ which converges to f in L_p. Then

$$\begin{aligned} &\lim_n \| f_n*g - f*g \|_p = 0 \text{ and, in view of (ii)}, \\ &\lim_n \| f_n*T(g) - f*T(g) \|_p = 0. \text{ Thus, by (i)}, \\ &f*T(g) = \lim_n f_n*T(g) = \lim_n T(f_n*g) = T(f*g) . \end{aligned}$$

LEMMA 1. *For each nonzero $f \in L_p$, there exists $g \in C_\infty$ for which $g*f \neq 0$.*

Proof. See [4] 20.15.

LEMMA 2. *For each $T \in m_p$ and $V \in A_p$,*

$$\sup\{\| T \circ V(h) \|_p: h \in L_p^t, \| h \|_p \leqq 1\} \leqq \| T \| \cdot \| V \| .$$

Proof. Write D for the set $\{W_f: f \in L_p^t, W_f \in A_p\}$. Then D is a dense subspace of A_p and, by (8), $\| W_f \| = \| f \|_p^t$ for all $W_f \in D$.

Hence, if $\rho' \mid D \to \mathfrak{B}p$ is defined by $\rho'(W_f) = W_{T(f)}$ for all $W_f \in D$, then ρ' is continuous. Let $\rho \mid A_p \to \mathfrak{B}_p$ be the unique continuous extension of ρ to A_p. The immediate object is to show that $\rho(V)$ and $T \circ V$ coincide on L_p^t.

Let $h \in L_p^t$ be such that $\|h\|_p \leqq 1$ and let $\{f_n\}$ be a sequence in L_p^t such the W_{f_n} is in D for each $n \in N$ and $\lim_n \|W_{f_n} - V\| = 0$. Since A_p is a subset of M_p, the operator V is in M_p and so, by Proposition 1.iii,

$$V \circ W_h(g) = V(g*h) = g * V(h) = W_{V(h)}(g)$$

for all $g \in L_p$; hence, $V \circ W_h = W_{V(h)}$. That $W_{W_{f_n}}(h) = W_{f_n} \circ W_h$ is easy to check. Thus, for each $n \in N$, (8) yields $\|W_{f_n}(h) - V(h)\|_p^t = \|W_{f_n} \circ W_h - V \circ W_h\|$. Hence,

$$\overline{\lim_n} \|W_{f_n}(h) - V(h)\|_p^t \leqq \overline{\lim_n} \|W_{f_n} - V\| \circ \|W_h\| = 0 .$$

Consequently,

$$\lim_n \| T(W_{f_n}(h)) - T(V(h)) \|_p^t = 0 . \tag{9}$$

For each $n \in N$ and $g \in L_p^t$, $W_{T(f_n)}(g) = g * T(f_n) = T(g*f_n) = T \circ W_{f_n}(g)$; hence, $\rho(W_{f_n}) = \rho'(W_{f_n}) = W_{T(f_n)} = T \circ W_{f_n}$. Consequently

$$\overline{\lim_n} \| T \circ W_{f_n} - \rho(V) \| = \lim_n \| \rho(W_{f_n}) - \rho(V) \| = 0 .$$

Thus

$$\lim_n \| T \circ W_{f_n}(h) - [\rho(V)](h) \|_p = 0 \qquad \text{and so}$$
$$\lim_n \| g*(T \circ W_{f_n}(h)) - g*[\rho(V)](h) \|_p = 0$$

for each $g \in C_\infty$. But, by (9),

$$\lim_n \| g*(T \circ W_{f_n}(h)) - g*(T(V(h))) \|_p = 0$$

for all $g \in C_\infty$. It follows that $g*[\rho(V)](h) = g*(T(V(h)))$ for all $g \in C_\infty$. By Lemma 1, this yields that

$$[\rho(V)](h) = T(V(h)) .$$

Now

$$\begin{aligned} \| T \circ V(h) \|_p &= \| [\rho(V)](h) \|_p = \lim_n \| [\rho(W_{f_n})](h) \|_p \\ &= \lim_n \| h * T(f_n) \|_p \leqq \| h \|_p \cdot \overline{\lim_n} \| T(f_n) \|_p^t \\ &\leqq \text{(since } \|h\|_p \leqq 1 \text{ and because of (8))} \\ &\qquad \| T \| \cdot \overline{\lim_n} \| f_n \|_p^t = \| T \| \cdot \overline{\lim_n} \| W_{f_n} \| = \| T \| \cdot \| V \| . \end{aligned}$$

PROPOSITION 2. *For each* $T \in \mathfrak{m}_p$, $V \in A_p$, *and* $f \in L_p^t$,

$$\| T(V(f)) \|_p \leqq \| T \| \cdot \| V(f) \|_p .$$

Proof. Let ε be any positive number. Since A_p is a Banach algebra with a minimal left approximate identity, Cohen's factorization theorem ([5] 32.26) implies that there exist P and S in A_p such that $\|P\| = 1$, $\|S - V\| < \varepsilon$, and $V = PS$. Thus, $\|S(f)\|_p \leqq \|V(f)\|_p + \varepsilon \cdot \|f\|_p$ and, by Lemma 2,

$$\begin{aligned} \| T(V(f)) \|_p &= \| T \circ P(S(f)) \|_p \\ &\leqq \|T\| \cdot \|P\| \cdot \|S(f)\|_p = \|T\| (\|V(f)\|_p + \varepsilon \|f\|_p) . \end{aligned}$$

It follows that $\| T(V(f)) \|_p \leqq \|T\| \cdot \|V(f)\|_p$.

LEMMA 3. *The set* $\{V(f): f \in L_p^t, V \in A_p\}$ *is a dense subspace of* L_p.

Proof. Let ε be a positive number and g be in L_p. Choose $f \in C_{00}$ such that $\|g - f\|_p < \varepsilon/2$. If $\{V_\alpha\}$ is a minimal left approximate identity for A_p, it follows from [6], Lemma 3, that $\lim_\alpha \|V_\alpha(f) - f\|_p = 0$. Thus, for some index α, $\|V_\alpha(f) - f\|_p < \varepsilon/2$ and so $\|V_\alpha(f) - g\|_p < \varepsilon$.

LEMMA 4. *Let* V *be in* $\mathfrak{B}_p$ *and* D *a dense subset of* L_p *such that* $V(h*f) = h*V(f)$ *for all* $h \in C_{00}$ *and* $f \in D$. *Then* V *is in* M_p.

Proof. Let x be in G. By [4] 20.15, there is a net $\{f_\alpha\}$ in C_{00} such that $\lim_\alpha \|{}_xh - f_\alpha * h\|_p = 0$ for all $h \in L_p$. It follows that $\lim_\alpha \|V({}_xh) - V(f_\alpha * h)\|_p = 0$ and $\lim_\alpha \|{}_xV(h) - f_\alpha * V(h)\|_p = 0$. Hence, for $h \in D$

$$\| V({}_xh) - {}_xV(h) \|_p = \lim_\alpha \| V(f_\alpha * h) - f_\alpha * V(h) \|_p = \lim_\alpha 0$$

by the hypothesis for V. Since D is dense in L_p, V is in M_p.

THEOREM 1. *Define* $\omega \mid M_p \to \mathfrak{m}_p$ *by letting* $\omega_T(f) = T(f)$ *for each* $T \in M_p$ *and* $f \in L_p^t$. *Then* ω *is an isometric isomorphism of* M_p *into* $\mathfrak{m}_p$. *Furthermore, if* T *is any operator in* $\mathfrak{m}_p$, *then there exists some* $S \in M_p$ *such that, for all* $V \in A_p$ *and* $f \subset L_p^t$, $\omega_S(V(f)) = T(V(f))$.

Proof. That ω is well-defined follows from Proposition 1. That ω is an isomorphism is evident when it is noted that L_p^t is a dense subset of L_p.

Let T be an arbitrary element of $\mathfrak{m}_p$. It follows from Proposition 2 and Lemma 3 that there exists a unique operator S in $\mathfrak{B}_p$ such that $S(V(f)) = T(V(f))$ for all $V \in A_p$ and $f \in L_p^t$. For $h \in C_{00}$, $V \in A_p$, and $f \in L_p^t$, Proposition 1 implies

$$S(h * V(f)) = S(V(h*f)) = T(V(h*f)) \\ = T(h * V(f)) = h * T(V(f)) = h * S(V(f)) \,.$$

By Lemmas 3 and 4, this implies that S is in M_p. Consequently, $\omega_S(V(h)) = S(V(h)) = T(V(h))$ for all $h \in L_p^t$ and $V \in A_p$.

To complete this proof, it will now suffice to show that ω is an isometry. Let T be in M_p. Let f be in L_p^t and ε a positive number. Choose $g \in L_p^t$ for which $\|g\|_p \leqq 1$ and $\|\omega_T(f)\|_p^t < \|g*\omega_T(f)\|_p + \varepsilon$. By Proposition 1.iii, $T(g*f) = g*T(f)$; this means that $T \circ W_f(g) = g*\omega_T(f)$. Hence,

$$\|\omega_T(f)\|_p^t < \|T \circ W_f(g)\| + \varepsilon \leqq \|T\| \cdot \|W_f\| + \varepsilon \,.$$

By (8), this implies $\|\omega_T(f)\|_p^t \leqq \|T\| \cdot \|f\|_p^t$. Hence

$$\|\omega_T\| \leqq \|T\| \,.$$

On the other hand, Proposition 2 and Lemma 3 imply

$$\|T\| = \sup \{\|T(V(h))\|_p \colon V \in A_p,\, h \in L_p^t,\, \|V(h)\|_p \leqq 1\} \\ = \sup \{\|\omega_T(V(h))\|_p \colon V \in A_p,\, h \in L_p^t,\, \|V(h)\|_p \leqq 1\} \leqq \|\omega_T\| \,.$$

This proves that $\|T\| = \|\omega_T\|$.

THEOREM 2. *Let ω be as in Theorem 1 and G be either commutative or compact. Then ω is surjective.*

Proof. Let T be any operator in $\mathfrak{m}_p$. By Theorem 1, there is an operator S in M_p for which $T(V(f)) = \omega_S(V(f))$ for all $V \in A_p$ and $f \in L_p^t$.

If G is compact, then $L_p^t = L_p$. It follows from the Hewitt-Curtis-Figa Talamanca factorization theorem ([5] 32.22) that each $h \in L_p^t$ is of the form $V(f)$ for some $V \in A_p$ and $f \in L_p^t$. Hence, $T = \omega_S$.

Suppose now that G is commutative (not necessarily compact). Assume that there existed $h \in L_p^t$ such that $\omega_S(h) \neq T(h)$. Then Lemma 1 implies that $g*(\omega_S - T)(h) \neq 0$ for some $g \in C_{00}$. Let $\{h_n\}$ be a sequence in C_{00} for which $\lim_n \|h_n - h\|_p = 0$. Then

$$\begin{aligned} &\|g*(\omega_S - T)(h)\|_p \\ &= \|(\omega_S - T)(g*h)\|_p = \|(\omega_S - T)(h*g)\|_p \\ &= \|h*(\omega_S - T)(g)\|_p = \lim_n \|h_n*(\omega_S - T)(g)\|_p \\ &= \|\lim_n \|(\omega_S - T)(h_n*g)\|_p = \lim_n \|(\omega_S - T)(W_{h_n}(g))\|_p = 0 \end{aligned}$$

a contradiction. Thus, $\omega_S = T$.

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