

## DIVISORIAL COMPLETE INTERSECTIONS

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A complete intersection in a commutative ring  $R$  with identity is a regular element  $f$  of  $R$  such that  $g \in R$  and  $g/f \in IR$  (the integral closure of  $R$  in its total quotient ring) imply that  $g/f \in R$ . It is assumed that  $R$  is nonimbedded and that  $IR$  is a noetherian  $R$ -module, and it is proven that the set of complete intersections in  $R$  is the set of regular elements of  $R$  not contained in any of a certain finite set of prime ideals of  $R$ , the nonnormal divisorial prime ideals of  $R$  together with the prime ideals which occur as an imbedded prime ideal of a proper principal ideal of  $R$ . This finite set of prime ideals contains the associated prime ideals of the conductor of  $IR$  in  $R$ , but it is shown that this is not always an equality.

In §§1-5 are developed properties of zero divisors relative to normalization and extensions of well known properties of integral domains in order to obtain the description of the set of complete intersections in a ring with zero divisors. The associated prime ideals of a proper principal ideal are then described, the set of complete intersections is shown to be closed, and some examples are given. *Ring* will mean throughout a non-null commutative ring containing an identity.

1. Valuations on a commutative ring. Let  $R$  be a commutative ring with identity, and let  $G$  be an ordered abelian group written additively. A valuation of  $R$  into  $G$  is a function  $v: R \rightarrow G \cup \{\infty\}$  for which  $v(\alpha\beta) = v(\alpha) + v(\beta)$ ,  $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$ ,  $v(0) = \infty$ , and if  $v(\alpha) = \infty$  then  $\alpha$  is a zero divisor in  $R$  for all  $\alpha, \beta \in R$  (where  $\infty + a = \infty$ ,  $\infty + \infty = \infty$ ,  $\infty > a$  for all  $a \in G$ ). With proof as usual,  $v(1) = 0$ ,  $v(-\alpha) = v(\alpha)$ , and if  $v(\alpha) < v(\beta)$  then  $v(\alpha + \beta) = v(\alpha)$ .

Let  $M$  be a multiplicative subset of  $R$ , let  $\psi: R \rightarrow M^{-1}R$  be the canonical homomorphism, and let  $w$  be a valuation of  $M^{-1}R$ . Then  $w\psi$  is a valuation of  $R$ . Conversely, let  $v$  be a valuation of  $R$  with  $v(M) \subset G$ , and define  $w(\psi\alpha/\psi\beta) = v(\alpha) - v(\beta)$  for all  $\alpha \in R$ ,  $\beta \in M$ . Then  $w$  is a valuation of  $M^{-1}R$ . This defines a bijective correspondence between the valuations of  $M^{-1}R$  and the valuations  $v$  of  $R$  for which  $v(M) \subset G$ . Let  $QR$  denote the total quotient ring of  $R$ , and let  $IR$  denote the integral closure of  $R$  in  $QR$ . Thus there is a natural bijective correspondence between the valuations of  $R$ , those of  $QR$ , and those of  $IR$ .

A valuation  $v$  of  $R$  is finite on  $R$  if  $v(\alpha) \geq 0$  for all  $\alpha \in R$ . If  $v$  is finite on  $R$ , then the extension of  $v$  to  $IR$  is finite on  $IR$ . If  $v$  is

finite on  $R$ , define the *center of  $v$  in  $R$*  to be the prime ideal  $\{\alpha \in R \mid v(\alpha) > 0\}$ .

Let  $v$  be a valuation of  $R$ . Let  $N = \{\alpha \in R \mid v(\alpha) = \infty\}$ , a prime ideal of  $R$  containing only zero divisors. Let  $\phi: R \rightarrow R/N$  and  $\psi: R/N \rightarrow Q(R/N)$  be the canonical homomorphisms, and let  $K_v$  denote  $Q(R/N)$ .  $v$  extends uniquely to a valuation of  $K_v$ ,  $\psi\phi\alpha/\psi\phi\beta \rightarrow v(\alpha + N) - v(\beta + N)$ , also denoted by  $v$  (where  $v\psi\phi$  is the valuation  $v$  of  $R$ ). The valuation ring of  $v$ ,  $R_v$  with maximal ideal  $\mathfrak{m}_v$  is defined to be the valuation ring of  $v$  in  $K_v$ . Let  $N_v = \{\alpha \in QR \mid v(\alpha) = \infty\}$ .

**2. Zero prime ideals.** Let  $R$  be a commutative ring with identity. A *zero prime ideal* of  $R$  is a prime ideal of  $R$  which contains only zero divisors of  $R$ . The zero prime ideals of  $R$ , of  $QR$ , and of  $IR$  are in a bijective correspondence. If  $N$  is a zero prime ideal of  $R$ , let  $QN = (QR)N$ , and let  $IN = QN \cap IR$ . A *maximal zero prime ideal* of  $R$  is a maximal element in the set of all zero prime ideals of  $R$ . Every zero divisor of  $R$  is contained in a zero prime ideal of  $R$ .  $R$  is *nonimbedded* if every zero prime ideal of  $R$  is minimal. A minimal prime ideal is a zero prime ideal ([1]). If  $R$  is nonimbedded then  $I(R_{\mathfrak{P}}) = R_{\mathfrak{P}}(IR)$  for every prime ideal  $\mathfrak{P}$  of  $R$  ([1], Cor. 2).

**PROPOSITION 1.** *Let  $N$  be a zero prime ideal of  $R$ . If the injection  $j: QR/QN \rightarrow Q(R/N)$  ( $j: (\alpha/\beta + QN) \rightarrow (\alpha + N)/(\beta + N)$ ) is surjective, then  $N$  is a maximal zero prime ideal of  $R$ .*

**PROPOSITION 2.** *Let  $R$  have only a finite number of maximal zero prime ideals, and let  $N$  be one. The injection  $j: QR/QN \rightarrow Q(R/N)$  is surjective, and  $j$  induces an isomorphism  $IR/IN \simeq I(R/N)$ .*

*Proof.* Let  $\alpha + N \in R/N$  be regular. Let  $N_1 = N, N_2, \dots, N_t$  be the maximal zero prime ideals of  $R$ , where  $\alpha \notin N_1 \cup \dots \cup N_s$  and  $\alpha \in N_{s+1} \cap \dots \cap N_t$ . Let  $\beta \in N_1 \cap \dots \cap N_s$  and  $\beta \notin N_{s+1} \cup \dots \cup N_t$ .  $\alpha + \beta$  is regular in  $R$ ,  $\alpha + \beta \in \alpha + N$ , and  $j$  is surjective.  $j(IR/IN) \subset I(R/N)$ . If  $(\alpha + N)/(\beta + N) \in I(R/N)$ , then the representative  $\beta$  can be chosen to be regular in  $R$ , and for some  $m \in N, a_i \in R$

$$\left(\frac{\alpha}{\beta}\right)^n (1 + m) + \left(\frac{\alpha}{\beta}\right)^{n-1} a_{n-1} + \dots + a_0 = 0.$$

Thus  $\alpha(1 + m)/\beta \in IR$  and  $j: IR/IN \rightarrow I(R/N)$  is surjective.

**PROPOSITION 3.** *Let  $R$  be a nonimbedded ring with a finite number,  $n$ , of zero prime ideals. Then  $IR \simeq R_1 \oplus \dots \oplus R_n$  where each ring  $R_i$  has a unique zero prime ideal and  $R_i$  is the integral closure*

of  $R/A_i$  for  $A_i = (R_1 + \dots + R_{i-1} + R_{i+1} + \dots + R_n) \cap R$ .

*Proof.* Let  $N_1, \dots, N_n$  be the zero prime ideals of  $R$ . Let  $\alpha_i \in \bigcap_{j \neq i} N_j$  and  $\alpha_i \notin N_i$  for  $i = 1, \dots, n$ , with  $\alpha_i \alpha_j = 0$  for  $i \neq j$ .  $\alpha = \alpha_1 + \dots + \alpha_n$  is a regular element of  $R$ . Let  $\beta_i = \alpha_i/\alpha$ . Now  $\beta_i^2 = \beta_i$  so  $\beta_i \in IR$ ,  $\beta_i \beta_j = 0$  for  $i \neq j$ , and  $\beta_1 + \dots + \beta_n = 1$ . Let  $R_i = (IR)\beta_i$ . If  $R_i$  had more than one zero prime ideal,  $R$  would have at least  $n + 1$  zero prime ideals.  $R_1 + \dots + R_{i-1} + R_{i+1} + \dots + R_n \subset IN_i$ , so  $A_i \subset N_i$ .  $QR \simeq QR_1 \oplus \dots \oplus QR_n$ .  $A_i = R \cap (QR_1 + \dots + QR_{i-1} + QR_{i+1} + \dots + QR_n)$  a contracted ideal, thus  $IA_i = (QR \cdot A) \cap IR = R_1 + \dots + R_{i-1} + R_{i+1} + \dots + R_n$ , and  $A_i \not\subset N_j$  for  $i \neq j$ . Now, by a proof entirely similar to the proof of Proposition 2, the injection  $Q(R/A_i) \rightarrow QR/QR \cdot A_i$  is surjective and induces an isomorphism  $I(R/A_i) \simeq IR/IA_i \simeq R_i$ .

Let  $\mathfrak{N}$  be the nil radical of  $R$ , let  $Q\mathfrak{N}$  be the nil radical of  $QR$ , and note that  $Q\mathfrak{N} \subset IR$ .

**PROPOSITION 4.** *Let  $R$  be a noetherian, nonimbedded ring. Then the injection  $i: QR/Q\mathfrak{N} \rightarrow Q(R/\mathfrak{N})$  ( $i: (\alpha/\beta + Q\mathfrak{N}) \rightarrow (\alpha + \mathfrak{N})/(\beta + \mathfrak{N})$ ) is surjective, and  $i$  induces an isomorphism  $IR/I\mathfrak{N} \simeq I(R/\mathfrak{N})$ .*

The *Proof* is similar to the proof of Proposition 2.

**3. Prime divisors.** Let  $v$  be a discrete rank one valuation of  $R$ , finite on  $R$ . Let  $\mathfrak{P}$  be the center of  $v$  in  $R$ , and let  $N = N_v \cap R$ .  $v$  is of the first kind on  $R$  if  $N$  is a minimal prime ideal of  $R$  and  $\mathfrak{P}/N$  is of height one in  $R/N$ . Such a discrete rank one valuation, finite on  $R$  of the first kind will be called a *prime divisor* of  $R$ .

**PROPOSITION 5.** *Let  $R$  be nonimbedded noetherian ring such that  $IR/Q\mathfrak{N}$  is a noetherian  $R$  module. If  $\alpha \in QR$  and if  $v(\alpha) \geq 0$  for every prime divisor  $v$  of  $R$ , then  $\alpha \in IR$ .*

*Proof.*  $I(R/\mathfrak{N}) \simeq IR/Q\mathfrak{N}$  (Prop. 4) is a noetherian  $R$  module. Let  $N_1, \dots, N_t$  be the zero prime ideals of  $R$ .  $I(R/\mathfrak{N}) \simeq \bigoplus_{i=1}^t I(R/\mathfrak{N})/I(N_i/\mathfrak{N})$  and  $Q(R/\mathfrak{N}) = \bigoplus_{i=1}^t Q(R/\mathfrak{N})/Q(N_i/\mathfrak{N})$ . Hence by Proposition 2 with the fact that the Proposition in question is valid for noetherian entire rings,  $\alpha + N_i \in I(R/N_i)$ , and  $\alpha + \mathfrak{N} \in Q(R/\mathfrak{N})$  is contained in  $I(R/\mathfrak{N})$ . By Proposition 4 the injection is surjective, and  $\alpha \in IR + Q\mathfrak{N} = IR$ .

Let  $R$  be a noetherian ring, let  $\alpha \in R$  be regular, and let  $\mathfrak{P}$  be an isolated prime ideal of  $(\alpha) = R\alpha$ . Let  $N$  be a minimal prime ideal contained in  $\mathfrak{P}$ . Then  $\mathfrak{P}/N$  is an isolated prime ideal of  $(\alpha + N)$ , and  $\mathfrak{P}/N$  is of height one in  $R/N$  ([4] vol. I, Th. 29, p. 238). The valua-

tions of  $R$  centered at  $\mathfrak{P}$  consist of a finite number of prime divisors of  $R$ . A prime ideal  $\mathfrak{P}$  of  $R$  is of *height one* if for each zero prime ideal  $N \subset \mathfrak{P}$ ,  $\mathfrak{P}/N$  is of height one in  $R/N$ .

**PROPOSITION 6 (THE PRINCIPAL IDEAL THEOREM).** *Let  $\alpha$  be a regular element of a noetherian ring  $R$ . The isolated prime ideals of  $(\alpha)$  are of height one. Conversely, if  $\mathfrak{P}$  is a prime ideal of height one of  $R$ , there is a regular element  $\alpha$  of  $R$  such that  $\mathfrak{P}$  is an isolated prime ideal of  $(\alpha)$ .*

A *divisorial prime ideal* of  $R$  is the center of some prime divisor of  $R$ . If  $\mathfrak{A}$  is an ideal of  $R$  and if  $\mathfrak{P}$  is a divisorial prime ideal which is also an associated prime ideal of  $\mathfrak{A}$ , call  $\mathfrak{P}$  a *divisorial prime ideal* of  $\mathfrak{A}$ . A prime ideal  $\mathfrak{P}$  of  $R$  is *normal* if  $R_{\mathfrak{P}}$  is normal. (A ring  $R$  is *normal* if  $R = IR$ .)

**PROPOSITION 7.** *Let  $R$  be a nonimbedded noetherian ring. If  $\mathfrak{P}$  is a normal prime ideal of  $R$ , then  $\mathfrak{P}$  contains a unique zero prime ideal of  $R$ .*

*Proof.* Let  $N_1, \dots, N_t$  be the distinct zero prime ideals of  $R_{\mathfrak{P}}$ .  $R_{\mathfrak{P}} = R_1 \oplus \dots \oplus R_t$  (Prop. 3),  $R_{\mathfrak{P}}$  is local, and therefore  $t = 1$ .

Note that a normal divisorial prime ideal of  $R$  is of height one in  $R$ .

**4. Factorial rings.** Let  $R$  be a commutative ring with identity. A regular element  $\alpha$  of  $R$  is *irreducible* if  $\alpha = \beta\gamma$  implies either  $\beta$  or  $\gamma$  is a unit of  $R$ .  $R$  is *factorial* if every regular element of  $R$  is a product of irreducible elements of  $R$  and if  $(\alpha)$  is a prime ideal of  $R$  for every irreducible element  $\alpha$  of  $R$ .

**PROPOSITION 8.** *If  $R$  is factorial then  $R$  is normal.*

The *Proof* is as usual.

**PROPOSITION 9.** *Let  $R = R_1 \oplus R_2$ ,  $R$  is factorial if and only if  $R_1$  and  $R_2$  are factorial.*

The *Proof* is an application of the definition.

**PROPOSITION 10.** *Let  $R$  have a unique zero prime ideal  $\mathfrak{N}$ .  $R$  is factorial if and only if  $R/\mathfrak{N}$  is factorial and  $\mathfrak{N} = Q\mathfrak{N}$ .*

**PROPOSITION 11.** *Let  $R$  be a noetherian ring.  $R$  is factorial if and only if every prime ideal of height one of  $R$  is principal.*

The *Proof* is the same as for integral domains, using the principal ideal theorem (Prop. 6).

5. Finiteness conditions on  $IR$  as an  $R$  module. For an overring  $S$  of  $R$  let  $\mathfrak{C}(R, S)$  be  $R: {}_R\bar{R}$ , the conductor of  $R$  in  $S$ , where  $\bar{R}$  is the integral closure of  $R$  in  $S$ . Let  $\mathfrak{N}$  be the nil radical of  $R$ .

PROPOSITION 12. *Let  $R$  be a finitely generated ring over a field.  $IR$  is a noetherian  $R$  module if and only if  $\mathfrak{N} = Q\mathfrak{N}$ .*

*Proof.* ( $\Rightarrow$ ): Let  $d \in \mathfrak{C}(R, IR)$  be regular, let  $\alpha \in \mathfrak{N}$ , and let  $\beta \in R$  be regular.  $\alpha/\beta = d(\alpha/d\beta) \in R$ . ( $\Leftarrow$ ): Let  $N_1, \dots, N_t$  be the minimal prime ideals of  $R$ .  $I(R/N_i)$  is a noetherian  $R$ , or  $R/N_i$ , module ([4] vol. I., Th. 9, p. 267).  $I(R/\mathfrak{N}) \simeq \bigoplus_{i=1}^t I(R/\mathfrak{N})/I(N_i/\mathfrak{N})$  (Prop. 3),  $I(R/\mathfrak{N})/I(N_i/\mathfrak{N}) \simeq I(R/N_i)$  (Prop. 2), and  $I(R/\mathfrak{N})$  is a noetherian  $R$  module.  $i: IR/I\mathfrak{N} \rightarrow I(R/\mathfrak{N})$  is injective, so  $IR$  is a noetherian  $R$  module.

PROPOSITION 13. *Let  $R$  be a noetherian ring which has a unique zero prime ideal. If  $IR$  is a noetherian  $R$  module then  $R$  is either reduced or artinian.*

*Proof.* First assume  $R = IR$ . Suppose  $n \in R$  is nilpotent and nonzero. Let  $\alpha \in R$  be regular. For some integer  $m$  and some  $\beta \in R$ ,  $\beta(n/\alpha^m) = n/\alpha^{m+1}$ . Thus  $n = \alpha\beta n$ ,  $1 - \alpha\beta$  is nilpotent,  $\alpha$  is a unit of  $R$ , and  $R$  is artinian. For the general case,  $IR$  is either reduced or artinian. If  $IR$  is reduced, so is  $R$ . If  $IR$  is artinian then  $IR = QR$ , by hypothesis  $\mathfrak{C}(R, IR)$  contains a regular element, and thus  $R = IR$  is artinian.

PROPOSITION 14. *Let  $R$  be a finitely generated ring over a field, and let  $R$  have a unique zero prime ideal.  $IR$  is a noetherian  $R$  module if and only if  $R$  is either reduced or artinian.*

EXAMPLE.  $R = k[x, y]_{(x, y)}/(x(x^2 - y^3), y(x^2 - y^3))$  is imbedded at  $(x, y)$ , is normal, and is neither reduced nor artinian.  $R$  is factorial, but  $R/\mathfrak{N} = k[x, y]/(x^2 - y^3)$  is not factorial.

6. Imbedded prime ideals of proper principal ideals. Let  $R$  be a noetherian ring. A *proper* principal ideal is a principal ideal of  $R$  generated by a regular nonunit of  $R$ . An *imbedded prime ideal* of an ideal  $\mathfrak{A}$  is an associated prime ideal of  $\mathfrak{A}$  which is not isolated.

THEOREM 1. *Let  $\mathcal{S}$  be the set of all prime ideals of  $R$  which occur as an imbedded prime ideal of a proper principal ideal. Let*

$\alpha$  be a regular nonunit of  $R$ . The following are equivalent

- (a)  $\mathfrak{P}$  is an imbedded prime ideal of  $(\alpha)$ .
- (b)  $\mathfrak{P} \in \mathcal{D}$  and  $\alpha \in \mathfrak{P}$ .

*Proof.* See [4] vol. II, Appendix 6 for the definitions of grade and prime sequence. Suppose that  $\mathfrak{P} \in \mathcal{D}$  and  $\alpha \in \mathfrak{P}$ . There is a regular nonunit  $\beta$  of  $R$  such that  $\mathfrak{P}$  is an imbedded prime ideal of  $(\beta)$ . Hence  $\text{grade}(R_{\mathfrak{P}}) = 1$ . If  $\mathfrak{P}$  were not an imbedded prime ideal of  $(\alpha)$ ,  $R_{\mathfrak{P}}/R_{\mathfrak{P}} \cdot \alpha$  would contain a regular nonunit, and  $\text{grade}(R_{\mathfrak{P}})$  would be greater than one.

7. Complete intersections in a ring. Let  $R$  be a commutative ring with identity. A regular element  $f$  of  $R$  is a *complete intersection in  $R$*  if and only if  $IR \cdot f \cap R = Rf$ . Equivalently if  $g \in R$  and  $g/f \in IR$  then  $g/f \in R$ . For a noetherian ring  $R$ , let  $\mathcal{A}\mathfrak{A}$  denote the set of associated prime ideals of an ideal  $\mathfrak{A}$  excluding the prime ideal  $R$  (so that  $\mathcal{A}R = \emptyset$ ).

**THEOREM 2.** *Let  $R$  be a nonimbedded ring such that  $IR$  is a noetherian  $R$  module. (Then  $R$  is noetherian. Let  $\mathcal{C} = \bigcap_d \mathcal{A}(d)$  where the intersection is over all regular elements  $d$  of  $\mathfrak{C}(R, IR)$ .)*

(a)  $\mathcal{C} = \mathcal{D} \cup \mathcal{A}\mathfrak{C}(R, IR)$ . Thus  $\mathcal{D}$  is a finite set. (The complement of  $\mathcal{D}$  in  $\mathcal{C}$  consists of the prime ideals in  $\mathcal{C}$  of height one, and  $\mathcal{C}$  contains the nonnormal divisorial prime ideals of  $R$ .)

(b) Let  $f$  be a regular element of  $R$ . The following are equivalent.

1.  $f$  is a complete intersection in  $R$ .
2.  $f$  is contained in no element of  $\mathcal{C}$ .
3. The divisorial prime ideals of  $(f)$  are normal, thus isolated, and  $(f)$  is unmixed.

The *Proof* follows. Let  $\mathfrak{C}$  denote  $\mathfrak{C}(R, IR)$ .  $IR_{\mathfrak{P}} = R_{\mathfrak{P}} \cdot IR$  ([1]).

**LEMMA 1.** *Let  $d \in \mathfrak{C}$  be regular, and let  $f \in R$  be regular. If  $f$  is contained in no element of  $\mathcal{A}(d)$ ,  $f$  is a complete intersection in  $R$ .*

*Proof.* Let  $g \in R$  with  $g/f \in IR$ . Then  $d(g/f) = c \in R$ , so  $c \in (d): (f) = (d)$ , and  $g/f = c/d \in R$ .

**LEMMA 2.** *If  $f$  is a complete intersection in  $R$ , then  $f$  is contained in no element of  $\mathcal{A}\mathfrak{C}$ .*

*Proof.* Let  $c \in \mathfrak{C}: (f)$ . Then  $cf(IR) \subset R$ , so  $c \in \mathfrak{C}$ , and  $\mathfrak{C}: (f) = \mathfrak{C}$ .

LEMMA 3. *If  $d \in \mathcal{C}$  is regular, then  $\mathcal{A}\mathcal{C} \subset \mathcal{A}(d)$ .*

*Proof.* Let  $\mathfrak{P}$  be a prime ideal of  $R$ ,  $\mathfrak{P} \in \mathcal{A}(d)$ . If  $\mathfrak{P}$  is a zero prime ideal, then  $\mathfrak{P} \in \mathcal{A}\mathcal{C}$ . Let  $f \in R_{\mathfrak{P}}\mathfrak{P}$  be regular with  $R_{\mathfrak{P}}d: R_{\mathfrak{P}}f = R_{\mathfrak{P}}d$ .  $d \in R_{\mathfrak{P}}\mathcal{C} = \mathcal{C}(R_{\mathfrak{P}}, R_{\mathfrak{P}}IR) = \mathcal{C}(R_{\mathfrak{P}}, IR_{\mathfrak{P}})$ .  $f$  is a complete intersection in  $R_{\mathfrak{P}}$  (Lemma 1), by Lemma 2  $R_{\mathfrak{P}}\mathfrak{P} \in \mathcal{A}\mathcal{C}(R_{\mathfrak{P}}, IR_{\mathfrak{P}})$ , and  $\mathfrak{P} \in \mathcal{A}\mathcal{C}$ .

LEMMA 4. *Let  $f$  be a complete intersection in  $R$ , and let  $g$  be a regular element of  $R$  with  $f$  contained in no divisorial prime ideal of  $(g)$ . Then  $f$  is contained in no element of  $\mathcal{A}(g)$ .*

*Proof.* Let  $c \in (g): (f)$ . For some  $h \in R$ ,  $cf = gh$ , and  $v(f) \leq v(h)$  for every prime divisor  $v$  of  $R$ . Then  $h/f \in IR$  (Prop. 5),  $h/f \in R$ , and  $c \in (g)$ .

LEMMA 5. *If  $\mathfrak{P}$  is a nondivisorial associated prime ideal of a principal ideal generated by a regular element, then  $\mathfrak{P} \in \mathcal{C}$ .*

*Proof.* Let  $\mathfrak{P}$  be a prime ideal containing a regular element with  $\mathfrak{P} \in \mathcal{C}$ , and let  $g$  be a regular element of  $R$ . By Lemma 1 there is a complete intersection  $f$  in  $R_{\mathfrak{P}}\mathfrak{P}$  contained in no divisorial prime ideal of  $R_{\mathfrak{P}}g$ . Apply Lemma 4.

LEMMA 6. *Let  $\mathfrak{P}$  be a divisorial prime ideal of  $R$ . If  $\mathfrak{P}$  contains a complete intersection of  $R$ , then  $\mathfrak{P}$  contains a unique zero prime ideal (and thus  $\mathfrak{P}$  is of height one in  $R$ ).*

*Proof.* Let  $f \in \mathfrak{P}$  be a complete intersection in  $R$ , let  $N$  be a minimal prime ideal of  $R$  contained in  $\mathfrak{P}$  with  $\mathfrak{P}/N$  of height one in  $R/N$ , and let  $v_1, \dots, v_s$  be the distinct prime divisors of  $R$  centered at  $\mathfrak{P}$  with  $v_i(N) = \{\infty\}$ . If there is a zero prime ideal distinct from  $N$  contained in  $\mathfrak{P}$ , it follows that there is a minimal prime ideal distinct from  $N$  contained in  $\mathfrak{P}$ . Let  $N_1 = N, N_2, \dots, N_t$  be the distinct minimal prime ideals of  $R$  which are contained in  $\mathfrak{P}$ . Let  $g \in N_2 \cap \dots \cap N_t$  and  $g \notin N_1$ . Let  $m_i = v_i(g) > 0$  and let  $n_i = v_i(f)$  for  $i = 1, \dots, s$ . By raising  $g$  to a power  $m_i \geq n_i$  for  $i = 1, \dots, s$  can be assumed. Assume  $m_1/n_1 \geq m_i/n_i$  for  $i = 2, \dots, s$ , and let  $m = m_1$ , and  $n = n_1$ . A factor may be introduced into  $g$  without destroying the above so that for all prime divisors  $w$  of  $R$  with  $N_w \cap R \neq N_1, \dots, N_t$ ,  $w(g) \geq mw(f)$ . Now for all prime divisors  $w$  of  $R$ ,  $w(f^m) \leq w(g^n)$ , so  $g^n/f^m \in IR$  (Prop. 5), and  $g^n/f^m \in R$ .  $v_1(g^n/f^m) = 0$ . However, if  $w$  is a prime divisor of  $R$  with  $N_w \cap R = N_2$ , then  $w(g^n/f^m) = \infty$ , so  $g^n/f^m \in N_2 \subset \mathfrak{P}$  which is a contradiction.

LEMMA 7. *Let  $d \in \mathcal{C}$  be regular. If a divisorial prime ideal  $\mathfrak{P}$  is not an associated prime ideal of  $(d)$ , then  $\mathfrak{P}$  is normal.*

*Proof.*  $d \in \mathcal{C}(R_{\mathfrak{P}}, IR_{\mathfrak{P}})$ , so in  $R_{\mathfrak{P}}$  there is a complete intersection  $f \in R_{\mathfrak{P}}\mathfrak{P}$ .  $\mathfrak{P}$  is of height one  $R$  (Lemma 6), and  $\mathcal{C} \not\subset \mathfrak{P}$ .

*Proof of a.*  $\mathcal{A}\mathcal{C} \subset \mathcal{C}$  by Lemma 3, and the non-divisorial prime ideals in  $\mathcal{D}$  are in  $\mathcal{C}$  by Lemma 5. A divisorial prime ideal in  $\mathcal{D}$  is nonnormal by Prop. 7, and by Lemma 7 it is in  $\mathcal{C}$ . Thus  $\mathcal{D} \cup \mathcal{A}\mathcal{C} \subset \mathcal{C}$ .  $\mathcal{D} \cup \mathcal{A}\mathcal{C} \supset \mathcal{C}$  for an isolated prime ideal of a principal ideal is of height one, and the height one prime ideals in  $\mathcal{C}$  are in  $\mathcal{A}\mathcal{C}$ . The last assertion is Lemma 7.

*Proof of b.* (1  $\Rightarrow$  2): By Lemmas 6 and 2 there is a regular element  $d$  of  $\mathcal{C}$  which is contained in no divisorial associated prime ideal of  $(f)$ . Apply Lemma 4. (2  $\Rightarrow$  3): If  $\mathfrak{P}$  is a divisorial associated prime ideal of  $(f)$   $\mathfrak{P}$  is normal (Lemma 7).  $(f)$  is unmixed (Lemma 5). (3  $\Rightarrow$  1):  $(f): \mathcal{C} = (f)$ . If  $g \in R$  and  $g/f \in IR$ , then  $g\mathcal{C} \subset (f)$ , and  $g/f \in R$ .

PROPOSITION 15. *Let  $R$  be a commutative nonnull ring with identity, and let  $f$  and  $g$  be complete intersections in  $R$ . Then  $fg$  is a complete intersection in  $R$ , and if  $f/g \in R$  it is a complete intersection in  $R$ . Let  $f \in R$  be such that  $f^n$  is a complete intersection in  $R$  for some integer  $n \geq 1$ . Then  $f$  is a complete intersection in  $R$ .*

*Proof.* Let  $h \in R$ . If  $h/gf \in IR$  then  $h/g \in IR$ ,  $h/g \in R$ , and  $h/gf \in R$ . If  $g/f \in R$  and  $hf/g \in IR$ , then  $hf/g \in R$ . If  $h/f \in IR$ , then  $hf^{n-1}/f^n \in IR$  and  $h/f \in R$ .

Let  $R$  be a nonimbedded noetherian ring, and let  $D$  be the divisor group of  $R$ , the free abelian group generated by the prime divisors of  $R$  (§3). If  $f$  is a regular element of  $QR$ , let  $(f) = \sum_v v(f) \cdot v$  where the sum is over all prime divisors  $v$  of  $R$ . Let  $P = \{(f) \mid f \in QR, f \text{ regular}\}$ , the subgroup of  $D$  of principal divisors, and let  $I$  be the subgroup of  $P$  generated by  $\{(f) \mid f \text{ is a complete intersection in } R\}$ .  $I = P$  if and only if  $R = IR$ , and thus  $I = D$  if and only if  $R$  is factorial (Prop. 11).  $P/I$  is torsion free (Prop. 15).

Let  $\mathfrak{A} \in D$  be a complete intersection, that is, let  $\mathfrak{A} \geq 0$  and  $\mathfrak{A} \in I$ . Then (Prop. 15) there is a complete intersection  $f$  in  $R$  with  $\mathfrak{A} = (f)$ . If  $h \in R$ ,  $(h) \in I$  if and only if  $h$  is a complete intersection in  $R$ .

8. Examples. 1. Let  $R$  be a reduced finitely generated ring over a field, of height one. The complete intersections in  $R$  are those regular elements of  $R$  which do not vanish at the nonnormal (singular)



primes of  $R$ . E.g., for the plane curve  $y^2 - x^3$ ,  $P/I = Z$ , the integers, and for  $y^2 - x^3 - x^2$ ,  $P/I = Z \oplus Z$ .

2. Let  $R$  be the coordinate ring of a reduced complete intersection in affine space. Then  $\mathcal{C}$  contains only prime ideals of height one of  $R$ . For  $R$  is Cohen-Macaulay, and in particular every proper principal ideal is unmixed. E.g., the surfaces  $y^2 - x^3$ ,  $y^2 - x^3 - zx^2$  in affine 3-space.

3. Let  $R = k[t^4, t^3u, tu^3, u^4] \subset k[t, u]$  the polynomial ring.  $IR = k[t^4, t^3u, t^2u^2, tu^3, u^4]$ . The two affine localizations  $k[1, u, u^3, u^4]$  and  $k[t^4, t^3, t, 1]$  of  $R$  are integrally closed. Hence  $\mathcal{C} = \{(t^4, t^3u, tu^3, u^4)\}$ .

4. Let  $R = k[t^6, t^4u^2, t^3u^3, tu^5, u^6] \subset k[t, u]$ .

$$IR = k[t^6, t^5u, t^4u^2, t^3u^3, t^2u^4, tu^5, u^6],$$

and

$$\mathfrak{C} = (t^4u^2, t^3u^3, tu^5, u^6) = \mathfrak{P}$$

a prime ideal.  $t^6 \notin \mathfrak{P}$ ,  $(t^4u^2)^2/t^6 \in IR$ ,  $\notin R$ , so  $(t^6)$  is imbedded at  $\mathfrak{M} = (t^6, \mathfrak{P})$ , and  $\mathcal{C} = \{\mathfrak{P}, \mathfrak{M}\}$ . Thus  $\mathcal{A}\mathfrak{C}$  may be strictly less than  $\mathcal{C}$ .

5. Let  $R = k[x, y, z]/(zx, zy)$ .  $(x, y, z)$  is the only nonnormal prime ideal of  $R$ , and it is divisorial.

6. Let  $R$  be a reduced, finitely generated ring over a field  $k$ . Then there exist  $f_1, \dots, f_s \in R$  such that  $(f_1, \dots, f_s)$  is of codimension  $s$  in  $R$ , is unmixed and self radical. Induct on  $s$ . For  $s = 1$ , choose  $f_1$  such that the locus of  $f_1$  is  $k$ -transversal to  $\text{Spec } R$ , and such that  $f_1$  is contained in no element of  $\mathcal{C}$ . The step from  $s - 1$  to  $s$  is the same as for  $s = 1$  on the ring  $R/(f_1, \dots, f_{s-1})$ . By affine localization, it follows that a projective reduced algebraic variety possesses a complete intersection of any codimension (which is locally a complete intersection of codimension  $s$  given globally by  $s$  forms contained in the homogeneous coordinate ring of the variety).

7. Let  $R$  be a nonimbedded ring such that  $IR$  is a noetherian  $R$  module, let  $\mathfrak{P} \in \mathcal{C}$ , and suppose that the order function  $w$  of  $R_{\mathfrak{P}}$  ([2], vol. II, p. 249) is a valuation. Let  $\mathfrak{m} = R_{\mathfrak{P}}\mathfrak{P}$ . Let  $f \in \mathfrak{m}$ , with  $f$  contained in no other element of  $\mathcal{C}(R_{\mathfrak{P}})$ . Let  $\mathfrak{q}$  be a primary component of  $(f)$  associated to  $\mathfrak{m}$ . For some integer  $n \geq 0$ ,  $\mathfrak{m}^{n+w(f)} \subset \mathfrak{q}$ . Thus if  $g \in R_{\mathfrak{P}}$ , if  $g/f \in IR_{\mathfrak{P}}$  and if  $w(g) \geq n + w(f)$ , then  $g/f \in R$ . Let  $f' \in \mathfrak{m}$  be contained in no other prime in  $\mathcal{C}(R_{\mathfrak{P}})$  with  $n'$  such that  $w(g) \geq n' + w(f')$  and  $g/f' \in IR_{\mathfrak{P}}$  imply  $g/f' \in R_{\mathfrak{P}}$ . Let  $g \in R_{\mathfrak{P}}$  with

$w(g) \geq n + w(f')$  and  $g/f' \in IR_{\mathfrak{p}}$ . There is an integer  $\ell \geq 0$  such that  $w(gf'^{\ell}) \geq n' + w(f')$ . Then  $gf'^{\ell}/f' \in R_{\mathfrak{p}}$ , and  $g/f' \in R_{\mathfrak{p}}$ . Let  $a(\mathfrak{P})$  be the subset of nonnegative integers  $n$  such that for any  $f \in \mathfrak{m}$  contained in no other prime in  $\mathcal{C}(R_{\mathfrak{p}})$ ,  $g \in R_{\mathfrak{p}}$ ,  $g/f \in IR_{\mathfrak{p}}$  and  $w(g) = n + w(f)$  imply  $g/f \in R_{\mathfrak{p}}$ . Thus  $a(\mathfrak{P})$  contains all but finitely many of the nonnegative integers. Let  $f$  be an element of  $\mathfrak{m}$  contained in no other prime in  $\mathcal{C}(R_{\mathfrak{p}})$ . If  $n \geq 0$  is such that  $g \in R_{\mathfrak{p}}$ ,  $g/f \in IR_{\mathfrak{p}}$  and  $w(g) \geq n + w(f)$  imply  $g/f \in R_{\mathfrak{p}}$ , and if  $n + w(f) \in a(\mathfrak{P})$ , then  $n \in a(\mathfrak{P})$ .

8. Let  $R$  be noetherian and have a unique nonzero zero prime ideal. Then the complete intersections in  $R$  are the units of  $R$ . Let  $n \neq 0$  be nilpotent, and let  $f$  be a complete intersection in  $R$ . Then  $n = cfn$  for some  $c \in R$ ,  $(1 - cf)n = 0$ , so  $1 - cf$  is nilpotent and  $f$  is unit.

9. Projective algebraic varieties and  $\mathfrak{o}$ -equivalence. Let  $V$  be a projective algebraic variety defined over a field  $k$ , such that  $I_k[V]$  is a noetherian  $k[V]$  module where  $k[V]$  is the homogeneous coordinate ring of  $V$  relative to a fixed projective embedding.  $\mathfrak{C} = \mathfrak{C}(k[V], I_k[V])$  is a homogeneous ideal of  $k[V]$ , and an affine localization of  $\mathfrak{C}$  is the conductor of the affine localization of  $k[V]$  in its integral closure. Affine localization also preserves primary decomposition. Thus the prime ideals in  $\mathcal{C} = \mathcal{C}(k[V])$  are homogeneous,  $\mathcal{C}$  can be defined locally, and  $\mathcal{C}$  is a biregular invariant over  $k$  if the irrelevant ideal  $\mathfrak{J}$  of  $k[V]$  is ignored. Let  $D$  be the group of divisors of  $V$  over  $k$ .  $\mathfrak{X} \in D$  is a complete intersection on  $V$  if and only if locally  $\mathfrak{X}$  is a complete intersection and  $\mathfrak{X} = (F)$  for some homogeneous element  $F$  of  $k[V]$ . Call  $F \in k[V]$  a complete intersection on  $V$  if  $F$  is homogeneous and if  $(F)$  is a complete intersection on  $V$ , which is true if and only if  $F$  is contained in no prime in  $\mathcal{C}$  other than  $\mathfrak{J}$  if  $\mathfrak{J} \in \mathcal{C}$ . The discussion of Example 7, §8 holds for  $\mathfrak{J}$  where degree is the order function  $w$ . Define  $a(V)$  similarly to  $a(\mathfrak{P})$ . By a Segré transformation a projective variety  $V'$  can be found which is biregularly equivalent to  $V$  and for which  $a(V')$  contains the positive integers.

Let  $F \in k[V]$  be regular, homogeneous, non-constant and contained in no prime of  $\mathcal{C}$  other than possibly  $\mathfrak{J}$ . Let  $S = k[V \sim V(F)] \sim \bigcup_{\mathfrak{p} \in \mathcal{C}} \mathfrak{P}$  and let  $\mathfrak{o} = S^{-1}k[V \sim V(F)]$ .  $\mathfrak{o}$  is a biregular invariant of  $V$  over  $k$ . Let  $\mathfrak{X}, \mathfrak{Y} \in D$ . Define  $\mathfrak{X} \rightarrow \mathfrak{Y}$  if and only if  $\mathfrak{Y} = \mathfrak{X} + (f)$  for some  $f \in \mathfrak{o}$ . Define  $\mathfrak{X} \sim \mathfrak{Y}$ ,  $\mathfrak{X}$  is  $\mathfrak{o}$ -equivalent to  $\mathfrak{Y}$ , if and only if  $\mathfrak{X} \rightarrow \mathfrak{Y}$  and  $\mathfrak{Y} \rightarrow \mathfrak{X}$  which is equivalent to  $\mathfrak{Y} = \mathfrak{X} + (f)$  for some unit  $f$  of  $\mathfrak{o}$ . (A unit of  $\mathfrak{o}$  is a quotient of two complete intersections on  $V$  in  $k[V]$  of equal degree.) The complete system of  $\mathfrak{X} \in D$  is  $|\mathfrak{X}| = \{\mathfrak{Y} \in D \mid \mathfrak{Y} \geq 0, \mathfrak{X} \rightarrow \mathfrak{Y}\}$ .  $\{f \in \mathfrak{o} \mid \mathfrak{X} + (f) \geq 0\}$  is a function module.  $\mathcal{L}_m = \{(F) \mid F \in k[V] \text{ homogeneous of degree } m\}$ .

**PROPOSITION 16.** *For all  $m$  sufficiently large there is a complete intersection on  $V$  in  $k[V]$  of degree  $m$  and  $\mathcal{L}_m$  is a complete system. If  $k$  is infinite and algebraically closed in  $k(V)$ , then  $\mathcal{L}_m$  is a complete system if and only if  $m \in a(V)$ .*

*Proof.* If  $k$  is infinite then there is a linear form  $L \in k[V]$  contained in no prime in  $\mathcal{C}$  other than possibly  $\mathfrak{S}$ . If  $k$  is finite, let  $p$  be a prime, and let  $k_i$  for  $i = 1, 2, \dots$  be an extension field of  $k$  such that  $k_i \subset k_{i+1}$  and  $[k_{i+1} : k_i] = p$ . For some  $k_i$ , there is a linear form  $L$  defined over  $k_i$  contained in no prime in  $\mathcal{C}$  other than  $\mathfrak{S}$ . The  $k_i/k$  norm of  $L$  is a complete intersection on  $V$  in  $k[V]$  of degree  $p^\ell$  for some  $\ell$ . Using two distinct primes, it follows that there is a complete intersection of degree  $m$  for all  $m$  sufficiently large.

Let  $m \in a(V)$ , and let  $F \in k[V]$  be a complete intersection of degree  $m$ .  $\mathcal{L}_m \subset |(F)|$ . Let  $\mathfrak{A} \in |(F)|$ . There are  $H, G \in k[V]$  homogeneous of equal degree,  $(G)$  a complete intersection on  $V$  with  $\mathfrak{A} = (H/G) + (F)$ . So  $FH/G \in k[V]$ , and  $\mathcal{L}_m = |(F)|$ .

Now suppose  $k$  is infinite and algebraically closed in  $k(V)$ . Suppose  $\mathcal{L}_m$  is a complete system, and let  $(F) \in \mathcal{L}_m$  where  $F$  is a complete intersection on  $V$  in  $k[V]$ . Then  $\mathcal{L}_m = |(F)|$ . Let  $G$  be regular homogeneous element of  $k[V]$  contained in no prime of  $\mathcal{C}$  other than  $\mathfrak{S}$ . Let  $H \in k[V]$  homogeneous with  $(H) \geq (G)$  and  $\deg H = \deg G + m$ . Then  $(F) \rightarrow (H) - (G)$ , there is an  $E \in k[V]$  with  $(E) = (H) - (G)$ ,  $H/G \in k[V]$ , and  $H/G \in k[V]$ . Thus  $m \in a(V)$ .

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