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CYCLIC VECTORS FOR REPRESENTATIONS ASSOCIATED WITH POSITIVE DEFINITE MEASURES: NONSEPARABLE GROUPS

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Let μ be any positive definite measure on a locally compact group, and let $(\pi^\mu, \mathscr{H}^\mu)$ be the associated unitary representation of G. Previous work of the authors' showed that a cyclic vector exists for π^μ if G is second countable; there is now a simple proof of this result, due to Hulanicki. Rather elementary conditions on the way μ is related to the geometry of G are examined which are necessary, or sufficient, for the existence of a cyclic vector. These conditions require μ to be "constant" on cosets (or double cosets) of certain subgroups of G. A conjectured necessary and sufficient conditions is presented. These results are adequate to decide whether or not π^u is cyclic for various nontrivial measures. As a special case it is shown that the left regular representation of G is cyclic $\Leftrightarrow$ G is first countable.

1. Notations. All groups are locally compact, not necessarily second or first countable. The space $C_c(G)$ of continuous functions with compact support is given the usual inductive limit topology. Convolutions $f*g$ of functions in $C_c(G)$ are defined in the usual way; we use the involution operation

$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})$$

(Δ the modular function) which makes $C_c(G)$ a $\|\cdot\|_1$-dense *-subalgebra of the convolution algebra $L^1(G)$. Positive definite measures μ are Radon measures (*not necessarily bounded*), so that $\mu \in C_c(G)^*$, that satisfy the condition

$$\langle \mu, f^* * f\rangle = \int_G (f^* * f)(x)d\mu(x) \geqq 0, \text{ all } f \in C_c(G) .$$

Positive definiteness is indicated by writing $\mu \succ 0$. The representation $(\pi^\mu, \mathscr{H}^\mu)$ associated with μ is defined by imposing the conjugate bilinear form

$$(f, g)_\mu = \int g^* * f d\mu \text{ for } f, g \in C_c(G)$$

on $C_c(G)$. Left translation $\lambda_x f(y) = f(x^{-1}y)$ preserves this form.

If we write $\|f\|_\mu = (f, f)_\mu^{1/2}$, and set $\mathscr{N}^\mu = \{f \in C_c(G): \|f\|_\mu = 0\}$, then the quotient map $j_\mu: C_c(G) \to \mathscr{H}_0^\mu = C_c/\mathscr{N}^\mu$ maps $C_c(G)$ into a pre-Hilbert space whose completion is denoted by $\mathscr{H}^\mu$. Left translation induces a unitary operator π_x^μ on $\mathscr{H}^\mu$; if we write $j_\mu(f) = [f]_\mu$, then $\pi_x^\mu[f]_\mu = [\lambda_x f]_\mu$ for all vectors in the dense subspace $\mathscr{H}_0^\mu$. Details concerning the representation $(\pi^\mu, \mathscr{H}^\mu)$ can be found in Dixmier [2], or Effros-Hahn [3]. For additional comments on positive definite measures, and their relationship to positive definite functions see [5] (introduction to §3).

The space $M(G) = C_c(G)^*$ of all Radon measures (bounded or not) has an involution defined by

$$\langle \mu^*, f\rangle = \overline{\int f(x^{-1})d\mu(x)} \text{ all } f \in C_c(G) .$$

To a certain extent convolution is defined in $M(G)$; thus $\mu*\nu$ is defined if one of the measures has compact support, or if both are bounded (finite total variation). Left Haar measure on a closed subgroup $K \subseteq G$ is denoted by m_K; $C_c(G)$ becomes a convolution subalgebra of $M(G)$ if we identify $f \in C_c(G)$ with $f\cdot m_G \in M(G)$. $M(G)$ includes point masses δ_x for $x \in G$, and the left translate $\lambda_x f$ of $f \in C_c(G)$ is just $\delta_x * f$.

We will be interested in measures "constant on cosets" of a closed subgroup K. Constancy on right cosets $K\backslash G = \{Kx: x \in G\}$ (resp. left cosets $G/K = \{xK: x \in G\}$) means that

$$(1) \qquad \delta_k*\mu = \mu \text{ (resp. } \mu*(\Delta_G(k)\delta_k)\text{), all } k \in K ;$$

here Δ_G is the modular function on G. If μ arises from a continuous function φ on G, so that $\mu = \varphi\cdot m_G$, this notion of constancy agrees with the usual one, in which

$$\varphi(kx) = \varphi(x) \text{ (resp. } \varphi(xk) = \varphi(x)\text{) for all } k \in K, x \in G .$$

By definition of convolution of measures (see Appendix), the modular function must appear in right-hand convolutions if constancy of measures is to agree with constancy of functions; indeed, if $x \in G$ then consider what happens to the right convolutes of Haar measure:

$$m_G*\delta_x = \Delta(x^{-1})\cdot m_G \text{ and } m_G*(\Delta(x)\cdot\delta_x) = m_G .$$

Let $P_c(K)$ be all probability measures $\left(\nu \geqq 0 \text{ in the usual sense; } \int d\nu = 1\right)$ with compact support and $\operatorname{supp}(\nu) \subseteq K$. By taking weak-$*$ limits of convex combinations it is easily seen that (1) holds $\Leftrightarrow$

$$(2) \qquad \nu*\mu = \mu \text{ (resp. } \mu*\Delta_G\cdot\nu) = \mu\text{) all } \nu \in P_c(K) ;$$

here

$$\int f(x)d[\Delta_G\cdot\nu](x) = \int f(x)\Delta_G(x)d\nu(x) \text{ for } f \in C_c(G) .$$

Special facts about convolutions, and constancy on cosets, are discussed in the Appendix.

2. A special case: left regular representation. We (briefly) deal with the left regular representation which acts in $L^2(G)$ via $\lambda_x f(g) = f(x^{-1}g)$.

THEOREM 2.1. *The left regular representation of a locally compact group G is cyclic $\Leftrightarrow$ G is first countable.*

Proof. ($\Leftarrow$) This is (2.6) of [5].

Proof. ($\Rightarrow$) If there is a cyclic vector $f \in L^2(G)$ its support $\operatorname{supp}(f)$ is σ-compact, so $\operatorname{supp}(f)$ lies within an *open* σ-compact subgroup G'. If $x \notin G'$ then $\lambda_x f$ is supported on xG'; thus $\operatorname{supp}(\lambda_x f) \cap \operatorname{supp}(f) = \varnothing$, so that $(\lambda_x f, f) = 0$ in $L^2(G)$. Evidently, the "reduced action" $G' \times L^2(G') \to L^2(G')$ is also cyclic, with cyclic vector f, because translations of f by $x \notin G'$ cannot contribute to the approximation of any vector in the subspace $L^2(G') \subseteq L^2(G)$. The following basic lemma (applied to G') shows that G' is first countable; thus G is first countable.

LEMMA 2.2. *If $G \times \mathscr{H} \to \mathscr{H}$ is faithful, cyclic representation of a σ-compact group G on a Hilbert space $\mathscr{H}$, then $\mathscr{H}$ is separable and G is first countable.*

Proof. Take compacta K_n containing the unit such that $K_n \subseteq \operatorname{int}(K_{n+1})$ and $\bigcup_{n=1}^{\infty} K_n = G$. If ζ is the cyclic vector the map $\varphi: g \to \pi_g(\zeta)$ is norm continuous, $\varphi: G \to \mathscr{H}$. Thus $\varphi(K_n)$ is norm compact, hence totally bounded, and there is a finite set of translates of ζ, $S_n \subseteq \varphi(K_n)$ such that all points in $\varphi(K_n)$ are within distance $1/n$ of S_n. Now the closed linear span

$$\overline{ls}\{\bigcup_{n=1}^{\infty} \varphi(K_n)\} = \overline{ls}\{\varphi(G)\} = ls\{\overline{\pi_g}(\zeta): g \in G\}$$

equals $\mathscr{H}$, and it is easy to see that $\overline{ls}\{\bigcup_{n=1}^{\infty} S_n\} = \mathscr{H}$ too. Thus $\mathscr{H}$ is *separable*.

Now let $\mathscr{U}(\mathscr{H})$ be the unitary operators on $\mathscr{H}$, with strong operator topology. The representation $\pi: G \to \mathscr{U}(\mathscr{H})$ is a continuous *isomorphism*. If K is a compact neighborhood of the unit in G, $\pi: K \to \pi(K)$ is a homeomorphism. But the strong operator topology is first countable since $\mathscr{H}$ is separable; thus the (relative) topology on K is first countable, and G is first countable.

This completes the discussion of regular representations.

3. Sufficient conditions. In [5] we showed that the representation $(\pi^\mu, \mathscr{H}^\mu)$ is cyclic for every positive definite measure on a second countable locally compact group. That proof, based on operator algebra methods, was nonconstructive and we raised the question of finding a construction which produces a cyclic vector using nothing but the geometric/algebraic features of the group. Even when π^μ is the left regular representation and $G = \boldsymbol{R}$ (real line) it is not completely obvious how this is to be done. In an elegant note [7] Hulanicki and Pytlik have proven:

THEOREM 3.1. *If $\mu \succ 0$ on a first countable group, the representation $(\pi^\mu, \mathscr{H}^\mu)$ is cyclic.*

The proof is totally constructive: take any countable basis of compact neighborhoods of the unit K_n and form the characteristic functions φ_n; taking scalars $a_n \geqq 0$ so that $\sum_{n=1}^\infty a_n \|\varphi_n^* * \varphi_n\|_\infty < \infty$, we get a function $\varphi = \sum_{n=1}^\infty a_n \varphi_n^* * \varphi_n \in C_c(G)$. The vector $\zeta = [\varphi]_\mu$ is cyclic. In particular, one can find cyclic vectors within $\mathscr{H}_0^\mu$; it is not necessary to turn to the completion $\mathscr{H}^\mu$.

Added in proof. R. Goodman has found a subtle, but serious, gap in the sections of the Effros-Hahn memoir [3] on which the results of Hulanicki and Pytlik [7] depend. To get a correct proof of Theorem 3.1, we start with the results of [5], which are unaffected by Prof. Goodman's discovery. A short, self-contained argument then yields Theorem 3.1. In [5] we used operator algebra methods to establish Theorem 3.1 for second countable groups; second countability was required so that operator methods could be applied freely. Below we show that the theorem may be established for arbitrary first countable groups without further reference to the theory of W^*-algebras. If the following proof is appended to Theorem 3.1, and references to [7] deleted, this paper remains unaffected by the gaps in [3]. Our references to [3] here, and in [5], are fairly superficial and do not use the afflicted section in [3] (p. 48, lines 9t—11t).

Incidentally, Goodman has justified the construction of Hulanicki and Pytlik for Lie groups (countable at infinity, but not necessarily connected) by considering the relation between representations and positive definite *distributions* on such groups. He shows that $(\pi^\mu, \mathscr{H}^\mu)$ has a cyclic vector for every positive definite distribution. His work will appear soon in an article: *Positive definite distributions and intertwining operators*, (Pacific J. Math.).

Proof of Theorem 3.1. Let G_0 be any open σ-compact (hence second countable) subgroup in G. The restricted measure $\nu = \mu \mid G_0$ is positive definite on G_0; the canonical injection $i\colon C_c(G_0) \to C_c(G)$ induces an isometric injection of $\mathscr{H}_0^{\nu}$ into $\mathscr{H}_0^{\mu}$, as indicated below, because if $f, g \in C_c(G_0)$ we have

$$(f, g)_\nu = \int_{G_0} g^* * f d\nu = \int_G g^* * f d\mu = (f, g)_\mu \,.$$

Therefore, $\mathscr{H}^{\nu}$ is identified with a closed subspace $\mathscr{H}^*$ in $\mathscr{H}^{\mu}$. Under

$$\begin{array}{ccccc} C_c(G_0) & \longrightarrow & \mathscr{H}_0^{\nu} & \longrightarrow & \mathscr{H}^{\nu} \\ {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow \\ C_c(G) & \longrightarrow & \mathscr{H}_0^{\mu} & \longrightarrow & \mathscr{H}^{\mu} \end{array}$$

this identification the operator π_x^{ν} is equivariant with $\pi_x^{\mu} \mid \mathscr{H}^*$ for every $x \in G_0$, because $(\delta_x * f, g)_\nu = (\delta_x * f, g)_\mu$ for all $x \in G_0$; $f, g \in C_c(G_0)$. Therefore, we may identify the action $G_0 \times \mathscr{H}^{\nu} \to \mathscr{H}^{\nu}$ with the restricted action $G_0 \times \mathscr{H}^* \to \mathscr{H}^*$ on the subspace $\mathscr{H}^* \subseteqq \mathscr{H}^{\mu}$.

Now let $\{e_n\} \subseteqq C_c(G)$ be a countable approximate identity with supports $\operatorname{supp}(e_n) \subseteqq G_0$ that decrease to the unit $\{e\}$: $e_n \geqq 0$, $\int e_n(x)dx = 1$ all n. As indicated in [5; Lemma 3.3], the vectors $[e_n]_\nu$ in $\mathscr{H}^{\nu}$ are asymptotically cyclic under the action of $\pi^{\nu}(G_0)$. Consequently, since the W^*-algebra $\mathfrak{A} = \pi^{\nu}(G_0)''$ in $\mathscr{L}(\mathscr{H}^{\nu})$ is separable, there is a cyclic vector $\zeta \in \mathscr{H}^{\nu}$ [5; Lemma 3.4]. Identifying ζ as a vector in $\mathscr{H}^* \subseteqq \mathscr{H}^{\mu}$, this means that if $\eta \in \mathscr{H}^*$, there are functions $\{h_n\} \subseteqq C_c(G_0)$ such that $\|\pi^{\mu}(h_n)\zeta - \eta\|_\mu \to 0$ as $n \to \infty$ (recall: $\pi^{\nu}(G_0)'' = \pi^{\nu}(C_c(G_0))''$).

We assert that ζ is a cyclic vector for the full action $G \times \mathscr{H}^{\mu} \to \mathscr{H}^{\mu}$. Because $\mathscr{H}_0^{\mu}$ is norm dense in $\mathscr{H}^{\mu}$, it is clearly sufficient to show that any vector of the form $\eta = [f]_\mu \in \mathscr{H}_0^{\mu}$ can be approximated by vectors $\pi^{\mu}(h)\zeta$ with $h \in C_c(G)$. Let $\{x(\alpha)\colon \alpha \in I\}$ be representatives for the distinct left cosets $G/G_0 = \{xG_0\colon x \in G\}$. If $f \in C_c(G)$, we may write (*finite* sum) $f = \sum_{\alpha \in I} \delta_{x(\alpha)} * f_\alpha$ where $f_\alpha \in C_c(G_0)$. Now there exist $\{h_{n,\alpha}\colon n = 1, 2, \cdots\}$ such that $\pi^{\mu}(h_{n,\alpha})\zeta \to [f_\alpha]$, for each $\alpha \in I$. Thus,

$$\pi^{\mu}_{x(\alpha)}\pi^{\mu}(h_{n,\alpha})\zeta = \pi^{\mu}(\delta_{x(\alpha)} * h_{n,\alpha})\zeta \to \pi^{\mu}_{x(\alpha)}[f_\alpha] = [\delta_{x(\alpha)} * f_\alpha]$$

as $n \to \infty$, and

$$\begin{aligned} \pi^{\mu}\left(\textstyle\sum_{\alpha \in I} \delta_{x(\alpha)} * h_{n,\alpha}\right)\zeta &= \textstyle\sum_{\alpha \in I} \pi^{\mu}_{n(\alpha)}\pi^{\mu}(h_{n,\alpha})\zeta \\ &\longrightarrow \textstyle\sum_{\alpha \in I} [\delta_{x(\alpha)} * f_\alpha] = [f] \end{aligned}$$

as $n \to \infty$. Clearly, ζ is cyclic.

If $\mu \succ 0$ the kernel of π^μ is a closed normal subgroup in G; now μ must be constant on (right = left) cosets of Ker (π^μ); moreover, this subgroup may be identified directly by examining how μ relates to the structure of G, without referring to the representation π^μ.

THEOREM 3.2. *Let μ be a positive definite measure on a locally compact group G. Then Ker π^μ is characterized as the largest normal subgroup $K \subseteqq G$ such that μ is constant on cosets of K.*

For a proof of (3.2) see Appendix A2. The result of [7] can be extended considerably using Theorem (3.2).

PROPOSITION 3.3. *Let μ be a positive definite measure. If there exists a closed, normal subgroup $K \subseteqq G$ such that*
(i) *G/K is first countable*
(ii) *μ is constant on cosets of K,*
then $(\pi^\mu, \mathscr{H}^\mu)$ is cyclic.

Proof. Clearly $K \subseteqq$ Ker π^μ; thus the map $x \rightarrow \pi^\mu_x$ is constant on K-cosets, and the representation $\pi^\mu: G \times \mathscr{H}^\mu \rightarrow \mathscr{H}^\mu$ is lifted back to G from a corresponding representation $\pi: (G/K) \times \mathscr{H}^\mu \rightarrow \mathscr{H}^\mu$ via the quotient homomorphism $\varphi: G \rightarrow G/K$.

Define $\varphi': C_c(G) \rightarrow C_c(G/K)$ so that $\varphi' f(xK) = \int_K f(xk) dm_K(k)$, where m_K = left Haar measure on K. It is a fairly routine matter to show that

$$\varphi'(f^*) = (\varphi' f)^* \qquad \varphi'(f * g) = (\varphi' f) * (\varphi' g) \text{ all } f, g \in C_c(G) ,$$

and that φ' is *surjective*, because K is normal in G. Somewhat more delicate calculations (presented in Appendix) show that the measure $\mu \succ 0$ on G corresponds to a unique measure μ' on G/K such that

$$\langle \mu', \varphi' f \rangle = \langle \mu, f \rangle \text{ all } f \in C_c(G) . \tag{3}$$

It follows from (3) that $\mu' \succ 0$ on G/K, since

$$\langle \mu', (\varphi' f)^* * (\varphi' f) \rangle = \langle \mu', \varphi'(f^* * f) \rangle = \langle \mu, f^* * f \rangle \geqq 0 \text{ all } f \in C_c(G)$$

(φ' is surjective!); furthermore, the map φ' induces an isometry $j: \mathscr{H}^\mu \rightarrow \mathscr{H}^{\mu'}$, since

$$\begin{aligned}(\varphi' f, \varphi' g)_{\mu'} &= \langle \mu', (\varphi' g)^* * (\varphi' f) \rangle = \langle \mu', \varphi'(g^* * f) \rangle \\ &= \langle \mu, g^* * f \rangle = (f, g)_\mu\end{aligned}$$

for $f, g \in C_c(G)$.

Finally, the diagram in Figure 1 commutes.

$$\begin{array}{ccccc} G \times \mathscr{H}^{\mu} & \xrightarrow{\varphi \times id} & (G/K) \times \mathscr{H}^{\mu} & \xrightarrow{id \times j} & (G/K) \times \mathscr{H}^{\mu'} \\ \downarrow \pi^{\mu} & & \downarrow \pi & & \downarrow \pi^{\mu'} \\ \mathscr{H}^{\mu} & \xrightarrow{\quad id \quad} & \mathscr{H}^{\mu} & \xrightarrow{\quad j \quad} & \mathscr{H}^{\mu'} \end{array}$$

FIGURE 1

The left-hand block commutes since π^{μ} is constant on cosets of K; the representation π (of G/K) is defined so that $\pi_{\varphi(x)}[f]_{\mu} = \pi_x[f]_{\mu} = [\delta_x * f]_{\mu}$ for $x \in G$, $f \in C_c(G)$. Next note that

$$\varphi'(\delta_x * f) = \delta_{\varphi(x)} * \varphi' f \quad \text{all } f \in C_c(G),\ x \in G\ ; \tag{4}$$

in fact,

$$\begin{aligned} \delta_{\varphi(x)} * \varphi' f(yK) &= \varphi' f(x^{-1}K \cdot yK) = \varphi' f(x^{-1}yK) \\ &= \int f(x^{-1}yk)\, dm_K(k) = \int (\delta_x * f)(yk)\, dm_K(k) \\ &= \varphi'(\delta_x * f)(yK)\ . \end{aligned}$$

Now compare images of a pair $(\varphi(x), [f]_{\mu}) \in (G/K) \times \mathscr{H}^{\mu}$ in the right-hand block:

$$\begin{aligned} &(\varphi(x), [f]_{\mu}) \longrightarrow \pi_{\varphi(x)}[f]_{\mu} = [\delta_x * f]_{\mu} \xrightarrow{\ j\ } [\varphi'(\delta_x * f)]_{\mu'} \\ &(\varphi(x), [f]_{\mu}) \longrightarrow (\varphi(x), [\varphi' f]_{\mu'}) \longrightarrow \pi^{\mu'}_{\varphi(x)}[\varphi' f]'_{\mu} = [\delta_{\varphi(x)} * \varphi' f]_{\mu'}\ . \end{aligned}$$

These agree, by (4).

Obviously π^{μ} is cyclic $\Leftrightarrow \pi$ is cyclic, since both representations produce the same operators on $\mathscr{H}^{\mu}$. Commutativity of the right-hand block shows (π^{μ} cyclic) $\Leftrightarrow$ (π cyclic) $\Leftrightarrow$ ($\pi^{\mu'}$ cyclic). By our hypotheses, Theorem 3.1 given above applies to the measure μ' on the first countable group G/K; we conclude that $\pi^{\mu'}$ (and π^{μ}) are cyclic.

Proposition 3.3 is the basis for a considerably stronger sufficient condition.

THEOREM 3.4. *Let μ be positive definite. Assume the existence of a σ-compact open subgroup $G' \subseteq G$, and a closed subgroup $K \subseteq G'$ that is normal in G' (but not necessarily normal in G) such that*

(i) G'/K is first countable

(ii) μ is constant on double cosets $K\backslash G/K$, (i.e., on both left and right cosets). Then $(\pi^{\mu}, \mathscr{H}^{\mu})$ is cyclic.

Proof. Note that supp (μ) need not be related to G'; we may have supp $(\mu) = G$. Taking G' as above, let $\mathscr{H}' = \overline{ls}\{[f]_{\mu}\colon f \in C_c(G)$ and

$\operatorname{supp}(f) \subseteqq G'\}$. Then $\mathscr{H}'$ is G'-invariant, so we get a "reduced action" $G' \times \mathscr{H}' \to \mathscr{H}'$.

If the reduced action is cyclic, so is the full action

$$G \times \mathscr{H}^\mu \to \mathscr{H}^\mu . \tag{5}$$

To prove (5) consider cosets $G/G' = \{xG' : x \in G\}$ and let $\{x_i : i \in I\}$ be coset representatives. Let $\mathscr{H}^i = \overline{ls}\{[f]_\mu : f \in C_c(G), \operatorname{supp}(f) \subseteqq x_i G'\}$. Obviously $\bigcup_{i \in I} \mathscr{H}^i$ spans $\mathscr{H}^\mu$, and $\pi_{x_i} : \mathscr{H}' \to \mathscr{H}^i$ is an *isometry onto* $(\forall i \in I)$. Let $\zeta_0 \in \mathscr{H}' = \mathscr{H}^{i_0}$ be the cyclic vector for the reduced action. If $\zeta \in \mathscr{H}^i$ (i arbitrary), then $\pi_{x_i}^{-1}(\zeta) \in \mathscr{H}'$ and there are finite linear combinations σ_n of the operators $\{\pi_x : x \in G'\}$ such that $\sigma_n(\zeta_0) \to \pi_{x_i}^{-1}(\zeta)$, which $\Rightarrow (\pi_{x_i}\sigma_n)(\zeta_0) \to \zeta$. Therefore $\mathscr{H}^\mu \supseteqq \overline{ls}\{G(\zeta_0)\} \supseteqq \overline{ls}\{\bigcup_{i \in I} \mathscr{H}^i\} = \mathscr{H}^\mu$ and ζ_0 is cyclic for the action $G \times \mathscr{H}^\mu \to \mathscr{H}^\mu$.

Now that (5) is proved, we need only demonstrate that the action $G' \times \mathscr{H}' \to \mathscr{H}'$ is cyclic. The restricted measure $\nu = \mu \mid G'$ is cleary positive definite on G', and it is clear that the actions $G' \times \mathscr{H}^\nu \to \mathscr{H}^\nu$ and $G' \times \mathscr{H}' \to \mathscr{H}'$ are equivalent ($\exists$ an obvious G'-equivariant isometry $\mathscr{H}^\nu \cong \mathscr{H}'$), so we are reduced to the situation in Proposition 3.3.

4. Necessary conditions; a conjecture.

We conjecture that the conditions of Theorem 3.4 are also necessary; that is, if G is any locally compact group and μ any positive definite measure, then

Conjecture 4.1. $(\pi^\mu, \mathscr{H}^\mu)$ is cyclic $\Leftrightarrow$ there exist a σ-compact open subgroup $G' \subseteqq G$ and a closed subgroup $K \subseteqq G'$ that is normal in G' (but not necessarily in G), such that

(i) G'/K is first countable

(ii) μ is constant on double cosets $K \backslash G / K$.

The conjecture will be supported in several ways. First we will show that these conditions really are necessary in two very different special cases.

THEOREM 4.2. *If $\mu \succ 0$ and the support* $\operatorname{supp}(\mu)$ *is σ-compact, then the conjecture is true.*

COROLLARY 4.3. *On any σ-compact, locally compact G the conjecture is valid for every $\mu \succ 0$ on G.*

Proof. The implication ($\Leftarrow$) in 4.1 is true for *every* $\mu \succ 0$ (Theorem 3.4). For ($\Rightarrow$) let ζ be the cyclic vector. Take $h_n \in C_c(G)$ such that $\|[h_n] - \zeta\|_\mu \to 0$; then there exists a σ-compact open subgroup $G' \subseteqq G$ such that

$$G' \supseteqq \operatorname{supp}(h_n), \text{ all } n, \text{ and } G' \supseteqq \operatorname{supp}(\mu) .$$

Obviously $\mathscr{H}' = \overline{ls}\{[f]: f \in C_c(G), \operatorname{supp}(f) \subseteqq G'\}$ is G'-invariant and $[h_n] \in \mathscr{H}'(\forall n)$; thus $\zeta \in \mathscr{H}'$ too. We claim that

(6) The reduced action $G' \times \mathscr{H}' \to \mathscr{H}'$ is cyclic (with cyclic vector ζ).

In fact, if $f, g \in C_c(G)$ are supported on G', and if $x \in G \sim G'$ (set-theoretic difference), then the double coset $(G'xG') \cap G' = \emptyset$. Thus,

$$(\pi_x^\mu[f], [g])_\mu = \int_G g^* * \delta_x * f d\mu = 0 ,$$

since $\operatorname{supp}(g^* * \delta_x * f) \subseteqq G'xG'$ and $\operatorname{supp}(\mu) \subseteqq G'$. Therefore, translates of ζ by $x \notin G'$ cannot contribute to an approximation of a vector $\xi \in \mathscr{H}'$. But ξ is approximated by G-translates of ζ, so it must be approximated using only G'-translates, which proves (6). Let $\nu = \mu | G'$. Since μ is *supported on* G', there is an obvious equivalence between the actions $G' \times \mathscr{H}' \to \mathscr{H}'$ and $G' \times \mathscr{H}^\nu \to \mathscr{H}^\nu$, so the latter is cyclic. If $K = \operatorname{Ker} \pi^\nu$ then K is normal in G' (and closed in G) and π^ν is obtained in the usual way from a representation π of G'/K; π and π^ν both act on $\mathscr{H}^\nu$, and

$$\pi_{\varphi(x)} = \pi_x^\nu \text{ all } x \in G' \ (\varphi: G' \to G'/K \text{ the quotient map}) .$$

Since we have factored out $\operatorname{Ker} \pi^\nu$, the representation $\pi: (G'/K) \times \mathscr{H}^\nu \to \mathscr{H}^\nu$ is *faithful*; it is also cyclic, since π^ν and π give the same operators in $\mathscr{H}^\nu$. Applying Lemma 2.2, we conclude that G'/K is first countable. Obviously ν is constant on cosets G'/K, by Theorem 3.2; μ is then constant on $K \backslash G/K$ cosets, being zero off G'.

In the next situation, we impose algebraic conditions on μ, but leave off any finiteness requirements.

THEOREM 4.4. *If $\mu \succ 0$ and μ is a central measure, the conjecture is valid.*

Proof. Again, we have only to prove (cyclic) $\Rightarrow$ $(\cdots)$. As in the proof above, we start by taking $h_n \in C_c(G)$ such that $\|[h_n] - \zeta\|_\mu \to 0$. Let G' be an open, σ-compact subgroup such that $\operatorname{supp}(h_n) \subseteqq G'$ for all n. Let $\mathscr{H}' = \overline{ls}\{[f]: f \in C_c(G), \operatorname{supp}(f) \subseteqq G'\}$; clearly $\zeta \in \mathscr{H}'$ and $\mathscr{H}'$ is G'-invariant. By (A. 11) of the Appendix there is a compact normal subgroup K of G' such that G'/K is second countable and each h_n is constant on cosets of K.

Now m_K = Haar measure on K is a finite measure and $m_K * h_n * m_K = h_n$ for $n = 1, 2, \cdots$. Let $\mu_K = m_K * \mu * m_K$; clearly μ_K is constant on cosets $K \backslash G/K$ and $\mu_K \succ 0$ on G. Form the submanifold $\mathscr{M} = ls\{\varphi * h_n$:

$\varphi \in C_c(G)$, $n = 1, 2, \cdots\}$ (*algebraic* linear span), let $j_\mu\colon C_c(G) \to \mathscr{H}^\mu$ be the canonical injection, and write $\mathscr{M}^\mu = j_\mu(\mathscr{M})$; then

(6) $$\mathscr{M}^\mu \text{ is } \|\cdot\|_\mu\text{-dense in } \mathscr{H}^\mu .$$

Indeed, if $\pi_\varphi^\mu = \int \varphi(x)\pi_x^\mu dx$ is the integrated form of π^μ for $\varphi \in L^1(G)$, it is obvious that

$$\pi_\varphi^\mu[f] = [\varphi * f] \text{ all } f \in C_c(G), \text{ all } \varphi \in C_c(G) ;$$

furthermore, $\{\pi_\varphi^\mu(\zeta)\colon \varphi \in C_c(G)\}$ is $\|\cdot\|_\mu$-dense in $\mathscr{H}^\mu$, since ζ is cyclic. Thus, given $\xi \in \mathscr{H}^\mu$, there exists a $\varphi \in C_c(G)$ such that $\|\pi_\varphi^\mu(\zeta) - \xi\| < 1/2n$, and there exists an h_k such that

$$\|[h_k] - \zeta\|_\mu < \frac{1}{\|\varphi\|_1} ,$$

consequently

$$\begin{aligned}\|[\varphi * h_k] - \xi\|_\mu &\leqq \|\pi_\varphi^\mu[h_k] - \pi_\varphi^\mu[\zeta]\|_\mu + \|\pi_\varphi^\mu[\zeta] - \xi\|_\mu \\ &\leqq \|\varphi\|_1 \cdot \|[h_k] - [\zeta]\|_\mu + \frac{1}{2n} < \frac{1}{n} ,\end{aligned}$$

so that $\mathscr{M}^\mu$ is $\|\cdot\|_\mu$-dense.

We would like to show that the mapping $A\colon C_c(G) \to C_c(G)$ defined via *right* convolution

$$A\colon f \to f * m_K \qquad f \in C_c(G)$$

induces a well-defined, bounded operator $\tilde{A}\colon \mathscr{H}^\mu \to \mathscr{H}^\mu$; that is, we want

$$\|f - g\|_\mu = 0 \Longrightarrow \|f * m_K - g * m_K\|_\mu = 0$$

(or, simply $\|f\|_\mu = 0 \Rightarrow \|f * m_K\|_\mu = 0$), and

$$\|f * m_K\|_\mu \leqq K \cdot \|f\|_\mu \qquad (K \text{ fixed constant}) ,$$

so that A can be transferred to $\mathscr{H}_0^\mu = C_c(G)/\mathscr{N}^\mu$. If this can be done, it follows directly that μ is constant on $K\backslash G/K$ cosets, because if $\tilde{A}$ exists it must equal the *identity* operator I on $\mathscr{H}^\mu$. This is so because

$$\tilde{A}[\varphi * h_n] = [\varphi * h_n * m_K] = [\varphi * h_n] \text{ all } \varphi \in C_c(G), \text{ all } n ,$$

(since $h_n * m_K = h_n$), so that $\tilde{A} = I$ on the dense submanifold $\mathscr{M}^\mu$. Once we know $\tilde{A} = I$, we observe that for any $f, g \in C_c(G)$,

$$\begin{aligned}\int g^* * f d\mu &= ([f], [g])_\mu = (\tilde{A}[f], \tilde{A}[g])_\mu \\ &= ([f * m_K], [g * m_K])_\mu = \int m_K * g^* * f * m_K d\mu .\end{aligned}$$

It is obvious that $m_K^* = m_K$, so that $(g*m_K)^* = m_K^* * g^* = m_K * g^*$ as above; in the appendix we verify that

$$\int m_K*\varphi*m_K \, d\mu = \int \varphi \, d[m_K*\mu*m_K]$$

for all $\varphi \in C_c(G)$; thus

$$\int g^**f \, d\mu = \int g^**f \, d\mu_K \text{ all } f, g \in C_c(G) .$$

Now the submanifold $\mathfrak{X} = \{g^**f: f, g \in C_c(G)\}$ is dense in the inductive limit topology on $C_c(G)$; μ and μ_K are continuous functionals, so we conclude that $\mu = \mu_K = m_K*\mu*m_K$. Therefore, pending the existence of $\tilde{A}$, we have shown that

$$(\pi^\mu, \mathscr{H}^\mu) \text{ cyclic} \Longrightarrow \mu \text{ constant on cosets } K\backslash G/K$$

where G' and K are defined as above.

We observe that, up to this point, we have used no special assumptions about μ; we have proved the *unrestricted* conjecture, modulo the existence of $\tilde{A}$. Unfortunately, it is difficult to see why we should have $\|f\|_\mu = 0 \Rightarrow \|f*m_K\|_\mu = 0$, let alone boundedness of $\tilde{A}$, unless μ is a central measure. Commentary on the unrestricted conjecture (following this proof) will be based on the constructions up to this point.

If μ is central we will show that $\|f*m_K\|_\mu \leqq \|f\|_\mu$ for $f \in C_c(G)$. By definition,

$$\mu \text{ central} \Longleftrightarrow \mu(f*g) = \mu(g*f), \text{ all } f, g \in C_c(G) ;$$

thus, if f is given and $g = f^*$, we have

$$\begin{aligned} \|f*m_K\|_\mu^2 &= \mu(m_K*f^**f*m_K) = \mu(f*m_K*m_K*f^*) \\ &= \mu((m_K*g)^**(m_K*g)) = \|m_K g\|_\mu^2 \\ &= \|\pi_{m_K}^\mu[g]\|_\mu^2 \leqq 1\cdot\|g\|_\mu^2 \\ &= \mu(f*f^*) = \mu(f^**f) = \|f\|_\mu^2 . \end{aligned}$$

The proof of Theorem 4.4 is complete.

5. Examples and further comment on the conjecture.

We are now in a position to demonstrate that certain positive definite measures μ do not give cyclic representations $(\pi^\mu, \mathscr{H}^\mu)$; to reach this conclusion we must use the necessary conditions obtained in §4.

EXAMPLE 5.1. Define $H = \Pi\{T_s: s \in \boldsymbol{R}\}$, product of uncountably many circle groups $T_s = \boldsymbol{R}/\boldsymbol{Z}$ (written additively). Let G be the space $\boldsymbol{R}_d \times H$ equipped with semidirect product structure

$$(r, \{p_s\})\cdot(t, \{q_s\}) = (r + t, \{p_s\} + \alpha_r\{q_s\}) \ ;$$

here $\boldsymbol{R}_d$ = discrete reals, and $\alpha_t \colon H \to H$ is the "right shift" mapping $h = \{p_s\}$ to $h' = \{q_s\}$ where $q_s = p_{s-r}$ (all $s \in \boldsymbol{R}$).

Obviously H is compact, connected. Let K_0 be the compact subgroup $K_0 = \{h = \{p_s\} \colon p_s = 0 \text{ for } s \neq 0\} \subseteqq H$, and let $\mu = m_{K_0}$ (normalized Haar measure). Obviously μ is positive definite on G; indeed, $\mu = m_{K_0} * \delta_e * m_{K_0}$ and $\delta_e \succ 0$ is obvious (see Lemma A. 6).

If $(\pi^\mu, \mathscr{H}^\mu)$ were cyclic, then by Theorem 4.2 there would exist open subgroup $G' \subseteqq G$ and closed subgroup $K \subseteqq G'$ (normal in G') such that

(i) μ is constant on double cosets $K\backslash G/K$,

(ii) G'/K first countable.

However, any open subgroup G' must contain all of H, while μ cannot be constant on left cosets of any subgroup larger than K_0. Since the quotient space H/K_0 is not first countable, the necessary G' and K cannot exist, and $(\pi^\mu, \mathscr{H}^\mu)$ cannot be cyclic.

Next we indicate the present status of the conjecture. If $\mu \succ 0$ and $(\pi^\mu, \mathscr{H}^\mu)$ has a cyclic vector ζ and if $\{h_n\} \subseteqq C_c(G)$ are chosen so that $\|[h_n] - \zeta\|_\mu \to 0$, we may construct open subgroup $G' \subseteqq G$ and compact subgroup $K \subseteqq G'$ (K normal in G', but not necessarily in G) such that

(i) G'/K first countable

(ii) $h_n * m_K = h_n$ for $n = 1, 2, \cdots$ (m_K = Haar measure).

If $j_\mu \colon C_c(G) \to \mathscr{H}^\mu$ is the canonical map,

$$\mathscr{M} = ls\{\varphi * h_n \colon \varphi \in C_c(G),\ n = 1, 2, \cdots\} \ ,$$

and $\mathscr{M}^\mu = j_\mu(\mathscr{M})$, we showed that $\mathscr{M}^\mu$ is $\|\cdot\|_\mu$-dense in $\mathscr{H}^\mu$. The "smoothed" measure $\mu_K = m_K * \mu * m_K$ is positive definite on G and, in addition, is constant on $K\backslash G/K$ cosets. In certain circumstances we have been able to show that $\mu = \mu_K$; the conjecture would be verified if this could be demonstrated in general. We have not been able to do this so far. However, there is an observation concerning the representations $(\pi^\mu, \mathscr{H}^\mu)$ and $(\pi^{\mu,K}, \mathscr{H}^{\mu,K})$ which strengthens our belief that the conjecture is valid.

THEOREM 5.2. *The representations* π^μ *and* $\pi^{\mu,K}$ *are unitarily equivalent.*

Proof. Consider the scheme of mappings in Figure 2.

$$\begin{array}{ccccc}
C_c(G) \supseteqq \mathscr{M} & \xrightarrow{j_\mu} & \mathscr{M}^\mu & \xrightarrow{id} & \mathscr{H}^\mu \\
\downarrow id & & \downarrow \tilde{J} & & \downarrow J \\
C_c(G) \supseteqq \mathscr{M} & \xrightarrow{j_{\mu,K}} & \mathscr{M}^{\mu,K} & \xrightarrow{id} & \mathscr{H}^{\mu,K}
\end{array}$$

FIGURE 2

For $f = \varphi * h_m$; $g = \psi * h_n$ in $\mathscr{M}$ we get

$$(f, g)_\mu = \int h_n^* * \psi^* * \varphi * h_m d\mu = \int m_k * h_n^* * \psi * h_m * m_K \, d\mu$$
$$= \int h_n^* * \psi^* * \varphi * h_m d[m_K * \mu * m_K] = (f, g)_{\mu_K} ,$$

so id: $\mathscr{M} \to \mathscr{M}$ induces an isometry $\tilde{J}$: $\mathscr{M}^\mu \to \mathscr{M}^{\mu,K}$. We noted that $\mathscr{M}^\mu$ is norm dense in $\mathscr{H}^\mu$; but $\mathscr{M}^{\mu,K}$ is also norm dense in $\mathscr{H}^{\mu,K}$. In fact, if $f \in C_c(G)$ and $\varepsilon > 0$, there exist $\varphi \in C_c(G)$ and h_n such that $\|[\varphi * h_n] - [f * m_k]\| < \varepsilon$, by density of $\mathscr{M}^\mu$ in $\mathscr{H}^\mu$. But,

$$\|f\|_{\mu,K}^2 = \int f^* * f \, d[m_K * \mu * m_K] = \int m_K * f^* * f * m_K \, d\mu$$
$$= \int (f * m_K)^* * (f * m_K) \, d\mu = \|f * m_K\|_\mu ,$$

all $f \in C_c(G)$, so that (recall $h_n = h_n * m_K$, all $n = 1, 2, \cdots$)

$$\|\varphi * h_n - f * m_K\|_\mu = \|\varphi * h_n * m_K - f * m_K\|_\mu = \|\varphi * h_n - f\|_{\mu_K} < \varepsilon ,$$

as required for density of $\mathscr{M}^{\mu,K}$. Thus $\tilde{J}$ extends to an onto isometry J: $\mathscr{H}^\mu \to \mathscr{H}^{\mu,K}$. It is obvious that $\tilde{J}$ is equivariant with respect to left translations, so the unitary equivalence $\pi \cong \pi^{\mu,K}$ is proved.

Let us say that two positive definite measures μ, ν are equivalent $\mu \sim \nu$ if $\pi^\mu \cong \pi^\nu$ (unitary equivalent representations).

COROLLARY 5.3. *If $\mu \succ 0$ and π^μ is cyclic, then there exists a $\nu \succ 0$ such that $\mu \sim \nu$ and ν satisfies the conditions set forth in the conjecture.*

COROLLARY 5.4. *If $\mu \succ 0$ and π^μ is cyclic, then there is an equivalent measure of the very special form $\mu_K = m_K * \mu * m_K$ (K a compact subgroup of G) that satisfies the conditions set forth in the conjecture.*

These results give necessary conditions for the existence of a cyclic vector for π^μ, but it does not seem to be very helpful to know only that some measure in the *equivalence class* of μ is related to the algebraic structure of G as stated in the conjecture. In particular, it does not seem possible to work out Example 5.1 using these results alone.

Two measures $\mu \succ 0$ and $\nu \succ 0$ can be distributed on G in very different ways and still produce the same unitary representation of G, but if ν is required to have the special form $\nu = m_K * \mu * m_K$ it seems unlikely that ν could produce the same representation as μ unless we actually have $\mu = \nu$. If so, this would prove the conjecture. We have not been able to decide this (isolated) question about positive definite measures in cases not already covered in Theorems 4.2 and 4.4.

6. APPENDIX. Convolution of two *finite* measures μ, ν is defined so that

$$\langle \mu * \nu, f \rangle = \iint f(st) d\mu(s) d\nu(t) = \iint f(st) d\nu(t) d\mu(s) ;$$

for $f \in C_c(G)$. If μ is arbitrary, the convolutions $\mu * \nu$ and $\nu * \mu$ are defined if ν has compact support. Then Fubini's theorem applies because

$$\begin{aligned}\iint |f(st)| d|\nu|(s) d|\mu|(t) &\leqq \int_{\mathrm{supp}(\nu)} \|f\|_\infty \cdot |\mu| (s^{-1} \cdot \mathrm{supp}\,(f) d|\nu|(s) \\ &\leqq \|f\|_\infty \cdot \|\nu\| \cdot |\mu| \,(\mathrm{supp}\,(\nu)^{-1} \cdot \mathrm{supp}\,(f));\end{aligned}$$

$|\mu|$ is applied to a compact set, whose measure is finite. If a measure $\mu \in M(G)$ is convolved with any function $f \in C_c(G)$, by identifying f with the Radon measure $f \cdot m_G$, the resulting measures $f * \mu$ and $\mu * f$ are given by continuous functions on G (multiplying Haar measure m_G):

$$\left.\begin{aligned} f * \mu(s) &= \int f(st^{-1}) \Delta_G(t)^{-1} d\mu(t) \\ \mu * f(s) &= \int f(t^{-1}s) d\mu(t) \end{aligned}\right\} \quad \text{all } s \in G .$$

In fact, for all $g \in C_c(G)$,

$$\begin{aligned}\langle \mu * f, g \rangle &= \iint g(ts) d\mu(t) f(s) ds = \iint f(t^{-1}s) g(s) d\mu(t) ds \\ &= \int g(s) \left[\int f(t^{-1}s) d\mu(t) \right] ds .\end{aligned}$$

Obviously $\varphi(s) = \int f(t^{-1}s) d\mu(t)$ is continuous, since μ is a regular Borel measure and $\mathrm{supp}\,(f)$ is compact. Similarly,

$$\langle f * \mu, g \rangle = \iint g(st) f(s) ds\, d\mu(t) = \int g(s) \left[\int f(st^{-1}) \Delta_G(t)^{-1} d\mu(t) \right] ds .$$

Constancy of measures on cosets. Constancy on left or right cosets are quite different conditions on a general measure $\mu \in M(G)$; however, if μ is *positive definite* they amount to the same thing.

THEOREM A. 1. *Let K be any closed subgroup of G. A positive definite measure $\mu \succ 0$ is constant on right cosets of $K \Leftrightarrow$ it is constant on left cosets ($\Leftrightarrow$ constant on double cosets $K\backslash G/K$).*

To prove this, we must assemble certain facts about involutions and convolutions of measures. First note that, if $\lambda, \nu \in M(G)$ and one of these has compact support (so that $\lambda * \nu$ is defined), then

$$\Delta_G \cdot (\lambda * \nu) = (\Delta_G \lambda) * (\Delta_G \nu) ;$$

because Δ_G is a homomorphism, we have

$$\begin{aligned}\int f(r) d[\Delta_G \cdot (\lambda * \nu)](r) &= \int f(r) \Delta_G(r) d[\lambda * \nu](r) \\ &= \iint f(st) \Delta_G(s) \Delta_G(t) d\lambda(s) d\nu(t) \\ &= \int f(r) d[(\Delta_G \lambda) * (\Delta_G \nu)](r)\end{aligned}$$

for all $f \in C_c(G)$. Below, we demonstrate that the adjoint μ^* of any positive definite measure μ is given by $\mu^* = \Delta_G^{-1} \cdot \mu$. A routine use of Fubini's theorem yields:

LEMMA A. 2. *If $\mu \in M(G)$ and $g \in C_c(G)$, let us define $\bar{g}(x) = \overline{g(x)}$ (complex conjugate), $\tilde{g}(x) = g(x^{-1})$ and $g^\vee(x) = g(x^{-1}) \Delta_G(x^{-1}) = \overline{g^*(x)}$. Then*

(i) $\langle \mu, g^* * f \rangle = \langle \bar{g} * \mu, f \rangle$

(ii) $\langle \mu, g * f \rangle = \langle g^\vee * \mu, f \rangle$ *all* $f \in C_c(G)$

(iii) $\langle \mu, f * g \rangle = \langle \mu * \tilde{g}, f \rangle$

LEMMA A. 3. *If $\mu \in M(G)$ and if $\{e_\alpha : \alpha \in I\}$ is any nonnegative approximate identity in $C_c(G)$, with* supp (e_α) *eventually within any neighborhood of the unity, let us define the "smoothed" measures $\mu_\alpha = \bar{e}_\alpha * \mu * \tilde{e}_\alpha$. Then*

(i) $\mu_\alpha \xrightarrow{(\sigma)} \mu$ $[(\sigma) =$ *weak-$*$ topology*$]$.

If μ is positive definite, the μ_α are also positive definite measures; furthermore,

(ii) $\mu^* = \Delta_G^{-1} \cdot \mu$ *if* $\mu \succ 0$.

Proof. Here (σ) is the $\sigma(M(G), C_c(G))$-topology. Clearly, $e_\alpha^* * f * e_\alpha \to f$ in the inductive limit topology of $C_c(G)$, so that

$$\langle \mu_\alpha, f \rangle = \langle \bar{e}_\alpha * \mu * \tilde{e}_\alpha, f \rangle = \langle \mu, e_\alpha^* * f * e_\alpha \rangle \longrightarrow \langle \mu, f \rangle ,$$

proving (i). It is also easy to show that $\mu \to \Delta_G^{-1} \cdot \mu$ and $\mu \to \mu^*$ are (σ)-continuous mappings on $M(G)$. Now take $\mu \succ 0$; the "smoothed" measures μ_α are represented by continuous functions $\varphi_\alpha(x)$ on G, so that $\mu_\alpha = \varphi_\alpha \cdot m_G$. Now $\mu_\alpha \succ 0$, because

$$\langle \mu_\alpha, f^* * f \rangle = \langle \mu, e_\alpha^* * f^* * f * e_\alpha \rangle = \langle \mu, g^* * g \rangle \geqq 0$$

where $g = f * e_\alpha$; thus φ_α is a continuous positive definite function on G. For such a function φ, it is well known that $\varphi(x) = \overline{\varphi(x^{-1})}$ on G;

thus $\varphi^*(x) = \overline{\varphi(x^{-1})}\Delta_G(x^{-1}) = \varphi(x)\cdot\Delta_G(x)^{-1}$, and $\mu_\alpha^* = \Delta_G^{-1}\cdot\mu_\alpha$ for all α. By continuity, we get

$$\mu^* = \lim_\alpha \mu_\alpha^* = \Delta_G^{-1}\cdot(\lim_\alpha \mu_\alpha) = \Delta_G^{-1}\cdot\mu \ .$$

In reaching this conclusion, it is essential that $\mu \succ 0$.

Proof of Theorem A. 1. Suppose that $\mu \succ 0$; then

$$\mu*(\Delta_G\nu) = \Delta_G\cdot((\Delta_G^{-1}\mu)*\nu) = \Delta_G\cdot(\mu^**\nu) = \Delta_G\cdot(\nu^**\mu)^* \ .$$

Since $P_c(K)^* = P_c(K)$, and $\mu^* = \Delta_G^{-1}\mu$, we easily conclude that

$$\nu*\mu = \mu \text{ (all } \nu\in P_c(K)) \Longleftrightarrow \mu*(\Delta_G\nu) = \mu \text{ (all } \nu \in P_c(K) \ .$$

Now let us consider the relationship between a measure $\mu \succ 0$ and $\operatorname{Ker}\pi^\mu$.

THEOREM A. 4. *Let K be a closed normal subgroup in G, and μ any positive definite measure. Then $K \subseteqq \operatorname{Ker}\pi^\mu \Leftrightarrow \mu$ is constant on right cosets $K\backslash G$ of $K \Leftrightarrow \mu$ is constant on left cosets G/K.*

COROLLARY A. 5. *Let $\mu \succ 0$ on G. Then $\operatorname{Ker}\pi^\mu$ is the largest closed, normal subgroup $K \subseteqq G$ such that μ is constant on left (or right) cosets of K.*

Proof. The last ($\Leftrightarrow$) follows immediately via Theorem A. 1, so we only prove the first one.

Note that $y \in \operatorname{Ker}\pi^\mu \Leftrightarrow \pi_y^\mu = I \Leftrightarrow$

$$(\pi_y^\mu[f], [g])_\mu = \mu(g^**\delta_y*f) = ([f], [g])_\mu = \mu(g^**f) \tag{7}$$

for all $f, g \in C_c(G)$, since vectors $[f]$ are dense in $\mathscr{H}^\mu$. Obviously $K \subseteqq \operatorname{Ker}\pi^\mu \Leftrightarrow$ (7) holds for all $y \in K$; however this happens $\Leftrightarrow$

$$\mu(g^**\nu*f) = \mu(g^**f) \text{ all } f, g \in C_c(G) \ , \tag{8}$$

where ν is any convex sum $\nu = \sum \lambda_i\delta_{y_i}$ of point masses ($y_i \in K$). Let $P_c(K)$ be the probability measures on K with compact support.

In the space $M_1(G)$ of all *finite* (i.e., finite total variation) measures ν on G we introduce a strong operator topology by letting measures ν act by left convolution on $L^1(G)$; thus, by definition,

$$\nu_\alpha \xrightarrow{(so)} \nu \Longleftrightarrow ||\nu_\alpha*f - \nu*f||_1 \to 0 \text{ all } f \in L^1(G) \ .$$

On any compact set $Q \subseteqq G$, the (so)-closed convex hull of the point masses $\{\delta_y: y \in Q\}$ is equal to the weak-$*$ closed hull, which is exactly the set of probability measures supported on Q. (Details: Greenleaf

[4], §1.) Let $\nu \in P_c(K)$, and $Q = \text{supp}(\nu)$; there is a net of finite convex combinations $\nu_\alpha = \sum\{\lambda(\alpha, k)\delta_k : k \in Q\}$ such that $\nu_\alpha \xrightarrow{(so)} \nu$, which implies that $\|\nu_\alpha * f - \nu * f\|_1 \to 0$ and $\|g^* * \nu * f - g^* * \nu * f\|_\infty \to 0$ for $f, g \in C_c(G)$. At the same time, all supports lie within a single compact set

$$\text{supp}(g^* * \nu_\alpha * f) \subseteq \text{supp}(g)^{-1} \cdot Q \cdot \text{supp}(f), \text{ all } \alpha ,$$

so that $g^* * \nu_\alpha * f \to g^* * \nu * f$ in the inductive limit topology on $C_c(G)$. Thus (7) holds for all $y \in K \Leftrightarrow$

$$(9) \qquad \mu(g^* * \nu * f) = \mu(g^* * f) \text{ all } \nu \in P_c(K), \text{ all } f, g \in C_c(G) .$$

Now examine left/right-hand sides in (9). We have

$$(10) \qquad \begin{aligned} \langle \mu, g^* * \nu * f \rangle &= \iint g^*(s)(\nu * f)(s^{-1}x)\,ds\,d\mu(x) \\ &= \iiint g^*(s) f(y^{-1}s^{-1}x)\,d\nu(y)\,ds\,d\mu(x) \\ &= \iiint \overline{g(s)} f(y^{-1}sx)\,d\nu(y)\,ds\,d\mu(x) \\ &= \int \overline{g(s)} \left[\iint f(y^{-1}sx)\,d\nu(y)\,d\mu(x)\right] ds , \end{aligned}$$

while

$$(11) \qquad \langle \mu, g^* * f \rangle = \int \overline{g(s)} \left[\int f(sx)\,d\mu(x)\right] ds .$$

Therefore, $\langle \mu, g^* * \nu * f \rangle = \langle \mu, g^* * f \rangle$ for all $f, g \in C_c(G)$, if and only if

$$\begin{aligned} \langle \delta_s * \mu, f \rangle &= \int f(sx)\,d\mu(x) = \iint f(y^{-1}sx)\,d\nu(y)\,d\mu(x) \\ &= \int \left[\int \overline{\overline{f(y^{-1}sx)}\,d\nu(y)}\right] d\mu(x) \\ &= \iint f(ysx)\,d\nu^*(y)\,d\mu(x) \\ &= \iint f(yx)\,d[\nu^* * \delta_s](y)\,d\mu(x) \\ &= \langle \nu^* * \delta_s * \mu, f \rangle \end{aligned}$$

for all $s \in G$, $f \in C_c(G)$, which is the same as saying that

$$(12) \qquad \begin{aligned} \delta_s * \mu &= \nu^* * \delta_s * \mu \\ \mu &= (\delta_{s^{-1}} * \nu^* * \delta_s) * \mu \end{aligned} \qquad \text{all } \nu \in P_c(K), \text{ all } s \in G .$$

Note that this equality holds for *all* $s \in G$ (not just m_G-a.e.), because the expression $[\cdots]$ in (10) and (11) vary continuously in s for each choice of f. The map $\nu \to \delta_{s^{-1}} * \nu * \delta_s$ is a convolution automorphism of

$P_c(K)$ since K is normal in G; furthermore, $\nu \in P_c(K) \Leftrightarrow \nu^* \in P_c(K)$. Thus, $\delta_{s^{-1}} * P_c(K)^* * \delta_s = P_c(K)$, so that (9) holds $\Leftrightarrow$

$$\mu = \nu * \mu \text{ all } \nu \in P_c(K) . \tag{13}$$

If K is a *compact* subgroup in G, then normalized Haar measure $m_K \in P_c(K)$. It is easily seen that a measure $\mu \in M(K)$ is constant on right (resp. left) cosets of $K \Leftrightarrow$

$$m_K * \mu = \mu \qquad (\text{resp. } \mu * m_K = \mu) . \tag{14}$$

Note that $\Delta_G(k) = 1$ on K since $\Delta_G \colon K \to \boldsymbol{R}$ is a homomorphism.

LEMMA A. 6. *If $K \subseteqq G$ is a compact subgroup, and if $\mu \succ 0$ on G, then $\mu_K = m_K * \mu * m_K$ is positive definite. For any $\mu \in M(G)$ we have*:

$$\langle m_K * \mu, f \rangle = \langle \mu, m_K * f \rangle$$
$$\langle \mu * m_K, f \rangle = \langle \mu, f * m_K \rangle$$
$$\langle \mu_K, f \rangle = \langle m_K * \mu * m_K, f \rangle = \langle \mu, m_K * f * m_K \rangle ;$$

moreover,

$$f * m_K(s) = \int_K f(st) dm_K(t)$$
$$m_K * f(s) = \int f(t^{-1}s) dm_K(t) .$$

Proof. To get the above formula for $f * m_K$, we note that $\Delta_G(k) = + 1$ on K, so that

$$f * m_K(s) = \int_K f(st^{-1}) \Delta_G(t^{-1}) dm_K(t) = \int_K f(st^{-1}) dm_K(t) = \int_K f(st) dm_K(t) .$$

Now the above formulas follow by routine calculations.

To see that $\mu_K \succ 0$, use these formulas, and the obvious fact $m_K = m_K^*$, to conclude that $\langle \mu_K, f^* * f \rangle = \langle \mu, m_K * f^* * f * m_K \rangle = \langle \mu, g^* * g \rangle \geqq 0$, where $g = f * m_K$.

Shifting measures to quotient groups. Suppose that K is a closed normal subgroup on G, and let left Haar measures dk and dg be fixed. Then left Haar measure $d\beta(xK)$ on G/K can be normalized so that

$$\int_G f(x) dx = \int_{G/K} \varphi' f(xK) d\beta(xK) \text{ all } f \in C_c(G)$$

where

$$\varphi' f(xK) = \int_K f(xk)dk \text{ all } x \in G .$$

The mapping $\varphi': C_c(G) \to C_c(G/K)$ is a surjection. (See Bourbaki [1], Ch. 7 §§2.2 and 2.8.)

THEOREM A. 7. *The map φ' is a $*$-homomorphism between convolution algebras with involution.*

Proof. Recall that $f^*(x) = \overline{f(x^{-1})}\Delta_G(x^{-1})$. Note that modular functions agree, $\Delta_G | K = \Delta_K$; furthermore, if $\alpha_x = K \to K$ is the automorphism $\alpha_x(k) = xkx^{-1}$ for $x \in G$, $k \in K$, this map alters Haar measure on K by a scalar, mod (α_x), which is given by $\Delta_{G/K}(xK) = \Delta_G(x) \cdot \text{mod }(\alpha_x)$ (Bourbaki [1], Ch. 7, §2.7). Thus $\varphi'(f^*) = (\varphi' f)^*$, because

$$\begin{aligned}
\varphi'(f^*)(xK) &= \int_K f^*(xk)dk = \int_K \overline{f(k^{-1}x^{-1})}\Delta_G(k^{-1})\Delta_G(x^{-1})dk \\
&= \Delta_G(x^{-1})\int_K \overline{f(x^{-1}\cdot \alpha_x(k^{-1}))}\Delta_K(k^{-1})dk \\
&= \Delta_G(x^{-1})\int_K \overline{f(x^{-1}\cdot \alpha_x(k))}dk \\
&= \Delta_G(x^{-1}) \text{ mod } (\alpha_x^{-1})\overline{\int_K f(x^{-1}k)dk} \\
&= \Delta_{G/K}(x^{-1}K)\overline{\varphi' f(x^{-1}K)} = (\varphi' f)^*(xK) .
\end{aligned}$$

As for convolution, let $\pi: G \to G/K$ be the quotient homomorphism; by routine calculations, $\varphi'(\delta_y * f)(xK) = \delta_{\pi(y)} * \varphi' f(xK) = \varphi' f(y^{-1}xK)$ for $x, y \in G$. Now represent convolution of function with compact support as a weak vector valued integral $f * g = \int_G f(y)\delta_y * g dy$; the continuity properties of φ' easily lead to the formula

$$\varphi'(f * g) = \int_G f(y)\varphi'(\delta_y * g)dy;$$

thus,

$$\begin{aligned}
\varphi(f * g) &= \int_G f(y)\delta_{yK} * \varphi'(g)dy \\
&= \int_{G/K}\left[\int_K f(yk)dk\right]\delta_{yK} * \varphi' f d\beta(yK) \\
&= \int_{G/K} \varphi' f(yK)(\delta_{yK} * \varphi' g)d\beta(yK) = (\varphi' f) * (\varphi' g) ,
\end{aligned}$$

for $f, g \in C_c(G)$.

THEOREM A. 8. *Defining* $\varphi'\colon C_c(G) \to C_c(G/K)$ *as above, let* μ *be any Radon measure constant on right K-cosets. There exists a unique Radon measure* μ' *on* G/K *such that*

$$\langle \mu', \varphi' f\rangle = \langle \mu, f\rangle \text{ all } f \in C_c(G) .$$

Furthermore, $\mu \succ 0$ *on* $G \Rightarrow \mu' \succ 0$ *on* G/K.

Proof. Uniqueness is clear since φ' is surjective; positivity, since φ' is a $*$-homomorphism between convolution algebras. The existence of μ' is proved (in a more general context) as Proposition 4, in Bourbaki [1], Ch. 7, §2.2.

Our final results in the appendix constitute a slight, but useful, generalization of a result of Glushkov [6] (G compactly generated) which in turn slightly generalizes the corresponding result of Montgomery and Zippin [8] (G/G_0 compact). It is to be hoped that this is the final generalization of this result.

THEOREM A. 9. *Let* G *be a locally compact group. If* G *is* σ-*compact and* $\{W_i\}$ *is a countable family of* nbds. *of* 1 *then there exists a compact normal subgroup* K *of* G *such that* $K \subseteqq \bigcap_i W_i$ *and* G/K *is second countable.*

Proof. Let G equal the union of compact sets F_m. By induction choose a sequence $\{U_m\}$ of compact symmetric nbds of 1 in G such that $U_m^2 \subseteqq U_{m-1} \cap W_m$ and (since each F_m is compact) such that $F_m \Delta U_m \subseteqq U_{m-1}$, where Δ is the action of G on itself by conjugation. Let $K = \bigcap_{m=1}^{\infty} U_m$. Then K is a compact subgroup of G contained in $\bigcap_i W_i$. By taking finite unions we may also assume the F_m's are nested. If $x \in G$ and $y \in K$ then $x \in F_{m_0}$ and therefore $x \in F_m$ for all $m \geqq m_0$. It follows that $x \Delta y \in U_m$ for $m \geqq m_0 - 1$. But the U_i are also nested so $x \Delta y \in U_m$, $i = 1, \cdots, m_0 - 2$; i.e, $x \Delta y \in K$. If V is an open set in G containing K then

$$\bigcap_m (U_m \cap (G - V)) = \bigcap_m U_m \cap (G - V) = K \cap (G - V) = \varnothing .$$

By the finite intersection property and the fact that the U_m are nested, it follows that $U_m \cap G - V = \varnothing$, i.e., $U_m \subseteqq V$. Thus $\{\pi(V_m)\}$ are a nbd. basis in G/K where $\pi\colon G \to G/K$ is the canonical epimorphism. Since G is σ-compact so is G/K. The metrization theorem [8] now completes the proof.

From A. 9 it follows immediately that 2nd countable groups are characterized as those which are σ-compact and first countable. (Although this can, of course, be seen in a more elementary way.) In addition, A. 9 enables one to characterize σ-compact groups.

COROLLARY A. 10. *Let G be a locally compact group; then G is σ-compact if and only if G is the projective limit of second countable groups.*

Proof. If G is σ-compact then simply take a single neighborhood W and apply A 9. Conversely, if G is the projective limit of second countable groups consider a single approximating group G/K. Since it is σ-compact and K is compact, the result follows.

COROLLARY A. 11. *Let G be a locally compact σ-compact group and $\{h_n\}$ be a sequence of continuous functions on G with values in the separable metric space (X, d). Then there exists a compact normal subgroup K of G such that G/K is* 2nd *countable and each h_n is constant on cosets of K.*

Proof. Let $G = \bigcup_{m=1}^{\infty} F_m$ where F_m are compact sets. By local compactness of G each F_m is contained in a compact neighborhood so we may, assume they are also nbds. As in the proof of A9 we may also assume the F_m's are increasing. Let $\{\mathcal{O}_j\}$ be a basis for open sets of X and consider the sets $A_{njm} = h_n^{-1}(\mathcal{O}_j) \cap F_m$ where $n, j, m = 1, 2, \cdots$, which are nonempty and then those pairs (n, j, m) and (n', j', m') for which $(A_{njm})^- \cap (A_{n'j'm'})^- = \varnothing$. For each such pair choose a neighborhood W of 1 in G such that $A_{njm}W \cap A_{n'j'm'}W = \varnothing$. From this one obtains a family of nbds. $\{W_i\}$ of 1 in G. Choose K as in A.9. We show $h_n(xy) = h_n(xz)$ for each $x \in G$, $y, z \in K$ and integer n. If not then for some choice of the variables $\delta = d(h_n(xy), h_n(xz)) > 0$. Since xK is compact and $\{F_m\}$ are nested, $xK \subseteqq F_m$ for m sufficiently large. Let $\mathcal{O}_j$ and $\mathcal{O}_{j'}$ be basis elements of diam $< \delta/3$ which contain $h_n(xy)$ and $h_n(xz)$ respectively. Then by continuity of h_n and the fact that the A_{njm} are precompact, A_{njm} and $A_{nj'm}$ have disjoint closures and contain xy and xz respectively. They therefore determine a W as above. It follows that $x \in A_{njm}\, y^{-1} \subseteqq A_{njm} \cdot K \subseteqq A_{njm} \cdot W$. Similarly x is also in $A_{nj'm} \cdot W$, a contradiction.

COROLLARY A. 12. *Let G a σ-compact locally compact group and ρ a continuous finite dimensional representation (not necessarily unitary) or even a weakly continuous representation of G on a separable Hilbert space. Then ρ actually lives on some* 2nd *countable quotient group of G. (This automatically extends the Mackey theory from* 2nd *countable groups to σ-compact groups.)*

Proof. Let $\{v_i\}$ be an orthonormal basis of $\mathscr{H}$ and $\rho_{ij}(x) = (\rho_x(v_i), v_j)$, $x \in G$; then $\{\rho_{ij}\}$ is a countable family of continuous functions. By all there exists a K such that ρ_{ij} are all constant on cosets of K. Since if $x \in G$ and $y \in K$, $\rho_{ij}(xy) = \rho_{ij}(x)$ for all i, j, we get $\rho(xy) = \rho(x)$.

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