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COMMUTATIVE ENDOMORPHISM RINGS

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Two problems of W. V. Vasconcelos are partially solved: (1) The total quotient ring of a commutative noetherian ring R is quasi-frobenius if and only if $\mathrm{End}_R(A)$ is commutative for each ideal A of R. (2) Let R be a commutative quasi-regular ring and E a finitely presented R-module. If E is faithful and $\mathrm{End}_R(E)$ is semi-prime, then E is isomorphic to an ideal of R. Only commutative rings with unit and unital modules are considered.

1. In [3] Vasconcelos considers problems concerning commutative endomorphism rings. Toward the end he asks for a characterization of rings R for which $\mathrm{End}(A)$ is commutative for each ideal $A \subset R$. He conjectures the following answer for noetherian rings.

THEOREM 1.1. *Let R be a noetherian ring with total quotient ring T. If $\mathrm{End}_R(A)$ is commutative for each ideal A of R then T is quasi-frobenius.*

Proof. It is sufficient to show that for each maximal ideal p of T the local ring $S = T_p$ has Krull dimension zero and $\mathrm{Ann}_S(q)$ is a one dimensional S/q-vector space where $q = pS$ [2, Theorem 221]. Each ideal I of S has a commutative endomorphism ring since we can select $J \subset R$ such that $J \otimes S = I$ and observe that the natural map $\mathrm{End}_R(J) \otimes_R S \to \mathrm{End}_S(J \otimes_R S)$ is a ring isomorphism [1, p. 39, Proposition 11]. Also because R is noetherian $q = \mathrm{Ann}_S(a)$ for some $a \in S$ [2, Theorem 82]. Since $\mathrm{End}_{S/q}(\mathrm{Ann}_S(q)) = \mathrm{End}_S(\mathrm{Ann}_S(q))$ is commutative and $\mathrm{Ann}_S(q) \neq 0$, then $\mathrm{Ann}_S(q)$ is a one dimensional S/q-vector space. It remains to show S is zero dimensional, i.e., q is nilpotent. Since $a \neq 0$, then there is an integer n such that $a \notin q^n$ [2, Theorem 79]. Since $\mathrm{Ann}_S(q)$ is simple we must have $Sa = \mathrm{Ann}_S(q)$ and $q^n \cap Sa = 0$. Now we show $q^n = 0$. Suppose not, choose $b \in q^n$, $b \neq 0$. Then $\mathrm{Ann}_S(b) \subset q = \mathrm{Ann}_S(a)$. Thus the correspondence $xb \to xa$ defines an S-homomorphism $f: Sb \to Sa$. Let $J = Sa + Sb$. This sum is direct since $Sa \cap Sb \subset Sa \cap q^n = 0$. Let $u, w \in \mathrm{End}_S(J)$ be the following composites:

$$u: J \longrightarrow Sa \subset J$$

$$w: J \longrightarrow Sb \xrightarrow{f} Sa \subset J .$$

Then $uw(b) = a$ and $wu(b) = 0$ contradicting the commutativity of $\mathrm{End}_S(J)$. Thus $q^n = 0$, and S is zero dimensional.

The converse to 1.1 is true also. This is because T is an injective T-module if T is quasi-frobenius. Indeed if R is any commutative ring whose total quotient ring T is T-injective then for $A \subset R$, $\mathrm{End}_R(A)$ can be viewed as a subring of $\mathrm{End}_T(AT)$ which is a homomorphic image of T and therefore commutative.

The next proposition gives a sufficient condition on $A \subset R$ for $\mathrm{End}(A)$ to be commutative.

PROPOSITION 1.2. *Let R be a commutative ring and A an ideal of R. If $A \cap \mathrm{Ann}(A) = 0$, then $\mathrm{End}_R(A)$ is commutative.*

Proof. Let $f, g \in \mathrm{End}_R(A)$ and $c = fg - gf$. For $a, b \in A$ we have $ac(b) = c(ab) = f(g(ab)) - g(f(ab)) = f(ag(b)) - g(bf(a)) = g(b)f(a) - f(a)g(b) = 0$. Hence $Ac(A) = 0$ implies $c(A) \subset A \cap \mathrm{Ann}(A) = 0$. Therefore $\mathrm{End}(A)$ is commutative.

An R-algebra will be called semi-prime if it has no non-zero nilpotent elements.

COROLLARY 1.3. *If R is semi-prime then* $\mathrm{End}(A)$ *is commutative and semi-prime for each ideal $A \subset R$.*

Proof. $A \cap \mathrm{Ann}(A)$ consists of nilpotents so $\mathrm{End}(A)$ is commutative by 1.2. If $f \in \mathrm{End}(A)$ is nilpotent, say $f^n = 0$, then for $x \in A$ $0 = f^n(x^n) = (f(x))^n$. Since R is semi-prime $f(x) = 0$ for $x \in A$. Thus $f = 0$.

If R is an integral domain, we can characterize the ideals of R as those torsionless R-modules E having $\mathrm{End}(E)$ commutative. For if $x \in E$ $x \neq 0$ there is $f: E \rightarrow R$ with $f(x) \neq 0$. Let $y \in E$. The two homomorphisms $z \rightarrow f(z)x$ and $z \rightarrow f(z)y$ commute. Hence $f(y)x = f(x)y$, so $f(y) = 0$ implies $y = 0$. Thus f is injective.

The next section is concerned with how well the property $\mathrm{End}(A)$ is semi-prime distinguishes the ideals of a semi-prime ring R from other R-modules.

2. In [3] Vasconcelos proves that when R is noetherian and semi-prime a finitely generated faithful R-module E with $\mathrm{End}_R(E)$ commutative and semi-prime is isomorphic to an ideal of R. He conjectures that the result may remain valid for a finitely presented E even if R is not noetherian. I could not resolve this but generalize his result to include those rings having an absolutely flat total quotient ring (called quasi-regular rings). The methods make no use of the commutativity of $\mathrm{End}(E)$. Thus in the situation considered (in 2.2 below) semi-prime implies commutativity. Although we are con-

sidering only commutative rings here, our generalization, unlike the original version of the theorem, can at least be conjectured for non-commutative rings.

This is the first step in the proof:

THEOREM 2.1. *Let R be a ring and E a finitely present R-module. If $x \in E$ is nonzero, then there exists $f \in \operatorname{End}(E)$ nonzero such that $f(E) \subset Rx$.*

Proof. First suppose R is noetherian. Let p be a prime minimal over $\operatorname{Ann}(x)$. Then there exists $y \in Rx$ such that $p = \operatorname{Ann}(y)$ [2, Theorem 86]. Localize at p. Let $K = R_p/p_p$. Since $E_p \neq 0$, $E_p \otimes K$ is at least one dimensional by the Nakayama lemma [2, Theorem 78]. Thus there is a surjection $h: E_p \otimes K \to K$. As an R_p-module, $K = (Ry)_p$. Let g be the composite $E_p \to E_p \otimes K \xrightarrow{h} (Ry)_p \subset (Rx)_p$. Since E is finitely presented, we have $\operatorname{Hom}_R(E, Rx)_p \cong \operatorname{Hom}_{R_p}(E_P, Rx_P)$ [1, p. 39, Proposition 11]. Hence $g = f/s$ for some $f: E \to Rx$ and $s \in R \backslash p$. Clearly, f has the required properties. Thus the result holds when R is noetherian. Since E is finitely presented we can use the following well known technique to reduce to the noetherian case: Let $R^m \xrightarrow{A} R^n \xrightarrow{B} E \to 0$ be a presentation of E, select bases f_i, e_j: let $A(f_i) = \sum a_{ij}e_j$ $B(e_j) = m_j$, $x = \sum x_j m_j$ with $a_{ij}, x_j \in R$. Let S be the subring generated by 1 and all the x's and a's. S is noetherian by the Hilbert Basis Theorem. Let F be the S-submodule generated by the m's. Then $F \otimes_S R = E$ and $x \in F$. Since S is noetherian there is nonzero $g: F \to Sx$. Tensoring with S yields a commutative diagram:

$$\begin{array}{ccccc} E & \longrightarrow & Sx \otimes R & \longrightarrow & Rx \\ \cup & & \uparrow & & \| \\ F & \xrightarrow[g]{} & Sx & \subset & Rx \end{array} .$$

Hence we can take f to be the composite of the maps on the upper row.

For an ideal I of R let $\operatorname{Min}(I)$ denote the primes of R minimal over I. For an R-module E let $\operatorname{Ass}(E)$ denote the Bourbaki associated primes of E. Thus $\operatorname{Ass}(E)$ is the union over $x \in E$ of the sets $\operatorname{Min}(\operatorname{Ann}(x))$.

THEOREM 2.2. *Let R be a semi-prime ring, E a finitely presented R-module. If* $\operatorname{End}(E)$ *is semi-prime, then* $\operatorname{End}(E)$ *is commutative and* $\operatorname{Ass}(E) = \operatorname{Min}(\operatorname{Ann}(E))$.

Proof. For any finitely presented R-module E $\operatorname{Min}(\operatorname{Ann}_R(E)) \subset \operatorname{Ass}(E)$ and the mapping $\operatorname{End}(E) \to \prod_{p \in Ass(E)} \operatorname{End}_{R_p}(E_p)$ induced by

$\text{End}(E) \to \text{End}_{R_p}(E_p)$ is an injective ring homomorphism. Thus it is sufficient to establish that if $p \in \text{Min}(\text{Ann}(x))$ and $x \in E$, then $E_p = (Rx)_p$. For then $[\text{End}_R(E)]_p \cong \text{End}_{R_p}(E_p)$ is commutative and $\text{Ann}_{R_p}(Rx_p) = \text{Ann}_{R_p}(E_p) = [\text{Ann}_R(E)]_p$. Thus by relationship between primes of R_p and primes of R contained in p we get $p \in \text{Min}(\text{Ann}(E))$. So let $p \in \text{Min}(\text{Ann}(x))$, $x \in E$. Put $T = R_p$, $q = pT$, $F = E_p$, $y = x/1 \in F$. T is quasi-local semi-prime with maximal ideal $q = \sqrt{\text{Ann}_T(y)}$. By 2.1 there is nonzero $f: F \to F$ with $f(F) \subset Ty$. Let $f(y) = ay$. Then $a \notin q$ else f is nilpotent and consequently zero since $\text{End}_T(F)$ is semi-prime. Let $b \in q$ and define $h = ba^{-1}f$. Then h is nilpotent since $b \in \sqrt{\text{Ann}(y)}$ and $h(F) \subset Ty$. Thus $h = 0$. Hence $0 = h(y) = by$. Therefore $q = \text{Ann}_T(y)$. Let $M = F/Ty$ and suppose $M \neq qM$. Then M/qM is at least one dimensional over T/q so there is a surjection $g: M \to T/q = Ty$. Let k be the composite of the natural map $F \to M$ followed by g. Then $k \in \text{End}(F)$ and $k^2 = 0$. Since $\text{End}(F)$ is semi-prime we get the contradiction $k = 0$ and $Ty \neq 0$. Therefore $M = qM$; and thus $F = Ty$ by Nakayama. Hence $E_p = (Rx)_p$ as required.

THEOREM 2.3. *Let R be quasi-regular ring and E a finitely presented R-module. If* $\text{End}(E)$ *is semi-prime and if there is an ideal $I \subset R$ such that* $\text{Ann}(I) = \text{Ann}(E)$, *then E is isomorphic to an ideal of R.*

Proof. By 2.2 $\text{Ass}(E) = \text{Min}(\text{Ann}(E)) = \text{Min}(\text{Ann}(I))$. Thus each associated prime of E consists of zero divisors of R [2, Theorem 84]. Therefore the natural map $E \to E \otimes T$ is injective where T denotes the total quotient ring of R. Let $F = E \otimes T$. $\text{End}_T(E) \cong \text{End}_R(E) \otimes T$ is semi-prime. Since T is absolutely flat, then F is a direct sum $Te_1 \oplus \cdots \oplus Te_n$ of ideals of T each of which is generated by an idempotent e_i of T [1, Exercise 18, p. 64]. Let $i \neq j$, $h \in \text{Hom}_T(Te_i, Te_j)$. Define $f: F \to F$ by $f(e_k) = 0$ for $k \neq i$ and $f(e_i) = h(e_i)$. Then $f^2 = 0$ and thus $h = 0$. Hence the idempotents e_i are mutually orthogonal and therefore $F = Te_1 + \cdots + Te_n$ is an ideal of T. Now multiplication by a suitable regular element will move the image of E in F inside R.

The hypothesis on $\text{Ann}_R(E)$ in 2.3 is satified when E is faithful.

There is some evidence that 2.3 may be valid for noncommutative rings. For example if R is an absolutely flat semi-prime ring and E a finitely presented right R-module (or more generally a projective right R-module) and if $\text{End}(E)$ is semi-prime then E is isomorphic to an ideal.

Added March 12, 1973. S. Alamelu has independently obtained Theorem 1.1. Her results will appear in the Proceedings of the American Mathematical Society.

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