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COMPLEMENTATION PROBLEMS FOR THE BAIRE CLASSES

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This paper initiates a study of the classes of Baire measurable functions on the unit interval I from the standpoint of the theory of spaces of continuous functions. For each countable ordinal α, the αth Baire class $\mathfrak{B}_\alpha$ has a representation as $C(\Omega_\alpha)$, where Ω_α is a certain compactification of the discrete set I. For $1 \leqq \alpha < \beta$, $\mathfrak{B}_\alpha$ is a closed subalgebra of $\mathfrak{B}_\beta$. The principal result proved here is the fact that $\mathfrak{B}_\alpha$ is always uncomplemented as a closed subspace of $\mathfrak{B}_\beta$. The method of proof relies on a detailed analysis on the canonical onto map $\phi: \Omega_\beta \to \Omega_\alpha$ induced by the imbedding of $\mathfrak{B}_\alpha$ in $\mathfrak{B}_\beta$, and consists of showing that this map admits no "averaging operator." It depends heavily on recent results in the theory of averaging operators due to S. Z. Ditor.

In this paper scalars and functions are real valued. However, the arguments extend easily to the complex case. In the last section we show how corresponding results may be obtained when I is replaced by any uncountable compact metric space.

1. **The Baire classes as function algebras.** We shall start by recalling classical definitions and facts concerning the Baire classes of functions on the unit interval I. Let $C(I)$ be the class of all real continuous functions on I with supremum norm. Denote by $\mathfrak{B}_1$ the class of all bounded functions which are pointwise limits of sequences of functions in $C(I)$, and for each countable ordinal α inductively define $\mathfrak{B}_\alpha$ to be the class of all bounded functions on I which are pointwise limits of sequences of functions in $\bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. We call $\mathfrak{B}_\alpha$ the class of Baire functions of order α.

There is another approach to $\mathfrak{B}_\alpha$. Each countable ordinal α is even or odd as follows: 1 is odd and each limit ordinal is even; the immediate successor of an even ordinal is odd, and of an odd ordinal is even. Let F_0 be the class of all closed subsets of I; and F_1 be the class of countable unions of sets in F_0. For each α

(i) F_α is the class of all countable unions of sets in $\bigcup_{\beta<\alpha} F_\beta$, if α is odd;

(ii) F_α is the class of all countable intersections of sets in $\bigcup_{\beta<\alpha} F_\beta$, if α is even.

Correspondingly, let G_0 be the class of all open subsets of I, and G_1 be all countable intersections of G_0 sets. For each α

(iii) G_α is the class of all countable intersections of sets in $\bigcup_{\beta<\alpha} G_\beta$,

if α is odd;

(iv) G_α is the class of all countable unions of sets in $\bigcup_{\beta<\alpha} G_\beta$, if α is even.

If α is odd (even) the class $F_\alpha(G_\alpha)$ is called the *additive class* of type α.

If α is odd (even) the class $G_\alpha(F_\alpha)$ is called the *multiplicative class* of type α.

For each α define $H_\alpha = F_\alpha \cap G_\alpha$. We call H_α the class of *ambiguous sets of type* α.

The classes $F_\alpha, G_\alpha, H_\alpha$ satisfy many simple relationships. In particular, the complement of an F_α set is in G_α, etc., and H_α is a field of sets (i.e., closed under finite unions and intersections and complements). For a complete discussion see [4] and [6].

Now for each α denote by $\mathfrak{A}_\alpha$ the class of all bounded functions f on I such that for every real λ, the sets

$$\{t \mid f(t) \leqq \lambda\}, \{t \mid f(t) \geqq \lambda\}$$

all belong to the multiplicative class of type α. The following classical result ([6], page 393) connects $\mathfrak{B}_\alpha$ and $\mathfrak{A}_\alpha$.

THEOREM 1.1. (*Lebesgue-Hausdorff*). *If α is a finite ordinal, $\mathfrak{B}_\alpha = \mathfrak{A}_\alpha$. If α is infinite, then $\mathfrak{B}_\alpha = \mathfrak{A}_{\alpha+1}$.*

We apply this theorem to characterize those subsets A of I whose characteristic functions k_A are in $\mathfrak{B}_\alpha$. Observe that $k_A \in \mathfrak{A}_\alpha$ iff $A \in H_\alpha$.

COROLLARY 1.2. *Let A be a subset of I. Then*

(a) *$k_A \in \mathfrak{B}_\alpha$ iff $A \in H_\alpha$, if α is finite.*

(b) *$k_A \in \mathfrak{B}_\alpha$ iff $A \in H_{\alpha+1}$, if α is infinite.*

To avoid constantly considering cases we also define the class $K_\alpha = H_\alpha$ if α is finite and $K_\alpha = H_{\alpha+1}$ if α is infinite.

In what follows we shall need a separation theorem ([6], page 351).

LEMMA 1.3. *Let R and S be disjoint subsets of I which are of multiplicative type α. There exist disjoint sets A and B belonging to H_α such that*

$$R \subseteqq A, \qquad S \subseteqq B.$$

Returning to $\mathfrak{B}_\alpha$, we note that it is an algebra of bounded functions on I which is closed under uniform convergence ([4], page 134). Thus $\mathfrak{B}_\alpha$ is a Banach algebra of bounded functions under the uniform norm satisfying the obvious condition.

$$\|f^2 + g^2\| \geqq \|f^2\|, f, g \in \mathfrak{B}_\alpha .$$

Let Ω_α be the compact set of nonzero multiplicative linear functionals on $\mathfrak{B}_\alpha$ with the weak star topology. Define

$$\hat{f}(\omega) = \omega(f), \omega \in \Omega_\alpha, f \in \mathfrak{B}_\alpha .$$

By the real form of the Gelfand-Naimark theorem (Arens [1]), the map $\chi: f \to \hat{f}$ is an isometric algebra isomorphism of $\mathfrak{B}_\alpha$ onto $C(\Omega_\alpha)$. Since evaluation at a point of I is multiplicative, there is a natural imbedding $\tau_\alpha: I \to \Omega_\alpha$ of I into Ω_α determined by

$$\hat{f}(\tau_\alpha(t)) = f(t), t \in I, f \in \mathfrak{B}_\alpha .$$

Moreover, $\tau_\alpha(I)$ is dense in Ω_α, since if $\hat{f}$ vanishes on $\tau_\alpha(I)$, f must vanish on I. Also $\tau_\alpha(I)$ is discrete in its relative topology as a subset of Ω_α, since the characteristic function of a point of I belongs to $\mathfrak{B}_\alpha$. Thus we may regard Ω_α as a certain compactification of the discrete unit interval.

Our next aim is to prove that Ω_α is totally disconnected. Note that the following statements are equivalent for a function $f \in \mathfrak{B}_\alpha$.

(a) f takes only the values 0 and 1 on I,

(b) $f = k_A$ where $A \in K_\alpha$,

(c) $\hat{f}$ takes only the values 0 and 1 on Ω_α,

(d) $\hat{f} = k_B$ where B is an open and closed subset of Ω_α. Moreover, we have

$$B = \overline{\tau_\alpha(A)}, A = \tau_\alpha^{-1}(B \cap \tau_\alpha(I)) .$$

The correspondence $A \to B$ defines an isomorphism of the Boolean algebra K_α of subsets of I onto the Boolean algebra K^α of all closed and open subsets of Ω_α. To prove that Ω_α is totally disconnected we must prove that K^α separates points of Ω_α.

THEOREM 1.4. *The space Ω_α is totally disconnected.*

Proof. Let ω_1 and ω_2 be distinct points of Ω_α. We can find $\hat{f} \in C(\Omega_\alpha)$ such that the sets

$$U_1 = \{\omega \,|\, \hat{f}(\omega) \geqq 1\}, U_2 = \{\omega \,|\, \hat{f}(\omega) \leqq 0\}$$

are closed neighborhoods of ω_1 and ω_2.

Define $f(t) = \hat{f}(\tau_\alpha(t))$, $t \in I$. Then the sets $A_i = \tau_\alpha^{-1}(U_i \cap \tau(I))$, $i = 1, 2$ satisfy

$$A_1 = \{t \,|\, f(t) \geqq 1\}, A_2 = \{t \,|\, f(t) \leqq 0\} ,$$

and thus are of multiplicative class $\alpha(\alpha+1)$ if α is finite (infinite). By Lemma 1.3 there exist disjoint sets W_1, W_2 in K_α such that $A_i \subseteqq W_i$. The sets $V_i = \overline{\tau_\alpha(W_i)}$ are open and closed neighborhoods of ω_i. They are disjoint, since otherwise their intersection contains a point of the dense set $\tau_\alpha(I)$, contradicting the fact that $W_1 \cap W_2 = \varnothing$.

2. **Topological tools.** In this section we collect some basic facts concerning spaces of continuous functions and projections onto subalgebras. Let S and T be compact Hausdorff spaces and $\phi: S \to T$ be a continuous onto map. We call the elements of the collection

$$\mathfrak{R} = \{\phi^{-1}(t) \mid t \in T\}$$

fibers. Then $\mathfrak{R}$ forms an upper semi-continuous closed set decomposition of S ([5], page 99). Define $\phi^0: C(T) \to C(S)$ by the formula

$$(\phi^0 f)(s) = f(\phi(s)), f \in C(T), s \in S .$$

The map ϕ^0 is an isometric algebra isomorphism of $C(T)$ onto the subalgebra of those functions in $C(S)$ which are constant on each set of $\mathfrak{R}$.

If the closed subalgebra $\phi^0(C(T))$ is the range of a bounded projection P from $C(S)$, we define the *projection constant* $p(\phi)$ to be the infimum of $\|P\|$ for all such P. We define $p(\phi) = +\infty$ if $\phi^0(C(T))$ is uncomplemented. We shall need a result of S. Z. Ditor [2] (formulated here in somewhat different terms) which relates $p(\phi)$ to the topological structure of S and ϕ. Suppose $\{t_\alpha\}$ is a net in T converging to t_0. We define the *cluster set* for $\{t_\alpha\}$ to be

$$\limsup \{\phi^{-1}(t_\alpha)\} = \left\{ \begin{matrix} s \mid \text{ for each } \alpha_0 \text{ and neighborhood } U \text{ of } s, \\ \text{there exists } \alpha \geqq \alpha_0 \text{ with } \phi^{-1}(t_\alpha) \cap U \neq \varnothing \end{matrix} \right\} .$$

The cluster set for $\{t_\alpha\}$ is a nonempty compact subset of the fiber $\phi^{-1}(t_0)$.

DEFINITION 2.1. Let $\phi: S \to T$ be continuous and onto. Define

$$M^{(1)} = \left\{ \begin{matrix} t \mid \text{ there exist nets } \{t_\alpha\}, \{t_\beta\} \text{ converging} \\ \text{to } t \text{ whose cluster sets are disjoint} \end{matrix} \right\} ;$$

and inductively define

$$M^{(n)} = \left\{ \begin{matrix} t \mid \text{ there exist nets } \{t_\alpha\}, \{t_\beta\} \text{ of points of } M^{(n-1)} \\ \text{converging to } t \text{ whose cluster sets are disjoint} \end{matrix} \right\} .$$

Clearly $M^{(1)} \supseteqq M^{(2)} \supseteqq \cdots$. The next theorem is due to S. Z. Ditor [2].

THEOREM 2.2. *If* $M^{(n)} \neq \emptyset$, *then* $p(\phi) \geqq n$. *If* $\bigcap_{n=1}^{\infty} M^{(n)} \neq \emptyset$, *then* $\phi^0(C(T))$ *is uncomplemented in* $C(S)$.

In our application of this result to the Baire classes, matters will be simplified by the fact that the map $\phi: S \to T$ appearing in that context will be irreducible (i.e., there is no proper closed subset S_1 of S such that $\phi(S_1) = T$). In the case of irreducible maps the set $M^{(1)}$ is just the set of multiple points for ϕ. We prove this. The necessary lemma is a consequence of the uppersemicontinuity of $\mathfrak{R}$.

LEMMA 2.3. *Let* $\phi: S \to T$ *be continuous and onto. The following are equivalent.*

(1) ϕ *is irreducible.*

(2) *For each open set* $U \subseteqq S$, *the set*

$$V = \{t \mid \phi^{-1}(t) \subseteqq U\}$$

is a nonempty open set in T.

(3) *For each open set* U *in* S, *the set*

$$W = \cup \{\phi^{-1}(t) \mid \phi^{-1}(t) \subseteqq U\}$$

is an open set dense in U.

PROPOSITION 2.4. *If* $\phi: S \to T$ *is an irreducible onto map, then*

$$M^{(1)} = \{t \mid \phi^{-1}(t) \text{ contains at least two points}\}.$$

Proof. Let s_1, s_2 be distinct points of $\phi^{-1}(t)$ with open neighborhoods U_1, U_2 with $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. By Lemma 2.3 (3), there is a net $\{t_\alpha\} \to t_0$ with $\phi^{-1}(t_\alpha) \subseteqq U_1$. Similarly, there is a net $\{t_\beta\} \to t_0$ with $\phi^{-1}(t_\beta) \subseteqq U_2$. Their cluster sets must be disjoint. Thus every multiple point lies in $M^{(1)}$. The converse inclusion is clear.

3. **Nonexistence of complements.** Let α and β be fixed countable ordinals with $1 \leqq \alpha < \beta$. Then $\mathfrak{B}_\alpha$ is a closed subalgebra of $\mathfrak{B}_\beta$. Under the representation of $\mathfrak{B}_\alpha$ as $C(\Omega_\alpha)$, $\mathfrak{B}_\beta$ as $C(\Omega_\beta)$, the natural imbedding of $\mathfrak{B}_\alpha$ in $\mathfrak{B}_\beta$ induces a canonical continuous onto map $\phi: \Omega_\beta \to \Omega_\alpha$. To show that $\mathfrak{B}_\alpha$ is uncomplemented in $\mathfrak{B}_\beta$ is equivalent to showing that $\phi^0(C(\Omega_\alpha))$ is uncomplemented in $C(\Omega_\beta)$. We do this by proving that the sets $M^{(n)} \subseteqq \Omega_\alpha$ for ϕ are all nonempty.

Since ϕ is one-to-one as a map of $\tau_\beta(I)$ onto $\tau_\alpha(I)$, and these sets are open and dense in Ω_β and Ω_α respectively, it follows from Lemma 2.3 that ϕ is irreducible. Thus by Proposition 2.4 $M^{(1)}$ is just the set of multiple points of ϕ. Since $\mathfrak{B}_\alpha$ is a proper subalgebra of $\mathfrak{B}_\beta$

(see [3]), $M^{(1)}$ is nonempty. We shall prove that if $2 \leqq \alpha < \beta$, then $M^{(1)} = M^{(2)} = \cdots$. The case when $\alpha = 1$ is more complicated.

As in Section 1, K^α denotes the Boolean algebra of open and closed sets in Ω_α. If $A \in K^\alpha$, $\phi^{-1}(A) \in K^\beta$, and ϕ^{-1} yields an isomorphism of K^α onto a subalgebra of K^β. The sets in $\phi^{-1}(K^\alpha)$ are just those sets in K^β which are unions of fibers for ϕ.

DEFINITIONS 3.1. (a) A set $A \in K^\beta$ is of *countable* or *uncountable type* depending on whether the Borel set $\tau_\beta^{-1}(A \cap \tau_\beta(A)) \in K_\beta$ is countable or uncountable.

(b) A point $\omega \in \Omega_\alpha$ is a *special multiple point* for ϕ if the fiber $\phi^{-1}(\omega)$ contains at least two distinct points all of whose open and closed neighborhoods are of uncountable type.

We shall prove the following results.

PROPOSITION 3.2. *If $1 \leqq \alpha < \beta$, then every set in K^α of uncountable type contains a special multiple point for ϕ: $\Omega_\beta \to \Omega_\alpha$.*

THEOREM 3.3. *If $1 \leqq \alpha < \beta$, the set Q of special multiple points is nonempty, and*

$$Q \subseteqq \bigcap_{n=1}^{\infty} M^{(n)} .$$

THEOREM 3.4. *If $1 \leqq \alpha < \beta$, then $\mathfrak{B}_\alpha$ is uncomplemented in $\mathfrak{B}_\beta$.*[1]

Theorems 2.2 and 3.3 clearly imply Theorem 3.4. The role of Proposition 3.2 is to ensure the existence of special multiple points. Its proof is complicated for $\alpha = 1$, so we assume it now and prove Theorem 3.3.

Proof of Theorem 3.3. Let ω_0 be a special multiple point in Ω_α for ϕ and let ψ_1, ψ_2 be distinct points of $\phi^{-1}(\omega_0)$ all of whose K^β neighborhoods are of uncountable type. Let B_1 and B_2 be in K^β with $\psi_i \in B_i$, $i = 1, 2$ and $B_1 \cap B_2 = \varnothing$, $B_1 \cup B_2 = \Omega_\beta$. Let A be any K^α neighborhood of ω_0 in Ω_α. The sets

$$C_i = B_i \cap \phi^{-1}(A), \; i = 1, 2 ,$$

are disjoint neighborhoods of ψ_i of uncountable type. Consider C_1. The uncountable Borel set

$$E_1 = \tau_\beta^{-1}(C_1 \cap \tau_\beta(I))$$

[1] It would suffice to suppose $\beta = \alpha + 1$, but this assumption does not simplify the proof.

contains a Cantor set F_1 ([6], page 446) which is necessarily in $K_1 \subseteqq K_\alpha$. Thus the set

$$D_1 = \overline{\tau_\alpha(F_1)} \subseteqq \Omega_\alpha$$

is of uncountable type, $D_1 \subseteqq A$ and $\phi^{-1}(D_1) \subseteqq C_1$. By Proposition 3.2 D_1 contains a special multiple point ω_1 and clearly $\phi^{-1}(\omega_1) \subseteqq C_1$. A similar argument applies to C_2. Since A was arbitrary, it follows that there exist two nets $\{\omega_\xi\}, \{\omega_\eta\}$ whose members are special multiple points, such that

$$\omega_\xi \longrightarrow \omega_0 , \qquad \omega_\eta \longrightarrow \omega_0$$
$$\phi^{-1}(\omega_\xi) \subseteqq C_1 , \qquad \phi^{-1}(\omega_\eta) \subseteqq C_2 .$$

Since $C_1 \cap C_2 = \varnothing$, the cluster sets $\limsup \phi^{-1}(\omega_\xi)$ and $\limsup \phi^{-1}(\omega_\eta)$ are nonempty disjoint subsets of $\phi^{-1}(\omega_0)$. Since $\omega_\xi, \omega_\eta \in M^{(1)}$, by Definition 2.1 $\omega_0 \in M^{(2)}$. However, now each ω_ξ, ω_η satisfies the hypotheses for ω_0, so by the same argument $\omega_\xi, \omega_\eta \in M^{(2)}$ and $\omega_0 \in M^{(3)}$, etc. Thus $\omega_0 \in \bigcap_{n=1}^{\infty} M^{(n)}$.

To prove Proposition 3.2 we first establish

LEMMA 3.5. *Let $1 \leqq \alpha < \beta$. Every set in K^α of uncountable type contains a multiple point.*

Proof. Let A be a set in K^α of uncountable type. The Borel set

$$E = \tau_\alpha^{-1}(A \cap \tau_\alpha(I))$$

contains a Cantor set F. We may write $F = G \cup H$, where G and H are both uncountable sets lying in $K_\beta \sim K_\alpha$, since the Cantor set contains Borel subsets of every Baire class relative to itself and hence also relative to I ([3] and [6], page 351). Thus

$$C = \overline{\tau_\beta(G)}, \; D = \overline{\tau_\beta(H)}$$

are disjoint sets in $K^\beta \sim \phi^{-1}(K^\alpha)$. Let $B = \overline{\tau_\alpha(F)}$. Then

$$\phi^{-1}(B) = C \cup D \subseteqq \phi^{-1}(A) .$$

If $\phi\colon \phi^{-1}(B) \longrightarrow B$ were one-to-one it would be a homeomorphism and C and D would lie in $\phi^{-1}(K^\alpha)$. Thus B (and hence A) contains a multiple point.

Proposition 3.2 for $\alpha \geqq 2$ now follows from

LEMMA 3.6. *Let $2 \leqq \alpha < \beta$ and ω_0 be a multiple point for $\phi\colon \Omega_\beta \longrightarrow \Omega_\alpha$. Each point of $\phi^{-1}(\omega_0)$ has all its K^β neighborhoods of uncountable type.*

Proof. Let $\psi \in \phi^{-1}(\omega_0)$ and C be a K^β neighborhood of ψ whose complement also intersects $\phi^{-1}(\omega_0)$. Suppose that

$$E = \tau_\beta^{-1}(C \cap \tau_\beta(I))$$

were countable. Then $E \in K_2 \subseteqq K_\alpha$ and

$$C = \phi^{-1}(\overline{\tau_\alpha(E)}),$$

which is a union of fibers. Since C intersects $\phi^{-1}(\omega_0)$ it must contain it, contradicting the fact C^c intersects $\phi^{-1}(\omega_0)$.

COROLLARY 3.7. *If $2 \leqq \alpha < \beta$, then every multiple point is a special multiple point, and*

$$Q = M^{(1)} = M^{(2)} = \cdots .$$

To prove Proposition 3.2 for $\alpha = 1$ we first need

LEMMA 3.8. *Let F be a Cantor set in I. There exists a subset B of F such that $B \triangle C \in K_2 \sim K_1$ for every countable set C in I.*[2]

Proof. We start by constructing a certain subset of I. Let $A = \bigcup_{n=1}^{\infty} K_n$, where each K_n is a Cantor set and A is dense in I. If U is any nonempty open set in I, $A^c \cap U$ is of second category in U. Thus $A \cap U$ and $A^c \cap U$ are both uncountable. Let $\psi: F \to I$ be any continuous irreducible onto map. (Such a map exists, since if $\Theta: K \to I$ is a continuous map from the Cantor set K onto I, there exists a closed subset T of K such that $\Theta \mid T$ is irreducible. Since T can have no isolated points, it is homeomorphic to K.) Now take $B = \psi^{-1}(A) \subseteqq F$. Suppose V is any nonempty relatively open subset of F. By Lemma 2.3 there exists a nonempty open set U in I such that $\psi^{-1}(U) \subseteqq V$. Thus $B \cap V$ and $B^c \cap V$ are both uncountable, since they contain the sets $\psi^{-1}(U \cap A)$ and $\psi^{-1}(U \cap A^c)$.

Now consider F as a subset of I and $B \subseteqq F$. Let C be any countable set in I. Then $B \triangle C = (B \cup D_1) \cap D_2^c$, where D_1 and D_2 are disjoint countable sets. It W is any open set in I which intersects F,

$$F \cap W \cap (B \triangle C) \supseteqq W \cap B \cap D_2^c$$
$$F \cap W \cap (B \triangle C)^c \supseteqq W \cap B^c \cap D_1^c$$

and these sets are uncountable. Thus the characteristic function of $B \triangle C$ is everywhere discontinuous on F, so $B \triangle C \notin K_1$ ([6], page 419). Since B is an F_σ set, $B \triangle C \in K_2$.

[2] $B \triangle C$ denotes the symmetric difference of B and C.

Proof of Proposition 3.2 *when* $\alpha = 1$. Let $\phi: \Omega_\beta \to \Omega_1$ and let E be any set in K^1 of uncountable type. Since $\tau_1^{-1}(E \cap \tau_1(I))$ contains a Cantor set, it follows from Lemma 3.8 that there exists a set $A \subseteqq \phi^{-1}(E) \subseteqq \Omega_\beta$ (necessarily of uncountable type) such that $A \Delta C \in K^2 \sim \phi^{-1}(K^1)$ for every set C in K^2 of countable type. Let $R = \phi(A) \cap \phi(A^c)$. Then R is a compact subset of E. We shall prove that R contains a special multiple point for ϕ. Note that for $\omega \in R$, $A \cap \phi^{-1}(\omega)$ and $A^c \cap \phi^{-1}(\omega)$ are both nonempty. If a point ω of R is not a special multiple point, then there is at most one point of $\phi^{-1}(\omega)$ all of whose K^β neighborhoods are of uncountable type. It follows by compactness of $A \cap \phi^{-1}(\omega)$ and $A^c \cap \phi^{-1}(\omega)$ that there exists a neighborhood B_ω of ω in K^1 such that either $\phi^{-1}(B_\omega) \cap A$ or $\phi^{-1}(B_\omega) \cap A^c$ is of countable type in K^2. Suppose now that R contains no special multiple point. By compactness, a finite number of such neighborhoods B_ω cover R. We may suppose them to be disjoint and divide them into two groups whose unions C_1 and C_2 satisfy

(1) $C_1 \cap C_2 = \varnothing$,

(2) $R \subseteqq C_1 \cup C_2$,

(3) $\phi^{-1}(C_1) \cap A$ and $\phi^{-1}(C_2) \cap A^c$ are both of countable type.

The open set

$$W = \{\omega \mid \phi^{-1}(\omega) \subseteqq A\}$$

is dense in A by Lemma 2.3. Because

$$\bar{W} \sim W \subseteqq R \subseteqq C_1 \cup C_2$$

the set

$$G = W \sim (C_1 \cup C_2)$$

is open and closed in Ω_1. Moreover

$$\phi^{-1}(G) = A \sim \phi^{-1}(C_1 \cup C_2) ,$$

because

$$\phi^{-1}(G) = \phi^{-1}(W) \sim \phi^{-1}(C_1 \cup C_2) \subseteqq A \sim \phi^{-1}(C_1 \cup C_2) \subseteqq \phi^{-1}(G) ,$$

since

$$A \sim \phi^{-1}(C_1 \cup C_2) \subseteqq A \sim \phi^{-1}(R) = \phi^{-1}(W) .$$

Thus the set $K = A \sim \phi^{-1}(G)$ is a subset of $\phi^{-1}(C_1 \cup C_2)$, and hence of countable type, and $A \Delta K = \phi^{-1}(G) \in \phi^{-1}(K^1)$. This contradiction proves the existence of a special multiple point for ϕ in R.

The method above may also be used to prove that $C(I)$ is uncomplemented in $\mathfrak{B}_\alpha$ for $\alpha \geqq 1$ and also that each $\mathfrak{B}_\alpha$ is uncomplemented in the Banach space $\mathfrak{B}_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathfrak{B}_\alpha$ of all bounded Borel measurable

functions. We shall not give the details. The fact that $C(I)$ is uncomplemented in $\mathfrak{B}_\alpha$ was proved earlier by B. B. Wells, Jr. in [11] using Phillips Lemma ([9], page 525). He also proved that $\mathfrak{B}_\alpha$ is uncomplemented in $l_\infty(I)$, assuming the continuum hypothesis.

4. Final remarks. We conclude with some observations and open problems.

1. The first observation is that our noncomplementary results for $\mathfrak{B}_\alpha$ on I hold equally well when I is replaced by any uncountable compact metric space. If M is such a space, then by a fundamental result of Milutin ([7], [8]) there exists a linear isomorphism ϕ: $C(M) \to C(I)$ of $C(M)$ onto $C(I)$. If $\{f_n\} \subseteqq C(M)$, then, by the Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty} f_n(s)$ exists for each $s \in M$ iff $\lim_{n\to\infty} \int f_n d\mu$ exists for each $\mu \in C(M)^* = \phi^* C(I)^*$ iff $\lim_{n\to\infty} (\phi f_n)(t)$ exists for each $t \in I$. It follows easily that ϕ extends to an isomorphism of $\mathfrak{B}_1(M)$ onto $\mathfrak{B}_1(I)$ and, inductively, of $\mathfrak{B}_\alpha(M)$ onto $\mathfrak{B}_\alpha(I)$, $\alpha < \omega_1$. Thus $\mathfrak{B}_\alpha(M)$ is uncomplemented in $\mathfrak{B}_\beta(M)$ for $1 \leqq \alpha < \beta$. Further, F. K. Dashiell has shown (using the Borsuk-Dugundji Theorem) that our results hold in any compact space containing a homeomorph of the Cantor set.

2. A Banach space X is *injective* if it is complemented in each Banach space Y into which it is imbedded. Theorem 3.4 and the first remark show that none of the Banach spaces $\mathfrak{B}_\alpha(M)$, M uncountable compact metric, are injective. This fact may also be proved as a simple consequence of a theorem of H. Rosenthal [10]. Suppose $\mathfrak{B}_\alpha(M)$ were injective. Since $\mathfrak{B}_\alpha(M)$ contains $c_0(M)$, it follows from Rosenthal's results that $\mathfrak{B}_\alpha(M)$ contains a subspace isomorphic to $l_\infty(M)$. However, this subspace has cardinality 2^c, while the cardinality of $\mathfrak{B}_\alpha(M)$ is c.

The argument just given shows also that $\mathfrak{B}_{\omega_1}(M)$, the space of bounded Borel functions on M, is not injective. This fact does not seem to follow from our methods.

3. We now give an example of a compact Hausdorff space Λ for which $C(\Lambda)$ is complemented in $\mathfrak{B}_1(\Lambda)$, even though $C(\Lambda)$ is not injective. We may represent $\mathfrak{B}_{\omega_1}(I) = C(\Lambda)$ where Λ is a σ-stonian space. Let τ: $I \to \Lambda$ be the natural imbedding of I as an open dense discrete subset of Λ. Let $g \in \mathfrak{B}_1(\Lambda)$ and $\{f_n\}$ be a bounded sequence in $C(\Lambda)$ converging pointwise to g on Λ. Then the functions $h_n \in \mathfrak{B}_{\omega_1}(I)$ defined by

$$h_n(t) = f_n(\tau(t)),\ t \in I,\ n = 1, 2, \cdots$$

converge pointwise to a function $k \in \mathfrak{B}_{\omega_1}(I)$. Its correspondent l in $C(\Lambda)$ agrees with g on $\tau(I)$ (but not on all of Λ unless g is continuous).

Clearly l is independent of the sequence $\{f_n\}$. The map $g \to l$ defines a norm one projection of $\mathfrak{B}_1(A)$ onto $C(A)$.

4. Information concerning the spaces Ω_α is very incomplete.

Question 1. Let $\alpha < \beta$ and $\phi: \Omega_\beta \to \Omega_\alpha$ be the canonical map. Is the set of multiple point for ϕ closed in Ω_α?

Question 2. For the map $\phi: \Omega_1 \to \Omega_1$ can one characterize the special multiple points?

Answers to these questions would greatly simplify the arguments in §3.

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