PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 2, 1973

NON-APOSYNDESIS AND NON-HEREDITARY DECOMPOSABILITY

H. E. SCHLAIS

Let M be a compact metric continuum. If $x \in M$ let $K(x)$ be the set to which an element y of M belongs if and only if M is not aposyndetic at x with respect to y. If, for all x in M, $x \in \operatorname{Int}(K(x))$, then M is the union of a countable collection of indecomposable subcontinua each of which is the closure of an open set. There exists a compact metric continuum M and a dense subset J of M such that for each $x \in J$, $K(x)=S$, but M contains no indecomposable subcontinua with nonvoid interior. It is the case, however, that if M has a point x such that $\operatorname{Int}(K(x)) = \varnothing$ then M contains an indecomposable subcontinua which intersects $\operatorname{Int}(K(x))$.

Professor F. Burton Jones, in [1], demonstrated that for compact metric continua relationships between nonaposyndesis and indecomposability exist. The examination of such relationships is continued in this paper. The primary concern is to show that certain non-aposyndetic properties, similar to but weaker than those of Theorems 9 and 10 of Jones' paper, insure the existence of indecomposable subcontinua.

1. DEFINITIONS. Let *space* be metric. For a set A, the *interior* of A and the *boundary* of A will be denoted by $\operatorname{Int}(A)$ and $\operatorname{Bd}(A)$ respectively. If A and B are sets, then by $A - B$ is meant $\{x \in A: x \notin B\}$. The *closure* of A will be denoted by $\bar{A}$. Given subsets A and B of a continuum M, M is said to be *aposyndetic at A with respect to B* if there is a subcontinuum H of M such that $A \subset \operatorname{Int}(H) \subset H \subset M - B$. For a subset A of a continuum M let $T_M(A) = \{x \in M: M$ is not aposyndetic at $\{x\}$ with respect to $A\}$ and $K_M(A) = \{x \in M: M$ is not aposyndetic at A with respect to $\{x\}\}$. $T_M(x)$ and $K_M(x)$ will be used for $T_M(\{x\})$ and $K_M(\{x\})$ and the subscript may be omitted where no confusion is likely to result (e.g. $T_M(x) = T(x)$).

2. Essentially indecomposable sets. Theorem 9 of [1] states that *a necessary and sufficient condition that the continuum M be indecomposable is that if x and y are points of M then M is non-aposyndetic at x with respect to y.* We obtain, from this condition, the following definition. A subset A of a continuum M is *essentially indecomposable* in M (often, in context, *essentially indecomposable*) if whenever x and y are points of A then M is not aposyndetic at x with respect to y. Thus the limit bar in the sin $1/x$ continuum is

essentially indecomposable in that continuum and any subset of an indecomposable continuum S is essentially indecomposable in S.

The following is immediate from the definition.

THEOREM 1. *An essentially indecomposable open subset of a continuum M is an essentially indecomposable subset of each subcontinuum of M which contains it.*

LEMMA 1. *If A is an essentially indecomposable open subset of a compact continuum M and H is a subcontinuum of M with $A - H = \emptyset$ then A is contained in a component C of $\overline{M - H}$.*

Proof. Assume the lemma false. Then $\overline{M - H}$ is the union of disjoint closed sets B_1 and B_2. Assume, without loss of generality, that B_1 contains an open subset U of A. Then if $x \in U$ and $y \in B_2, \cap A$, $H \cup B_1$ is a continuum with $x \in \operatorname{Int}(H \cup B_1) \subset H \cup B_1 \subset M - y$. This is a contradiction.

If F is a collection of sets then by F^* is meant the union of the elements of F.

THEOREM 2. *If A is an essentially indecomposable open subset of a compact continuum M and F is a finite collection of subcontinua of M such that for each $f \in F$, $A - f \neq \emptyset$, then A is contained in one component of $\overline{M - F^*}$.*

Proof. The theorem follows from Lemma 1 and Theorem 1 using finite induction.

THEOREM 3. *If A is an essentially indecomposable open subset of a continuum M, then there is an open subset U of M, containing A, such that U is maximal with respect to being essentially indecomposable and open. Further if $U \cap V \neq \emptyset$ with V open and essentially indecomposable, then $V \subset U$.*

Proof. Let $B = \{W \subset M\colon W$ is open, essentially indecomposable, and $W \supset A\}$. Let $U = B^*$. Clearly, a subcontinuum K of M, containing a point of U in its interior, contains A. Thus if $W \in B$, there is a point x_W such that $x_W \in \operatorname{Int}(K)$; so $K \supset W$. Hence $U \subset K$ and U is essentially indecomposable. If an essentially indecomposable open set R contains U then $R \supset A$ and $R \in B$; so $R \subset U$.

If V is an essentially indecomposable open set such that $U \cap V \neq \emptyset$, then it follows that $U \cup V$ is essentially indecomposable and open. Thus $U \cup V \in B$ and so $V \subset U$.

A subset A of a continuum M which is maximal with respect to being open and essentially indecomposable in M will be called a *maximal essentially indecomposable open* subset of M.

COROLLARY 3.1. *There exist at most countably many maximal essentially indecomposable open subsets of a separable continuum M.*

THEOREM 4. *If an essentially indecomposable open subset A of a compact continuum M is such that $M - A$ has at most countably many components, then $\bar{A}$ is connected.*

Proof. Assume the theorem false. Then there exists an essentially indecomposable open set A with $\bar{A} = H \cup K$ where H and K are nonempty disjoint closed subsets of M and $M - A$ has at most countably many components. Let $\{C_i: i = 1, 2, \cdots\}$ be a counting of the components of $M - A$. For $i = 1, 2, \cdots$, let H_i be the collection of all components in $\bar{A}$ which contain a point of C_i and let $K_i = \overline{C_i \cup H_i^*}$. Since $A \subset \cup \{K_i\}$ there is a j such that K_j contains an open subset of A and thus K_j contains A. Let $A_1 = \{h \in H_j: h \subset H\}$. Then $\overline{A_1^* \cup C_j}$ is a subcontinuum which contains an open subset of A but does not contain A. This is a contradiction.

EXAMPLE 1. The closure of a maximal essentially indecomposable open set need not be connected. Let M' be a compact plane indecomposable continuum which contains the points $(0, 0)$ and $(1, 0)$. Let $M = \{(x, y, z): (x, y) \in M'$ and $z = 0$ or $x > 1/2$ and $z = x - 1/2\}$. Let U be the open subset of M which is $\{(x, y, z) \in M: x < 1/2\}$. If x and y are points of U, it follows that M is not aposyndetic at x with respect to y and U is essentially indecomposable. If V is an open subset of M which properly contains U then one of $A = \{(x, y, z) \in M: x > 1/2, z = 0\}$ or $B = \{(x, y, z) \in M: x > 1/2, z = x - 1/2\}$ contains an open subset of V. Assume without loss of generality it is A. Then the subcontinuum of M which is $M - A$ contains U but does not contain V and V is essentially decomposable. Thus U is a maximal essentially indecomposable open set. Clearly $\bar{U}$ is not connected.

LEMMA 2. *In a continuum M, if $y \in \operatorname{Int} K(x)$ then $K(y) \subset K(x)$ and, hence, $\operatorname{Int} K(y) \subset \operatorname{Int} K(x)$.*

For a continuum M, $\operatorname{Int} I(M)$ denote $\{x \in M: x \in \operatorname{Int}(K_M(x))\}$.

THEOREM 5. *For a continuum M, $I(M)$ is an F_σ set.*

Proof. For each $\varepsilon > 0$ let $I_\varepsilon = \{x \in I(M): d(x), M - \operatorname{Int}(K(x)) \geqq \varepsilon\}$.

To see that I_ε is a closed set let p be a limit point of I_ε and suppose $\{x_i: i = 1, 2, \cdots\}$ is a sequence of points of I_ε which converges to p. There exists an integer N such that if $i \geqq N$ and $j \geqq N$, then $d(x_i, x_j) \leqq \varepsilon/2$. For such i and j $\operatorname{Int}(K(x_i)) = \operatorname{Int}(K(x_j))$ by Lemma 2. If H is a subcontinuum of M containing p in its interior, then for some $i \geqq N$ the point x_i is in $\operatorname{Int}(H)$ and therefore $\operatorname{Int}(K(x_i)) \subset H$. But $\operatorname{Int}(K(x_i)) = \operatorname{Int}(K(x_N))$, so $\operatorname{Int}(K(x_N)) \subset K(p)$. Since $d(x_N, p) \leqq \varepsilon/2$, $p \in \operatorname{Int}(K(x_N))$ and $p \in I(M)$. If $x \in M$ is such that $d(x, p) < \varepsilon$ then there is an $e > 0$ such that $d(x, p) + e < \varepsilon$. If $i > N$ is such that $d(x_i, p) < e$, then $d(x_i, x) \leqq d(x_i p) + d(x, p) < \varepsilon$ and $x \in \operatorname{Int}(K(x_i))$. As above, $\operatorname{Int}(K(x_i)) \subset \operatorname{Int}(K(p))$ and so $d(p, S - \operatorname{Int} K(p)) \geqq \varepsilon$. Thus $p \in I_\varepsilon$.

Since $I(M) = \bigcup_{n=1}^\infty I_{1/n}$, $I(M)$ is an F_σ set

THEOREM 6. *$I(M)$ intersects an open subset U of a complete continuum M in a second category subset if and only if U contains an essentially indecomposable open set.*

Proof. If the open subset U of M contains a second category subset which is a subset of $I(M)$, then for some $M = 1, 2, \cdots,$ the set $I_{1/n}$ (as defined in the proof of Theorem 5) contains an open subset V. If W is an open subset of V of diameter less than $1/2n$, then, for each $x \in W$, $K(x) \supset W$. Thus W is essentially indecomposable.

Conversely, if the complete continuum M contains an essentially indecomposable open set U, it then follows that for each $x \in U$, $x \in I(M)$.

COROLLARY 6.1. *If, in the compact continuum M, $I(M)$ intersects the open subset U of M in a second category set, then U intersects an indecomposable subcontinuum of M which has a nonvoid interior.*

Proof. From Theorem 6, U contains an essentially indecomposable open subset of M. A subcontinuum of M irreducible about U is indecomposable.

COROLLARY 6.2. *If $I(M)$ intersects each open subset of the compact continuum M in a second category set then the collection of indecomposable subcontinua of M which have nonvoid interior has a dense union in M.*

EXAMPLE 2. A compact plane continuum M in which $I(M)$ is dense which contains no indecomposable subcontinua with nonvoid interiors.

Let D be the closed square disk in E^2 whose opposite vertices are $(-1, -1)$ and $(1, 1)$. Let D_1^1 and D_1^2 be subsets of D homeomorphic to $D - \{(1, 0)\}$ which spiral out to $\operatorname{Bd}(D)$, as indicated in Figure 1, so that $\operatorname{Bd}(D)$ is the limiting set of each of the spirals.

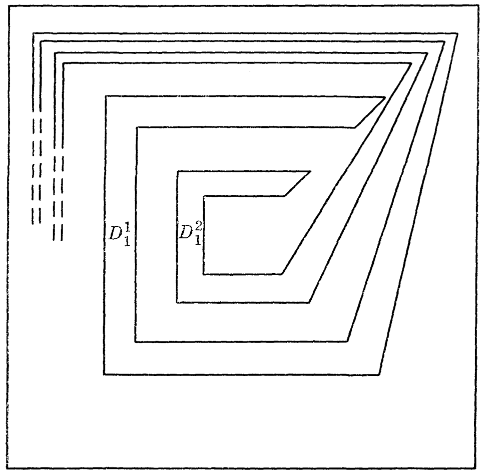

FIGURE 1

Define the spaces M_n inductively as follows. Let $M_1 = \overline{D_1^1 \cup D_1^2}$. Let f_1^1 and f_1^2 be homeomorphisms of $D - \{(1, 0)\}$ onto D_1^1 and D_1^2 respectively. Let $M_2 = \overline{f_1^1(M_1 - \{(1, 0)\})} \cup \overline{f_1^2(M_1 - \{(1, 0)\})}$. For $n > 2$, assume M_{n-1} has been defined and let $D_{n-1}^1, D_{n-1}^2, \cdots, D_{n-1}^{2(n-1)}$ be a counting of the images in M_{n-1} of D_1^1 and D_1^2 respectively. Let $f_{n-1}^1, f_{n-1}^2, \cdots, f_{n-1}^{2(n-1)}$ be homeomorphisms of $D - \{(1, 0)\}$ onto

$$D_{n-1}^1, D_{n-1}^2, \cdots, D_{n-1}^{2(n-1)}, \text{ such that for each } x \in D - \{(1, 0)\}$$

$$d(f_{n-1}^j(x), D - f_n^j\{D - \{1, 0\}\}) < \frac{1}{n-1}\,. \tag{1}$$

Let $M_n = \overline{\cup f_{n-1}^j(S_1 - \{(1, 0)\})}$, $j = 1, 2, \cdots, 2^{(n-1)}$, and then define M to be $\cap M_n$, $n = 1, 2, \cdots$.

If n is a positive integer, $i \leqq 2^n$ and p and q are points in $\operatorname{Bd}(D_n^i)$, then it is clear that M is not aposyndetic at p with respect to q. Since, if $K \geqq n$ and $j \leqq 2^K$ is such that $D_K^j \subset D_n^i$, each point of $\operatorname{Bd}(D_n^i)$ is a boundary point of D_K^j, we have for each $x \in \operatorname{Bd}(D_n^i)$, $K(x) \supset \{\operatorname{Bd}(D_k^j)\colon D_k^j \subset D_n^i\}^*$. It follows from (1) that, for such x, $K(x)$ is dense in $D_n^i \cap M$ and, since $K(x)$ is closed, it thus contains $\overline{D_n^i} \cap M$. If for some n and i, $x \in \operatorname{Bd}(D_n^i)$ then $x \in \operatorname{Int}(K(x))$. Since, as above, $\{x\colon x \in \operatorname{Bd}(D_n^i), n = 1, 2, \cdots, i \leqq 2^n\}$ is dense, $I(M)$ is dense.

If K is a subcontinuum of M with interior, then there are integers N, i, and j, with $i \neq j$, such that $\operatorname{Int}(K) \cap D_N^i \neq \emptyset$ and $\operatorname{Int}(K) \cap D_N^j \neq \emptyset$. The subcontinua $\overline{K \cap D_N^i}$ and $K \cap (M - D_N^j)$ of K decompose K.

LEMMA 3. *Let A be a maximal essentially indecomposable open* *subset of a compact continuum M and $x \in I(M)$. Then if $x \notin \bar{A}$, M is aposyndetic at x with respect to A.*

Proof. If $x \notin \bar{A}$ and U is an open subset of $M - A$ such that $U \subset \operatorname{Int}(K(x))$, then $A \cup U$ is not essentially indecomposable. It follows from Lemma 1 that there is a subcontinuum H of M such that $H \supset A$ and $(A \cup U) - H \neq \varnothing$ — i.e., M is aposyndetic at A with respect to $U - H$. Since U is any open subset of $\operatorname{Int}(K(x)) - A$,

$$B = \{y \in \operatorname{Int}(K(x)) \colon T(y) \cap A = \varnothing\}$$

is dense in $\operatorname{Int}(K(x)) - A$. Let $K = \{T(y) \colon y \in B\}^*$. Since $T(y)$ is connected [1, Theorem 3] and $x \in T(y)$ for each $y \in B$, $\bar{K}$ is a continuum. Since $x \notin \bar{A}$, $x \in \operatorname{Int}(\bar{K})$ and, since $\bar{K} \cap A = \varnothing$, we have the lemma.

LEMMA 4. *If, for a compact continuum M, $I(M) = M$ and A is a maximal essentially indecomposable open set, then $\bar{A}$ is connected.*

Proof. Assume, on the contrary, that A is a maximal essentially indecomposable open set and yet $\bar{A} = A_1 \cup A_2$ where A_1 and A_2 are separated subsets of M. Let U be an open subset of $M - A_2$ containing A, with $\operatorname{Bd}(U) \cap \bar{A} = \varnothing$. By Lemma 3, M is aposyndetic at each $x \in \operatorname{Bd}(U)$ with respect to A. Hence there is a finite collection $\{H_1, H_2, \cdots, H_n\}$ of subcontinua whose interiors cover $\operatorname{Bd}(U)$ such that for each $i = 1, 2, \cdots, n$, $H_i \cap A = \varnothing$. By Theorem 2, there is a subcontinuum C of M with $C \subset \overline{M - \cup H_i}$ such that $A \subset C$. But then $C \subset M - \operatorname{Bd}(U)$ and so $\operatorname{Bd}(U)$ does not separate M between A_1 and A_2. This is a contradiction.

THEOREM 7. *If M is a compact continuum with $I(M) = M$, then M is the union of a countable collection of indecomposable continua each of which is the closure of a member of a closure preserving collection of maximal essentially indecomposable open subsets of M.*

Proof. It follows from Theorem 6 and Corollary 3.1 that there is a countable collection of maximal essentially indecomposable open subsets of M whose union is dense in M. Let $\{U_n \colon n = 1, 2, 3, \cdots\}$ be a counting of this collection. Since for each i, $\bar{U}_i$ is connected (Lemma 4) it follows that $\bar{U}_i$ is an indecomposable continuum.

Let $U = \{U_{n_i} \colon i = 1, 2, \cdots\}$ be a subcollection of $\{U_n \colon n = 1, 2, \cdots\}$ and x be an element of M such that $x \notin \overline{U_{n_i}}$ for any i. Then, by Lemma 3, for each i, M is aposyndetic at x with respect to U_{n_i}. Thus $\operatorname{Int}(K(x)) \cap U_{n_i} = \varnothing$ for each i and since $x \in \operatorname{Int}(K(x))$, $x \notin \overline{\cup U_{n_i}}$. Therefore $\{U_n \colon n = 1, 2, \cdots\}$ is closure preserving and $M = \cup \bar{U}_i$.

EXAMPLE 3. There exists a compact metric continuum M such that $I(M) = M$ and M is not the union of the closures of any finite collection of maximal essentially indecomposable open sets.

For each $i = 1, 2, \cdots$ let M_i be a planar indecomposable subcontinuum of E^3 of diameter less than $1/i$ such if $i \neq j$ $M_i \cap M_j = (0, 0, 0)$. Let $M = \bigcup M_i$.

Clearly $M = I(M)$. The maximal essentially indecomposable open subsets of M are $\mathrm{Int}\,(M_j)$ relative to M. No finite collection of the closures of $\mathrm{Int}\,(M_j)$ has union M.

3. **Non-aposyndesis and the existence of indecomposable subcontinua.** An indecomposable subcontinuum M is $K(x)$ for each $x \in M$ [1, Theorem 9], i.e., M is $I(M)$. In Example 2, $I(M)$ is dense. Note also that if one takes a nested sequence E of the D_n^i's used in the construction of Example 2, then the subspace $\bar{M}_1$ of M (where M_1 is $\bigcup \{D_n^i \cap M \colon D_n^i \in E\}$) is an indecomposable continuum. This is seen by observing that if x and y are points of $\bar{M}_1$ then $\bar{M}_1$ is not aposyndetic at x with respect to y and applying Theorem 9 of [1] again. Since in each of the above examples there exists an indecomposable subcontinuum and $I(M)$ is infinite the following two considerations seem natural:

(a) If $I(M) \neq \varnothing$ then must it necessarily be infinite; and

(b) If $I(M) \neq \varnothing$ then must M contain an indecomposable subcontinuum?

Consideration (a) is answered negatively by Example 4 while Theorem 9 shows that the answer to consideration (b) is yes.

EXAMPLE 4. There exists a compact plane continuum M which has exactly one point p such that $p \in \mathrm{Int}\,(K(p))$ (i.e., $I(M) = \{p\}$). For this point $K(p) = M$.

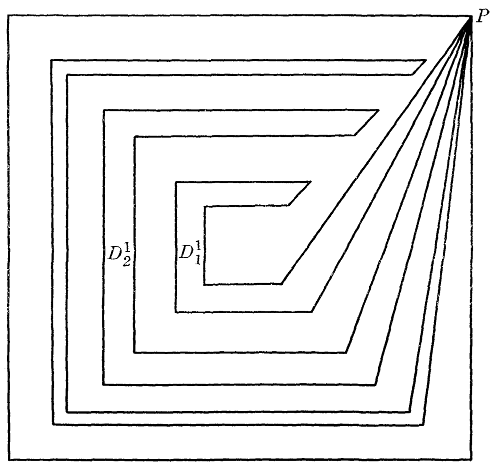

FIGURE 2

Let C be the unit square in the plane, $p \in C$, and, as indicated in Figure 2, $D_1^1, D_2^1, \cdots$ be a sequence of closed topological disks in the plane having C as its limit set, such that $D_i \cap D_j = \{p\}$ if $i \neq j$. Let $M_1 = \{D_i^1 : i = 1, 2, \cdots\} \cup C$. Assume M_n to be defined and let $\{D_i^n : i = 1, 2 \cdots\}$ be a counting of the maximal topological disks in M_n. Let f_i^n be a homeomorphism of M_1 into D_i^n such that $f_i^n(p) = p$, $f_i^n(C) = \mathrm{Bd}\,(D_i^n)$ and, for $x \in M_1$

$$d(f_i^n(x), E_2 - f_i^n(x)) < \frac{1}{n} . \tag{2}$$

Let $M_{n+1} = \cup \{f_i^n(M_1) : i = 1, 2, 3, \cdots\}$ and $M = \cup \{M_n : n = 1, 2, \cdots\}$. If C_i^n is the simple closed curve which is the boundary of D_i^n, then it is clear that for any i and n, M is not aposyndetic at p with respect to any point of C_i^n. Thus

$$K(p) \supset \cup \{C_i^n : n = 1, 2, \cdots, i = 1, 2, \cdots\} .$$

It follows from condition (2) that $K(p) = M$. Clearly, if an element x of M is different from p and $y \in K(x)$, there exists a sequence $\{y_i : i = 1, 2, 3, \cdots\}$ converging to y such that, for each i, $y_i \notin K(x)$. Thus $K(x)$ is nowhere dense for $x \neq p$ and p is the only point of M with $\mathrm{Int}\,(K(p)) \neq \emptyset$.

THEOREM 8. *A necessary and sufficient condition that the compact continuum M not be hereditarily decomposable is that, for some subcontinuum K of M and point p of K, there exists a sequence $U_1, U_2, \cdots$, of pairwise disjoint open subsets of M which converges to p such that, for each i, $U_i \cap K \neq \emptyset$, $p \notin U_i$ and, if $i < j$, the p-component of $K - U_i$ does not intersect $\bar{U}_j$.*

Proof. The sufficiency is established first. Assume K, p, and $U_1, U_2, \cdots$ are as in the statement of the theorem. Let H be a subcontinuum of K irreducible with respect to intersecting each $\bar{U}_j$. Assume H is decomposable. Then $H = A \cup B$ where A and B are proper subcontinua of H. Assume without loss of generality $A \cap \bar{U}_j \neq \emptyset$ for infinitely many j. Then $p \in A$. It follows, that for each i there is a $j > i$ such that $\bar{U}_j \cap A \neq \emptyset$. Hence for each i, $A \cap U_i \neq \emptyset$. But A is a proper subcontinuum of H. This contradicts the choice of H.

To establish the necessity, let K be an indecomposable subcontinuum of M and $p \in K$. Let U_1 be an open set intersecting K such that $p \notin \bar{U}_1$. Since K is indecomposable, p is not an interior point, relative to K, of P_1, the p-component of $K - U_1$. Thus there is an open set U_2 of diameter less than 1/2, which intersects K, such that

$\bar{U}_2 \cap P_1 = \emptyset$, $d(p, U_2) < 1/2$ and $\bar{U}_2 \cap \bar{U}_1 \neq \emptyset$.

If the set U_n has been defined, let P_n be the p-component of $K - U_n$. Since $P_n = \cup \{P_i: i = 1, 2, \cdots, n\}$ contains no interior relative to K, p is not an interior point, relative to K, of P_n. Let U_{n+1} be an open set of diameter less that $1/(n+1)$ which intersects K, such that $\overline{U_{n+1}} \cap P_n = \emptyset$, $d(p, U_{n+1}) < 1/(n+1)$ and $\overline{U_{n+1}} \cap \cup \{\bar{U}_i: i = 1, 2, \cdots, n\} = \emptyset$. Clearly K, p and $U_1, U_2, \cdots$ are as required.

THEOREM 9. *If M is a compact continuum and for some $x \in M$, $\operatorname{Int}(K(x)) \neq \emptyset$, then M is not hereditarily decomposable.*

Proof. Let U_0 be a nonempty open subset of $\operatorname{Int}(K(x))$ with $\bar{U}_0 \subset \operatorname{Int}(K(x))$ such that $x \notin \bar{U}_0$. Let B be the set of points in $\bar{U}_0$ which are points of C_x, the x-component of $M - U_0$, and U_1 be an open subset of U_0 with $\bar{U}_1 \subset U_0$. Let ε_1 be a positive real number less than $1/2$ such that $N_{\varepsilon_1}(B) \cap U_1 = \emptyset$ and $N_{\varepsilon_1}(B) \subset \operatorname{Int}(K(x))$.

Since M is not aposyndetic at x with respect to any point of U_1, it can be seen as follows, that C_1, the x-component of $M - U_1$, does not contain $N_{\varepsilon_1}(B)$. Assume the statement false. Since C_1 does not contain U_1, $x \in \overline{M - C_1}$. Clearly, x is not in any component of $\overline{M - C_1}$ which intersects U_1. Thus for each $p \in U_1$, there exists mutually separated sets A_p and B_p such that $A_p \cup B_p = \overline{M - C_1}$, $x \in A_p$ and $p \in B_p$. For $p \in U_1$, $C_1 \cup A_p$ is a continuum. But $x \in \operatorname{Int}(C_1 \cup A_p)$ and $C_1 \cup A_p \subset M - \{p\}$. This is a contradiction. Let U_2 be a nonempty open subset of $N_{\varepsilon_1}(B)$ such that $\bar{U}_2 \subset M - C_1$ and $\bar{U}_2 \subset N_{\varepsilon_1}(B) - B$. Let C_2 be the x-component of $M - U_2$. Note, as follows, that $\operatorname{Int}(C_2)$ does not contain B. Assume that $B \subset \operatorname{Int}(C_2)$. Since $U_2 \subset K(x)$, $x \in \overline{M - C_2}$. Further, C, the x-component of $\overline{M - C_2}$, does not intersect U_2 since $C \subset C_1$ and $U_2 \cap C_1 = \emptyset$. Thus there is a separation of $\overline{M - C_2}$ between x and a point of U_2. As above, this is a contradiction.

If the set U_n has been defined, let C_n be the x-component of $M - U_n$ and ε_n be a positive real number less than $1/2^n$ such that $N_{\varepsilon_n}(B) \cap (\bigcup_{i=1}^{n} U_i) = \emptyset$ and $N_{\varepsilon_n}(B) \subset \operatorname{Int}(K(x))$. It follows, as above, that C_n does not contain $N_{\varepsilon_n}(B)$. Let U_{n+1} be an open subset of $N_{\varepsilon_n}(B)$ such that $\overline{U_{n+1}} \cap C_n = \phi$ and $\overline{U_{n+1}} \subset N_{\varepsilon_n}(B) - B$.

Assume without loss of generality that $U_1, U_2, \cdots$ converges to a point b of B. Suppose $\bar{U}_i \cap C_x \neq \emptyset$ for some i. Since $\bar{U}_1 \cap C_x = \emptyset$, there is a smallest $i > 1$ such that $\bar{U}_i \cap C_x \neq \emptyset$. But then $\bar{U}_i \cap C_{i-1} \neq \emptyset$. This contradicts our choice of U_i. It now follows that M, b, and $U_1, U_2, \cdots$ satisfy the condition of Theorem 8.

THEOREM 10. *If an element x of a compact continuum M has $\operatorname{Int}(K(x)) \neq \emptyset$ and K is the collection of indecomposable subcontinua* *of* M, *then* K^* *is dense in* $\mathrm{Int}\,(K(x))$.

Proof. The choice for U_0 in the proof of Theorem 9 can be made in such a way that U_1 is any open subset of $\mathrm{Int}\,(K(x))$ such that $x \notin \bar{U}_1$ and $\bar{U}_1 \subset \mathrm{Int}\,(K(x))$. From the proof of Theorem 8, there exists an indecomposable subcontinuum at M which intersects U_1. Since every open subset of $\mathrm{Int}\,(K(x))$ contains such an open set, K^* is dense in $\mathrm{Int}\,(K(x))$.

REFERENCE

1. F. B. Jones, *Concerning non-aposyndetic continua*, Amer. J. Math., **70** (1948), 403-413.

Received December 1, 1971.

THE VIRGINIA MILITARY INSTITUTE