PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 1, 1973

A FACTORABLE BANACH ALGEBRA WITHOUT BOUNDED APPROXIMATE UNIT

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This note gives an example of a Banach algebra all of whose elements can be factored, but which has no bounded left or right approximate unit.

We first state P. J. Cohen's factorization theorem for Banach algebras and the generalization thereof obtained by E. Hewitt and, independently, P. C. Curtis and A. Figà-Talamanca. (See 32.22 and 32.26 of [1].)

THEOREM 1. *Let A be a Banach algebra with bounded left approximate unit. Then for each $z \in A$ and $\delta > 0$, there exist elements $x, y \in A$ such that*:

(i) $z = xy$,
(ii) $y \in \overline{Az}$,
(iii) $||z - y|| < \delta$.

THEOREM 2. *Let A be a Banach algebra with left approximate unit bounded in norm by D, and $(L, ||\cdot||_L)$ a left Banach A-module. Let z belong to the closed linear span of $A\cdot L$. Then for every $\delta > 0$ we can find $a \in A$, $y \in L$ such that*:

(i) $z = a\cdot y$,
(ii) $||a|| \leqq D$,
(iii) $y \in \overline{A\cdot z}$,
(iv) $||y - z||_L \leqq \delta$.

In particular, $A\cdot L$ is a closed linear subspace of L.

Theorem 1 may be abbreviated to read: "If a Banach algebra has either a bounded left or right approximate unit, then each of its elements can be factored." The purpose of this note is to show by example that the converse of this abbreviated version of Cohen's result is false. We remark here that partial converses to Theorems 1 and 2, employing suitable "factorability" hypotheses, have been obtained in [2], [3], and [4].

The general idea of our construction is set forth in the following lemma.

LEMMA. *Let B be a Banach algebra with 1, and suppose $a \in B$ satisfies*:

(i) $||a|| = 1$,

(ii) *for* $b \in B$, $ab = 0$ *implies* $b = 0$,

(iii) aB *is not closed in* B,

(iv) $b_1ab_2 = 1$ *for some* $b_1, b_2 \in B$.

Let $A = aB$. *Then* A *can be renormed to be a Banach algebra each of whose elements can be factored, but which has no bounded right or left approximate unit.*

Proof. Define $\|\cdot\|_0$ on A by $\|ab\|_0 = \|b\|$ for $b \in B$. This is a well-defined Banach space norm on A by (ii). For $b, c \in B$, we have $\|(ab)(ac)\|_0 = \|bac\| \leqq \|b\|\,\|c\| = \|ab\|_0\|ac\|_0$, so $\|\cdot\|_0$ is an algebra norm on A as well. The algebra B is naturally a left A-module, and in fact a Banach A-module, since for $b, c \in B$, $\|(ab)c\| \leqq \|b\|\,\|c\| = \|ab\|_0\|c\|$. By (iii), $A \cdot B = aB$ is not a closed subspace of B; we conclude from Theorem 2 that A has no bounded left approximate unit. Suppose that A has a right approximate unit (bounded or otherwise) $\{ae_\alpha\}$. Then in particular $\lim_\alpha \|ae_\alpha - 1\| = \lim_\alpha \|a^2e_\alpha - a\|_0 = 0$, showing that $\overline{aB} = B$ and hence $aB = B$, since a Banach algebra with 1 can have no dense proper right or left ideals. This contradicts (iii), so A has no right approximate unit. Finally, we note that (iv) allows us to factor any element ab of A as $ab = (abb_1)(ab_2)$.

We now give an example of a situation in which the hypotheses of the lemma are satisfied. Let l^2 denote the space of all absolutely square-summable sequences of complex numbers, normed in the usual way, and let $\mathfrak{B}$ be the Banach algebra of all bounded linear operators on l^2. Let $\{\lambda_n\}$ be a sequence of real numbers with $1 \geqq \lambda_n > 0$ for $n = 1, 2, \cdots$, and $\lim_n \lambda_n = 0$. Letting x_j denote the jth coordinate of the element $x \in l^2$, we define operators $T, V \in \mathfrak{B}$ by

$$Tx = (x_1, \lambda_1x_2, x_3, \lambda_2x_4, x_5, \cdots)$$
$$Vx = (x_1, 0, x_2, 0, x_3, 0, \cdots)$$

Clearly $\|T\| = 1$ and T is one-to-one, so for $W \in \mathfrak{B}$, $TW = 0$ implies $W = 0$. For $n = 1, 2, \cdots$, let $E_n \in \mathfrak{B}$ be the orthogonal projection on the $(2n)$th coordinate. We have $\lim_n \|TE_n\| = \lim_n \lambda_n = 0$. Since $\|E_n\| = 1$ for each n, we see from the open mapping theorem that $T\mathfrak{B}$ cannot be closed in $\mathfrak{B}$. A direct computation shows that $V^*TV = I$. We may now invoke the lemma (with V^*, T, and V playing the roles of b_1, a, and b_2, respectively) to see that $T\mathfrak{B}$, appropriately normed, is a Banach algebra all of whose elements can be factored, but which has no bounded left or right approximate unit.

If we set $X = T(l^2)$, then it is clear that the natural representation of $T\mathfrak{B}$ on X is irreducible. X is dense in l^2, and from this it follows that said representation is faithful, i.e., $T\mathfrak{B}$ is a primitive algebra. One also checks that the map $TW \to T(W^*)$ is an (isometric) involution on $T\mathfrak{B}$. The example which we have introduced is therefore not particularly pathological from an algebraic standpoint.

The author wishes to thank Professor Kenneth A. Ross for suggesting the problem of deciding whether or not an algebraically factorable Banach algebra must have a bounded approximate unit.

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Received March 30, 1972.

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