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THE COHOMOLOGICAL DESCRIPTION OF A TORUS ACTION

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The theorem proved in this paper is an example of a "regularity" theorem in the study of topological group actions—that is, it shows that a general topological action of a group continues to have certain properties of "linear" actions. Consider an action of a torus T on a cohomology n-sphere X, with fixed point set the cohomology r-sphere F. Consider the map $H^n(X_T; Z) \to H^n(F_T; Z)$, and let $c\eta$ be the image of the generator of $H^n(X; Z)$, considered as lying in $H^{n-r}(BT; Z)$, where c is an integer and η has no nontrivial integer divisors. The polynomial part η is well understood. The theorem will evaluate the integer part c in the following sense: in the linear case, c can be easily expressed in terms of the dimensions of the fixed point sets of various non-connected subgroups of T. It is shown that this formula continues to hold in the general topological case, given some weak assumptions. There is also a corresponding result for the case $F = \varnothing$.

The main tool will be the fibration $\pi: X_T \to B_T \equiv BT$, where X_T is as usual $E_T \times_T X$. We will use the usual limit arguments to allow ourselves to pretend that E_T is compact. Cohomology will be sheaf cohomology with compact supports (which will not usually be indicated). The spectral sequence of $X_T \to BT$ with coefficients in A will be denoted $E_r(X_T; A)$. The fixed point set of T acting on X will be denoted $F(T, X) \equiv F(T)$. $X \sim_Z Y (X \sim_p Y)$ will mean that X is a compact Z-cohomology (Z_p-cohomology) manifold with $Z(Z_p)$ cohomology ring the same as that of Y. $\dim_p(X)$ or $\dim_Z(X)$ will be the usual cohomological dimension of X over Z_p or Z. See [1] or [2] for details. For an abelian group A, let $\mathscr{F}A$ be $A/\mathrm{Torsion}(A)$.

If a torus T acts on a space X, a subtorus H of T is said to be *distinguished* if $F(H) \supsetneqq F(K)$ for any subtorus K which has $K \supsetneqq H$. In particular, the distinguished corank one subtori of T are those subtori H of corank one in T that have $F(H) \supsetneqq F(T)$. Recall that given a corank one subtorus of T, there is a corresponding integer-valued linear functional on the Lie algebra of T, a corresponding element of $H^1(T; Z)$ and a corresponding element (not divisible by any integer) in $H^2(BT; Z)$.

Now consider a torus T acting on $X \sim_Z S^n$. Let $F(T) \sim_Z S^r$, and look at $F_T \subsetneqq X_T$. Consider the cases $r > 0$, $r = 0$, and $r = -1$ $(F(T) = \varnothing)$ separately.

In case $r > 0$, the map

$$H^n(X; Z) \cong E_\infty^{0,n}(X_T; Z) \longrightarrow E_\infty^{n-r,r}(F_T; Z) \cong H^{n-r}(BT; Z)$$

takes the generator of $H^n(X; Z)$ to $c\eta$ where c is an integer, and η is $\Pi g_i^{(n_i - r)/2}$. (Here the g_i's correspond to the distinguished corank one subtori U_i of T, and $n_i = \dim_Z F(U_i)$.)

In case $r = 0$, $F(T) \sim {}_Z S^0$, we have $\pi\colon F_T \cong F \times B_T \to BT$, and the inclusion $F_T \to X_T$ induces

$$H^n(X; Z) \longrightarrow \tilde{H}^0(F; Z) \otimes H^{n-r}(BT; Z)$$

which takes g to (generator) $\otimes\, c\eta$ as before.

In case $r = -1$, $F(T) = \varnothing$, the transgression

$$H^n(X; Z) \longrightarrow H^{n+1}(BT; Z)$$

takes the generator to $c\eta$.

The theorem below will identify the integer c.

Let p be any prime. (Several of the objects below will depend on p, although this dependence will not be explicitly indicated.) (The letter p will also be used as one of indices of a spectral sequence, but hopefully no confusion will result.) For $i = 1, 2, \cdots$ let $S(i)$ be the subgroup of elements t of T such that $p^i t = 1$, the identity element of T. Let $S(0)$ be the subgroup of T consisting of 1 only. Clearly $S(i) \cong (Z_{p^i})^k$, where k is the rank of T. Each $F(S(i))$ is a Z_p-cohomology n_i-sphere for some n_i.

THEOREM. *Suppose that for any prime p and $i = 1, 2, \cdots$ that $F(S(i))$ has finitely generated Z-cohomology. Let p^a be the largest power of p that divides c. Then*

$$\sum_{i=1}^{\infty} [\dim_p F(S(i)) - \dim_Z F] = 2a\ .$$

Further, $F(S(i)) = F$ for $i > a$.

Proof. The second claim follows from the first and the fact that $\dim_p F(S(i)) - \dim_Z F$ is always even. (See [1], Chapter IV.)

We will first do the case $r > 0$ and then reduce the other two cases to this case.

Consider the spectral sequence of $F(S(i))_T \to BT$. Because $F(S(i)) \sim {}_p S^{n_i}$ and has finitely generated integral cohomology, it is easy to see from the universal coefficient theorem that $H^*(F(S(i)); Z)$ has no p-torsion, that $H^0(F(S(i)); Z) = Z$, and that $\mathscr{F} H^*(F(S(i)); Z) = H^*(S^{n_i}; Z)$. Because $r > 0$, the Z_p spectral sequence of $F(S(i))_T \to B_T$

collapses. It is then easy to verify the following facts about $E_\infty(F(S(i))_T; Z)$:

(i) $E_\infty(F(S(i))_T; Z)$ has no p-torsion.

(ii) $\mathscr{F} E_\infty^{p,q}(F(S(i))_T; Z) \cong H^p(BT; Z)$ if $q = 0$ or n_i, and $= 0$ otherwise. (As abelian groups with no reference to the multiplicative structure).

(iii) The bottom row $E_\infty^{*,0}(F(S(i))_T; Z)$ is isomorphic to $H^*(BT; Z)$, as a ring.

(iv) Let h be a generator of $\mathscr{F} E_\infty^{0,n_i}(F(S(i))_T; Z)$. Multiplication by h defines a map

$$H^p(BT; Z) \cong E_\infty^{p,0}(F(S(i))_T; Z) \longrightarrow \mathscr{F} E_\infty^{p,n_i}(F(S(i))_T; Z) \cong H^p(BT; Z)$$

This map is monomorphism, and its cokernel is finite with non-p order.

We know $F(S(i)) \sim {}_pS^{n_i}$ for $i = 0, 1, 2, \cdots$ where $n_0 = n$, and $F(S(\ell + 1)) = F$ for ℓ large enough, so $n_{\ell+1} = r$. We have

$$X = F(S(0)) \supseteqq F(S(1)) \supseteqq \cdots \supseteqq F(S(\ell)) \supseteqq F(S(\ell + 1)) = F .$$

We can consider the inclusion map $F_T \to X_T$ to be the composition

$$F(S(\ell + 1))_T \longrightarrow F(S(\ell))_T \longrightarrow \cdots \longrightarrow F(S(1))_T \longrightarrow F(S(0))_T .$$

Let h_i be a generator of $\mathscr{F} E_\infty^{0,n_i}(F(S(i))_T; Z)$. We will show below that the induced map φ: $\mathscr{F} E_\infty(F(S(i))_T; Z) \to \mathscr{F} E_\infty(F(S(i + 1))_T; Z)$ has $\varphi(h_i)$ divisible by $p^{i(n_i-n_{i+1})/2}$ and by no higher power of p. Using this and the facts (i)-(iv), we can show that $2a = \Sigma_{i=0}^{\ell} i(n_i - n_{i+1})$, which equals $\Sigma_{i=1}^{\ell}(n_i - n_{\ell+1})$, which is our conclusion. Thus we only have to prove our claim about the number of factors of p dividing $\varphi(h_i)$.

Consider the following diagram:

$$\begin{array}{ccc}
\mathscr{F} E_\infty(F(S(i))_T; Z) & \xrightarrow{\varphi} & \mathscr{F} E_\infty(F(S(i + 1))_T; Z) \\
\uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\
\mathscr{F} E_\infty(F(S(i))_{T/S(i)}; Z) & \xrightarrow{\psi} & \mathscr{F} E_\infty(F(S(i + 1))_{T/S(i)}; Z) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\
E_\infty(F(S(i))_{T/S(i)}; Z_p) & \longrightarrow & E_\infty(F(S(i + 1))_{T/S(i)}; Z_p) \\
\downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\zeta} \\
E_\infty(F(S(i))_{S(i+1)/S(i)}; Z_p) & \xrightarrow{\tau} & E_\infty(F(S(i + 1))_{S(i+1)/S(i)}; Z_p)
\end{array}$$

Let k_i be the generator of $\mathscr{F} E_\infty^{0,n_i}(F(S(i))_{T/S(i)}; Z)$. It is easy to see that $\alpha(k_i)$ is a non-p multiple of h_i. Now the map β on the

$E^{2,0}$ terms is $Z^k \cong H^2(BT/S(i); Z) \to H^2(BT; Z) \cong Z^k$, which is multiplication by p^i. Since $\psi(k_i)$ lies in filtration degree $n_i - n_{i+1}$, we can see using (iv) above that $\beta(\psi(k_i))$ contains precisely $i(n_i - n_{i+1})/2$ more factors of p than $\psi(k_i)$ does. Thus it is sufficient to show that $\psi(k_i)$ is not divisible by p.

The map δ is reduction mod p, so it suffices to show that $\delta\psi(k_i) \neq 0$. Therefore it suffices to show that $\tau\varepsilon\gamma(k_i) \neq 0$. But $S(i+1)/S(i) \cong (Z_p)^k$ acts on $F(S(i)) \sim {}_pS^{n_i}$, with fixed point set $F(S(i+1)) \sim {}_pS^{n_{i+1}}$, and $\varepsilon\gamma(k_i)$ is the generator of $E_\infty^{0,n_i}(F(S(i))_{S(i+1)/S(i)}; Z_p)$. In these circumstances, we must have $\tau\varepsilon\gamma(k_i)$ nonzero (see [1], Chapter XIII, and [3]) which finishes the proof in the case $r > 0$.

The cases $r = 0$ and $r = -1$ are handled by replacing X by SX and S^2X respectively, where SX denotes the nonreduced suspension. The action of T on SX (or S^2X) then has a nonempty connected fixed point set, so the problem is reduced to the previous case. It is not hard to see that if $X \sim {}_ZS^n$, then $SX \sim {}_ZS^{n+1}$. Thus we only need to show: (1) Suppose T acts on $X \sim {}_ZS^n$ with $F(T) = \varnothing$. Consider the actions of T on X $(F(T, X) = \varnothing, r = -1)$ and on SX $(F(T, SX) \sim {}_ZS^0, r = 0)$. One gets an integer c from each action. We need to show that the two c's are the same, at least up to sign. And (2) Suppose T acts on $X \sim {}_ZS^n$ with $F(T) \sim {}_ZS^0$. Consider the actions of T on $X(F(T, X) \sim {}_ZS^0, r = 0)$ and on SX $(F(T; SX) \sim {}_ZS^1, r = 1)$. Again, one gets two c's which we need to prove are the same up to sign.

The second case, going from $r = 0$ to $r = 1$, is easy; one merely uses the naturality of the suspension map.

In the first case, going from $r = -1$ to $r = 0$, we have an action of T on $X \sim {}_ZS^n$ with $F = \varnothing$, so n is odd. In the spectral sequence of $p\colon X_T \to B_T$, the generator of $H^n(X)$ transgresses to $c\eta \in H^{n+1}(BT)$, $c\eta \neq 0$. On the other hand, the spectral sequence of $q\colon (SX)_T \to BT$ collapses. Then it is easy to check that $H^n(X_T) = 0$, $H^{n+1}(X_T) = H^{n+1}(BT)/\langle c\eta\rangle$, $H^{n+2}((SX)_T) = 0$, and there is a split short exact sequence

$$0 \longrightarrow H^{n+1}(BT) \xrightarrow{q^*} H^{n+1}((SX)_T) \xrightarrow{j^*} H^{n+1}(SX) \longrightarrow 0\ ,$$

where $j\colon SX \to (SX)_T$ is the inclusion of the fiber. One can consider $(SX)_T$ to be the space $(X_T \times I)/\sim$, where $(a, 1) \sim (b, 1)$ and $(a, 0) \sim (b, 0)$ iff $p(a) = p(b)$. The map $q\colon (SX)_T \to BT$ is given by $[a, t] \to p(a)$. The sets $E_0 = \{[a, 0] \in (SX)_T\}$ and $E_1 = \{[a, 1] \in (SX)_T\}$ are each mapped homeomorphically onto BT by q. Let i_0 and $i_1\colon BT \to (SX)_T$ be the corresponding two sections of q. Showing that the action on SX gives rise to the same c as that on X reduces to showing that in

$$H^{n+1}(SX) \xleftarrow{j^*} H^{n+1}((SX)_T) \xrightarrow{i_0^* - i_1^*} H^{n+1}(BT) ,$$

$(i_0^* - i_1^*)\circ(j^*)^{-1}$ takes the generator of $H^{n+1}(SX)$ to $\pm c\eta$.

Let $U = \{[a, t] \in (SX)_T \mid t > 1/4\}$; let $V = \{[a, t] \in (SX)_T \mid t < 3/4\}$. U and V have E_0 and E_1 as strong deformation retracts, $U \cup V = (SX)_T$, and $U \cap V$ has $\{[a, 1/2] \in (SX)_T\} \cong X_T$ as a strong deformation retract. Consider the Meyer-Vietoris sequence of pair (U, V):

$$\begin{array}{ccccccccc}
H^n(U\cap V) & \longrightarrow & H^{n+1}(U\cup V) & \longrightarrow & H^{n+1}(U)\oplus H^{n+1}(V) & \longrightarrow & H^{n+1}(U\cap V) & \longrightarrow & H^{n+2}(U\cup V) \\
\| & & \| & & \| & & \| & & \| \\
H^n(X_T) & & H^{n+1}((SX)_T) & & H^{n+1}(BT)\oplus H^{n+1}(BT) & & H^{n+1}(X_T) & & H^{n+2}((SX)_T) \\
\| & & & & & & \| & & \| \\
0 & & & & & & \dfrac{H^{n+1}(BT)}{\langle c\eta\rangle} & & 0
\end{array}$$

so we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{n+1}(BT) & = & H^{n+1}(BT) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle \Delta} & & \downarrow & & \\
0 & \longrightarrow & H^{n+1}((SX)_T) & \longrightarrow & H^{n+1}(BT)\oplus H^{n+1}(BT) & \longrightarrow & \dfrac{H^{n+1}(BT)}{\langle c\eta\rangle} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j^*} & \searrow{\scriptstyle i_0^*-i_1^*} & \downarrow{\scriptstyle D} & & \| & & \\
0 & \longrightarrow & H^{n+1}(SX) & \longrightarrow & H^{n+1}(BT) & \longrightarrow & \dfrac{H^{n+1}(BT)}{\langle c\eta\rangle} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where Δ is the diagonal map and D is given by $D(a, b) = a - b$. The top two rows and all three columns are exact, so that bottom row is also exact. The horizontal map on the bottom left is $(i_0^* - i_1^*)\circ(j^*)^{-1}$, so we can see that this map takes the generator of $H^{n+1}(SX)$ to $\pm c\eta \in H^{n+1}(BT)$, as was to be shown.

This finishes the proof of the theorem.

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