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UNIQUENESS FOR THE CAUCHY PROBLEM FOR DEGENERATE PARABOLIC EQUATIONS

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Consider a second order degenerate parabolic operator L. The present paper is concerned with the uniqueness of solutions of the Cauchy problem: $Lu = f$ in a strip $0 < t \leqq T$, $u(0, x) = \phi(x)$ for all x in R^n. It is proved that there is at most one solution subject to a growth condition which depends on the degeneracy of L. In the special case where L is ultraparabolic, uniqueness is proved under only onesided growth condition. The methods used involve the construction of comparison functions in suitable sequences of domains.

In § 1 we state the main results on uniqueness of regular solutions. The proofs are given in § 2. In § 3 we derive uniqueness results for weak solutions defined by means of stochastic differential equations. Finally, in § 4, a uniqueness theorem is proved in case L is ultraparabolic, for solutions satisfying only a onesided growth condition.

Results of the same nature as in § 1 were obtained in very special cases in [4], [10]. Results overlapping with those of § 3 have recently been obtained by Sonin [12]; for more details see Remark 1 of § 3.

1. Uniqueness of regular solutions.

Let

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(t, x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t, x)\frac{\partial u}{\partial x_i} + c(t, x)u - \frac{\partial u}{\partial t}$$

where (a_{ij}) is a symmetric positive semidefinite matrix. Let k_i $(1 \leqq i \leqq p)$ and m be positive intergers such that $1 \leqq k_1 < k_2 < \cdots < k_p = m \leqq n$, and set

$$A_j = \{k_{j-1} + 1, k_{j-1} + 2, \cdots, k_j\},\ 1 \leqq j \leqq p,\ \text{where } k_0 = 0 .$$

Write

$$x = (x', x''),\ x' = (x'_1, \cdots, x'_p),\ x'_j = (x_{k_{j-1}+1}, \cdots, x_{k_j}) .$$

Let $d_i(r)$ $(1 \leqq i \leqq p)$ be positive value functions for $r \geqq 0$, having two continuous derivatives in $\sqrt{r}$ and satisfying

$$(1.1) \qquad \begin{cases} \left|\dfrac{r d'_i(r)}{d_i(r)}\right| \leqq C,\ (d_i(r))^2 \left|\dfrac{d}{dr}\dfrac{d'_i(r)}{(d_i(r))^2}\right| \leqq C . \\ \dfrac{r}{d_i(r)} \text{ is monotone increasing in } r\ (1 \leqq i \leqq p) . \end{cases}$$

Here and in what follows, various positive constants will be denoted by the same symbol C. Note that (1.1) holds if

$$d_i(r) = (1 + r^2)^{\rho_i}\ , \qquad \rho_i \leqq \frac{1}{2}\ . \tag{1.2}$$

We shall assume:

$$a_{ii}(t, x) \leqq C d_k(|x'_k|) \text{ if } i \in A_k,\ 1 \leqq k \leqq p\ , \tag{1.3}$$

$$a_{ij}(t, x) = 0 \text{ if } m + 1 \leqq i \leqq n,\ 1 \leqq j \leqq n\ , \tag{1.4}$$

$$|b_i(t, x)| \leqq C\Big(1 + |x'_k| + \sum_{l \neq k} |x'_l|^{\mu_{lk}} + C|x''|^{\gamma_k}\Big) \quad \text{if } i \in A_k,\ 1 \leqq k \leqq p\ , \tag{1.5}$$

$$|b_j(t, x)| \leqq C\Big(1 + \sum_{k=1}^{p} |x'_k|^{\delta_k} + |x''|\Big) \qquad \text{if } m + 1 \leqq j \leqq n\ , \tag{1.6}$$

$$c(t, x) \leqq C\Big(1 + \sum_{k=1}^{p} \frac{|x'_k|^2}{d_k(|x'_k|)} + |x''|^{\lambda}\Big) \tag{1.7}$$

where $\mu_{lk}, \gamma_k, \delta_k, \lambda$ are nonnegative numbers subject to the following conditions:

$$r^{2\mu_{kl}} d_l(r) \leqq C(1 + r^2) d_k(r^{\mu_{kl}})\ , \tag{1.8}$$

$$r^{2\gamma_k} \leqq C(1 + r^{\lambda}) d_k(r^{\gamma_k})\ , \tag{1.9}$$

$$d_k(r) \leqq C(1 + r^{2-}\delta_k),\ d_k(r) \leqq C(1 + r^{2-\lambda\delta_k})\ . \tag{1.10}$$

If

$$\frac{r^2}{d_i(r)} = R^2, \text{ then write } r = e_i(R)\ .$$

(By (1.1), $e_i(R)$ is uniquely defined.) Set

$$e(R) = \max_{1 \leqq i \leqq p} [e_i(R)]^{\delta_i}\ . \tag{1.11}$$

A function $u(t, x)$ will be called *regular* if it is continuous in $[0, T] \times R^n$ and if its partial derivatives u_t, u_x, u_{xx} exist in $(0, T] \times R^n$.

THEOREM 1. *Assume that* (1.3)–(1.7) *and* (1.1), (1.8)–(1.10) *hold. Let $u(t, x)$ be a regular function satisfying*

$$Lu(t, x) = 0 \ \textit{in}\ (0, T] \times R^n\ , \tag{1.12}$$

$$u(0, x) = 0 \ \textit{in}\ R^n\ , \tag{1.13}$$

$$|u(t, x)| \leqq C \exp\left\{\beta\left[\sum_{k=1}^{p} \frac{|x'_k|^2}{d_k(|x'_k|)} + |x''|^\lambda\right]\right\} \qquad (\beta > 0) \tag{1.14}$$

in $[0, T] \times R^n$, *provided*

$$|x''| \leqq e(R), \text{ where } R^2 = \sum_{k=1}^{p} \frac{|x'_k|^2}{d_k(|x'_k|)} .$$

Then $u(t, x) \equiv 0$ *in* $[0, T] \times R^n$.

Stronger assertions can be made in case the δ_k vanish. Suppose

$$|b_i(t, x)| \leqq C\left(1 + |x'_k| + \sum_{l \neq k} |x'_l|^{\mu_{kl}}\right)\gamma(t, x'') \quad (1 \leqq i \leqq m) , \tag{1.15}$$

$$|b_j(t, x)| \leqq C(1 + |x''|) \qquad (m + 1 \leqq j \leqq n) , \tag{1.16}$$

$$c(t, x) \leqq C\left(1 + \sum_{k=1}^{p} \frac{|x'_k|^2}{d_k(|x'_k|)}\right)\gamma(t, x'') \tag{1.17}$$

where $\gamma(t, x'')$ is an arbitrary continuous function. Then we can state:

THEOREM 2. *Assume that* (1.3), (1.4), (1.15)-(1.17), *and* (1.1), (1.8) *hold. Let* $u(t, x)$ *be a regular function satisfying* (1.12), (1.13), *and let*

$$|u(t, x)| \leqq C \exp\left\{\beta \sum_{k=1}^{p} \frac{|x'_k|^2}{d_k(|x'_k|)}\right\}\mu(t, x'') \qquad (\beta > 0) \tag{1.18}$$

where $\mu(t, x'')$ *is a continuous function. Then* $u(t, x) \equiv 0$ *in* $[0, T] \times R^n$.

2. Proofs of Theorem 1.2. The proofs of Theorems 1, 2 are based on the construction of (i) a comparison function $H(t, x)$, and (ii) a suitable family of increasing domains D_R, $R > 0$. We begin with the proof of Theorem 1. We take

$$H(t, x) = \exp\left\{\frac{k}{1 - \mu t}\left[\sum_{k=1}^{p} \frac{|x'_k|^2}{d_k(|x'_k|)} + \alpha(x'')\right] + \nu t\right\}$$

where $k > 2\beta$ and

$$\alpha(x'') = (1 + |x''|^2)^{\lambda/2} .$$

By direct computations one finds that

$$LH(t, x) < 0$$

provided μ, ν are positive and sufficiently large, and

$$(2.1) \qquad 0 \leqq t \leqq \frac{1}{2\mu} .$$

Here we make use of (1.3)–(1.7), (1.1), (1.8)–(1.10), and the inequality

$$|a_{ij}| \leqq \sqrt{a_{ii}}\sqrt{a_{jj}} .$$

We shall take D_R to be a product $D'_R \times D''_R$ where

$$D'_R = \left\{x'; \sum_{k=1}^{p} \frac{|x'_k|^2}{d_k(|x'_k|)} < R^2\right\}$$

and D''_R the interior of a cone in the (t, x'')-space with base

$$S_\rho = \{(t, x''); t = 0, |x''| < \rho\}$$

and vertex $(\eta, 0)$. Clearly

$$\partial D_R = [\partial D'_R \times \overline{D''_R}] \cup [D'_R \times \partial D''_R]$$

where $\partial\Omega$, $\bar{\Omega}$ denote respectively the boundary and closure of a set Ω. Also, $\partial D''_R = S_\rho \cup \Gamma$ where Γ is spanned by the generators of the cone. Note that

$$(2.2) \qquad \begin{gathered} |b_j(t, x)| \leqq C(1 + e(R) + |x''|) \leqq C(1 + e(R) + \rho) \\ \text{if } m + 1 \leqq j \leqq n, \ (t, x) \in D'_R \times \Gamma . \end{gathered}$$

We wish to choose $\rho = \rho(R)$, η such that the partial derivative

$$(2.3) \qquad -\frac{\partial w}{\partial t} + \sum_{j=m+1}^{n} b_j(t, x', x'')\frac{\partial w}{\partial x_j}$$

at the points of $D'_R \times \Gamma$ is a derivative in a direction pointing into D_R. This direction is determined by the trajectories

$$(2.4) \qquad \begin{cases} \dfrac{dt}{ds} = -1 , \\ \dfrac{dx_j}{ds} = b_j(t, x', x'') \qquad (m + 1 \leqq j \leqq n) \end{cases}$$

initiating on $D'_R \times \Gamma$. Thus we want to choose $\rho = \rho(R)$, η so that

$$\frac{d}{ds}\left\{\left[\sum_{j=m+1}^{n} x_j^2(s)\right]^{1/2} - \frac{\rho}{\eta}(\eta - t)\right\} < 0$$

at the points $s = s_0$ where $\sum_{j=m+1}^{n} x_j^2(s) = (\rho^2/\eta^2)(\eta - t)^2$. Since, by (2.2), (2.4),

$$\left\{\sum_{j=m+1}^{n}\left(\frac{dx_i}{ds}\right)^2\right\}^{1/2} \leqq C(1 + e(R) + \rho)$$

at $s = s_0$, it suffices to choose ρ, η such that

$$C(1 + e(R) + \rho) < \frac{\rho}{\eta} . \tag{2.5}$$

Taking

$$\rho = e(R) , \qquad \eta < \frac{1}{3C} , \tag{2.6}$$

(2.5) is satisfied for all sufficiently large R. We further restrict η by: $\eta \leqq 1/(2\mu)$.

Consider now, for any $\varepsilon > 0$, the function

$$v(t, x) = [\varepsilon H(t, x) \pm u(t, x)]e^{\gamma t}$$

in $\bar{D}_R$, where $\gamma \geqq c(t, x)$ in $\bar{D}_R$. It satisfies

$$(L - \gamma)v(t, x) < 0 \text{ in } \bar{D}_R \cap \{t > 0\} .$$

The function v cannot assume a negative minimum at points of D_R, for at such points $(L - \gamma)v(t, x) \geqq 0$. Next, v cannot attain a negative minimum in $\bar{D}_R$ at points $D'_R \times \Gamma$, for at such points

$$-\frac{\partial v}{\partial t} + \sum_{j=m+1}^{n} b_j(t, x)\frac{\partial v}{\partial x_j} \geqq 0$$

by the choice of ρ, η in (2.6), and

$$\sum_{i,j=1}^{m} a_{ij}(t, x)\frac{\partial^2 v}{\partial x_i \partial x_j} \geqq 0, \ \frac{\partial v}{\partial x_i} = 0 \qquad (1 \leqq i \leqq m) ,$$

so that again $(L - \gamma)v \geqq 0$, which is impossible.

Since $v > 0$ on $D'_R \times S_\rho$ (by (1.13)) and $v > 0$ on $\partial D'_R \times D''_R$ if R is large (by (1.14)), we conclude that $v > 0$ in D_R. Hence

$$| u(t, x) | \leqq \varepsilon H(t, x) \tag{2.7}$$

at each point (t, x) in D_R. Taking $R \to \infty$ and noting that each point (t, x) with $0 < t < \eta$ is contained in D_R if R is sufficiently large, it follows that (2.7) holds in the whole strip $0 < t < \eta$. Taking $\varepsilon \to 0$ we conclude that $u(t, x) \equiv 0$ in the strip $0 \leqq t \leqq \eta$. We can now proceed step by step to prove that $u(t, x) \equiv 0$ in the strip $0 \leqq t \leqq T$.

To prove Theorem 2 we note that (2.5) now becomes

$$C(1 + \rho) < \frac{\rho}{\eta} .$$

We take ρ a fixed positive number >1 and independent of R, and

$$\eta < \frac{1}{2C} .$$

We also take $\alpha(x'') \equiv 0$ in the definition of $H(t, x)$. Then, if μ, ν are sufficiently large (depending on ρ) and $\eta \leqq 1/(2\mu)$, then $LH < 0$ in $\bar{D}_R$. We can now proceed as before to establish the inequality

$$| u(t, x) | \leqq \varepsilon H(t, x')$$

in D_R. Taking $R \to \infty$ (ρ is fixed) and then $\varepsilon \to 0$, we conclude that

$$u(t, x', x'') = 0 \text{ if } x' \in R^m, \ (t, x'') \in D''_R ;$$

the cone D''_R is independent of ρ. Since this is true for any cone D''_R with base S_ρ, $u(t, x) \equiv 0$ in the strip $0 \leqq t \leqq \eta$. A step-by-step argument gives $u(t, x) \equiv 0$ in the strip $0 \leqq t \leqq T$.

REMARK 1. If the $d_i(r)$ are given by (1.2), then (1.8)-(1.10) reduce to

$$\rho_l + 2\mu_{kl} \leqq 2 + \mu_{kl}\rho_k , \qquad (2 + \rho_k)\gamma_k \leqq \lambda ,$$
$$\lambda\delta_k \leqq 2 + \rho_k , \qquad \delta_k \leqq 2 + \rho_k .$$

We can take

$$e(R) = C \max_{m+1 \leqq i \leqq n} \{1 + R^{2\delta_i/(2+\rho_i)}\} .$$

REMARK 2. Using different comparison functions for H (such as

$$\left[\sum_{k=1}^{p} \frac{| x'_k |^2}{d_k(| x'_k |)} + Kt\right]^q e^{\alpha t} \qquad (K, q, \alpha \text{ positive})$$

cf. [3; p. 56]) one can obtain variants of Theorems 1, 2 where the growth conditions on a_{ij}, b_i, c and on the solution u are modified.

3. **Uniqueness for weak solutions.** We assume

(A) The coefficients a_{ij}, b_i are independent of t and $c(t, x) \equiv 0$.

(B) There is an $m \times m$ matrix $\sigma(x) = (\sigma_{ij}(x))$ such that

$$2a_{ij}(x) = \sum_{k=1}^{m} \sigma_{ik}(x)\sigma_{jk}(x) \qquad (1 \leqq i, \ j \leqq m)$$

and the functions $\sigma_{ik}(x)$, $b_j(x)$ ($1 \leqq i, \ k \leqq m, \ 1 \leqq j \leqq n$) are Lipschitz continuous in x, uniformly in compact subsets of R^n.

(C) $$\sum_{i,j=1}^{m} | \sigma_{ij}(x) | + \sum_{i=1}^{n} | b_i(x) | \leqq C(1 + | x |) .$$

Recall [2], [9] that if (1.4) holds and the $a_{ij}(x)$ have continuous second derivatives then there exists a matrix $\sigma(x)$ satisfying (B).

Consider the stochastic differential system (see [5], [6] for the relevant theory)

$$(3.1)\qquad \begin{aligned} d\xi_i(t) &= b_i(\xi(t))dt + \sum_{j=1}^{m} \sigma_{ij}(\xi(t))dw_j(t) && (1 \leqq i \leqq m)\,, \\ d\xi_j(t) &= b_j(\xi(t))dt && (m+1 \leqq j \leqq n) \end{aligned}$$

where the $w_j(t)$ are independent Brownian motions. Let G be any bounded domain in R^n and denote by τ the exit time of $(s, \xi(s))$ from the cylinder $[0, t) \times G$, where $\xi(s)$ $(0 \leqq s \leqq t)$ is the solution of (3.1) with $\xi(0) = x$.

DEFINITION. A continuous function $u(t, x)$ in the strip $[0, T] \times R^n$ is a *weak solution* of the equation $Lu = 0$ if for any bounded domain G in R^n and for every $t \in (0, T]$,

$$(3.2)\qquad u(t, x) = E_x u(t - \tau, \xi(\tau))\,.$$

A slightly different definition was used in [12]. If u is a classical solution $Lu = 0$ in the strip $(0, T]$, continuous in $[0, T]$ then, by Ito's formula, it is a weak solution. Conversely, a smooth weak solution is a classical solution. A weak solution satisfying $u(0, x) = \phi(x)$ in R^n is called a weak solution of the Cauchy problem

$$(3.3)\qquad \begin{aligned} Lu(t, x) &= 0 \text{ in } (0, T] \times R^n\,, \\ u(0, x) &= \phi(x) \text{ in } R^n\,. \end{aligned}$$

If H is a smooth function satisfying $LH \leqq 0$ in the strip $[0, T]$ then, by Ito's formula,

$$H(t, x) \geqq E_x H(t - \tau, \xi(\tau))\,.$$

From this and (3.2) we obtain, for any $\varepsilon > 0$,

$$(3.4)\qquad \varepsilon H(t, x) \pm u(t, x) \geqq E_x\{\varepsilon H(t - \tau, \xi(\tau)) \pm u(t - \tau, \xi(\tau))\}\,.$$

Suppose now that the conditions of Theorem 1 are satisfied. Modify the definition of the domain $D_R = D'_R \times D''_R$ used in the proof of Theorem 1, by taking

$$D''_R = \{(t, x'');\ 0 < t < T,\ |x''| < \rho\}\,.$$

Denote by τ the exit time of $(s, \xi(s))$ from the cylinder $D_R \cap \{s < t\}$; $t \leqq \eta$. For $m + 1 \leqq i \leqq n$, $0 < s < \tau$,

$$\left|\frac{d\xi_i(s)}{ds}\right| = |\,b_i(s, \xi(s))\,| \leqq C\Big(1 + \sum_{k=1}^{p} |\,\xi'_k(s)\,|^{\delta_k} + \sum_{i=m+1}^{n} |\,\xi_i(s)\,|\Big)$$
$$\leqq C(1 + e(R) + \rho) .$$

Hence, if $\xi(0) = x^0$ is fixed, R is large and $\rho = e(R)$, $\eta < 1/(3C)$, then $\xi''(s) = (\xi_{m+1}(s), \cdots, \xi_n(s))$ does not leave the ball $|\,x''\,| < \rho$ at $s = \tau$. Next, if $u(0, x) \equiv 0$ then $u(t - \tau, \xi(\tau)) = 0$ if $\tau = t$. Using (3.4) with the same H as in the proof of Theorem 1, we conclude that

$$\varepsilon H(t, x^0) \pm u(t, x^0) \geqq E_{x^0}\{\chi[\varepsilon H(t - \tau, \xi(\tau)) \pm u(t - \tau, \xi(\tau))]\}$$

where $\chi = 1$ if $\xi(\tau)$ lies on $\partial D'_R \times D''_R$ and $\tau < t$, and $\chi = 0$ otherwise. Since on the set where $\chi = 1$,

$$(\varepsilon H \pm u) \longrightarrow 0 \text{ if } R \longrightarrow \infty ,$$

it follows that $|\,u(t, x^0)\,| \leqq \varepsilon H(t, x^0)$. Taking $\varepsilon \to 0$ we conclude that $u(t, x^0) = 0$. Hence $u(t, x) \equiv 0$ in the strip $0 \leqq t \leqq \eta$. We thus obtain:

THEOREM 3. *Let the conditions* (A)-(C), (1.3)-(1.6), (1.8)-(1.10) *hold. If $u(t, x)$ is a weak solution of the Cauchy problem* (1.12), (1.13) *and if it satisfies* (1.14) *where $|\,x''\,| \leqq e(R)$ ($R, e(R)$ as in Theorem* 1), *then $u(t, x) \equiv 0$ in* $[0, T] \times R^n$.

Similarly one can prove:

THEOREM 4. *Let the conditions* (A)-(C), (1.3), (1.4), (1.15), (1.16), *and* (1.1), (1.8) *hold. If $u(t, x)$ is a weak solution of the Cauchy problem* (1.12), (1.13) *and if it satisfies* (1.18) *where $\mu(t, x'')$ is a continuous function, then $u(t, x) \equiv 0$ in* $[0, T] \times R^n$.

Split the coordinates of x'' into q sets:

$$x'' = (x''_1, \cdots, x''_q) , \qquad x''_j = (x''_{\sigma_{j-1}+1}, \cdots, x''_{\sigma_j}) \quad (1 \leqq j \leqq q)$$

where $\sigma_0 = m + 1 < \sigma_1 < \cdots < \sigma_q = n$, and let $B_j = \{\sigma_{j-1} + 1, \cdots, \sigma_j\}$. We can refine Theorems $1 - 4$ by imposing different growth conditions on the b_i with respect to the variable x''_j. We give here one example.

Suppose

$$a_{ii}(x) \leqq C \qquad (1 \leqq i \leqq m) , \tag{3.5}$$

$$|\,b_i(x)\,| \leqq C(1 + |\,x'\,| + |\,x''\,|^{\gamma}) \qquad (1 \leqq i \leqq m) , \tag{3.6}$$

and (1.4) holds, and replace (1.6) by

$$(3.7) \qquad |b_j(x)| \leqq C(1 + |x'|^{\delta_l} + |x''|) \text{ if } j \in B_l\,, \ 1 \leqq l \leqq q\,.$$

We can then proceed as in the proof of Theorem 3, but with modified $D_R = D'_R \times D''_R$:

$$D'_R = \{x'; |x'| < R\}\,,$$
$$D''_R = D_{R,1} \times \cdots \times D_{R,q}\,, \qquad D_{R,j} = \{x''_j; |x''_j| < R^{\delta_j}\}\,.$$

We also take a modified function H:

$$H(t, x) = \exp\left\{\frac{k}{1 - \mu t}\left[|x'|^2 + \sum_{l=1}^{q}(1 + |x''_l|^2)^{\lambda_l/2}\right] + \nu t\right\}$$

where $\lambda_l = 2/\delta_l$. If

$$(3.8) \qquad \gamma \leqq \min_{1 \leqq l \leqq q} \frac{1}{\delta_l}$$

then we can again prove that $LH < 0$ provided μ, ν are sufficiently large and $0 \leqq t \leqq 1/(2\mu)$. We sum up:

THEOREM 5. *Let the conditions* (A)-(C), (1.4) *and* (3.5)-(3.8) *hold. If u is a weak solution of* (1.12), (1.13) *and*

$$(3.9) \qquad |u(t, x)| \leqq C \exp\left\{\beta\left[|x'|^2 + \sum_{l=1}^{q}|x''_l|^{2/\delta_l}\right]\right\} \qquad (\beta > 0)$$

then $u(t, x) \equiv 0$ in $[0, T] \times R^n$.

REMARK 1. Sonin [12] has proved theorems which overlap with Theorems 3-5. His method is entirely probabilistic; our method is much simpler. In the growth condition on $u(t, x)$ he allows $|x'|^2$ to be replaced by a slightly more general function, namely, $|x'|\,h(|x'|)$ where

$$\int^{\infty} \frac{dr}{h(r)} = \infty\,.$$

However he imposes more restrictive growth conditions on the $b_i(t, x)$ $(1 \leqq i \leqq n)$.

REMARK 2. Under some smoothness and growth conditions on $\phi(x)$, one can easily establish the existence of a regular solution for the Cauchy problem (3.3) (with coefficients depending also on t) by means of a probabilistic formula. It is convenient to write the Cauchy problem in the form

$$\frac{\partial v}{\partial t} + \sum_{i,j=1}^{n} a_{ij}(t, x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t, x)\frac{\partial v}{\partial x_i} + c(t, x)v = 0 \quad \text{in } [0, T) \times R^n , \tag{3.10}$$
$$v(T, x) = \phi(x) \text{ in } R^n .$$

This form is obtained from (3.3) by the transformation $t = T - s$ upon denoting $a_{ij}(T - s, x)$, $b_i(T - s, x)$, $c(T - s, x)$ by $a_{ij}(s, x)$, $b_i(s, x)$, $c(s, x)$ respectively. Suppose there is a matrix $(\sigma_{ij}(t, x))$ $(i, j = 1, \cdots, m)$ such that

$$2a_{ij}(t, x) = \sum_{k=1}^{m} \sigma_{ik}(t, x)\sigma_{jk}(t, x) \qquad (1 \leqq i, j \leqq m) ,$$

and suppose $\sigma_{ij}(t, x)$, $b_l(t, x)$ satisfy a uniform Lipschitz condition in t, x and are bounded by $C(1 + |x|)$. Denote by $\xi_{x,t}(s)$ the solution of

$$d\xi_i(s) = b_i(s, \xi(s))ds + \sum_{j=1}^{m} \sigma_{ij}(s, \xi(s))dw_j(s) \quad (1 \leqq i \leqq m) ,$$
$$d\xi_k(s) = b_k(s, \xi(s))ds \qquad (m + 1 \leqq k \leqq n) ,$$
$$\xi_{x,t}(t) = x .$$

Let

$$v(t, x) = E_x\left\{\phi(\xi_{x,t}(T)) \exp\left[\int_t^T c(s, \xi_{x,t}(s))ds\right]\right\} . \tag{3.11}$$

If

$$|D_x^i \phi(x)| \leqq C(1 + |x|^{2r}) \text{ for } i = 0, 1, 2 \ (r > 0) , \tag{3.12}$$

if $c(s, x) \leqq 0$, and if the derivatives

$$D_t\sigma_{ij},\ D_t b_i,\ D_x^\alpha \sigma_{ij},\ D_x^\alpha b_i,\ D_x^\alpha c \qquad (\alpha = 1, 2)$$

are continuous and bounded then, by [6], $v(t, x)$ is a regular solution of (3.10), and $|v(t, x)| \leqq C(1 + |x|^r)$.

If $m < n$ and if $\delta_k = 0$ in the condition (1.6), then the last assertion is valid under weaker growth conditions on $\phi(x)$, namely, (3.12) may be replaced by

$$|\phi(x', x'')| + |D_x\phi(x', x'')| + |D_{xx}\phi(x', x'')| \leqq C(1 + |x'|^r)g(x'') \qquad (r > 0) \tag{3.13}$$

where $g(x'')$ is a continuous function. The regular solution $v(t, x)$ is bounded by $C(1 + |x'|^r)h(x'')$, $h(x'')$ a continuous function.

REMARK 3. Suppose (A)–(C) hold and suppose $\sigma_{ij}(x)$, $b_k(x)$ satisfy a uniform Lipschitz condition in R^n. If $\phi(x)$ is a continuous function satisfying

$$(3.14) \qquad |\phi(x)| \leqq C(1 + |x|^r) \qquad (r > 0)$$

then the function

$$u(t, x) = E_x\{\phi(\xi(t))\}$$

is well defined. Using the Strong Markov property of the solutions of (3.1) one can easily show that u is a weak solution of the Cauchy problem (3.3). If $m < n$ and $\delta_k = 0$ in (1.6), then the last assertion remains true if (3.14) is replaced by

$$|\phi(x)| \leqq C(1 + |x'|^r)g(x'')$$

where $g(x'')$ is a continuous function.

4. Uniqueness for solutions satisfying a lower growth condition. From the proof of Theorem 1 we obtain the following maximum-principle-type result:

LEMMA 1. *Assume that* (1.3)-(1.7) *and* (1.6), (1.8)-(1.10) *hold. Let* $u(t, x)$ *be a regular function satisfying*

$$Lu(t, x) \leqq 0 \text{ in } (0, T] \times R^n ,$$
$$u(0, x) \geqq 0 \text{ in } R^n ,$$
$$u(t, x) \geqq -C \exp\left\{\beta\left[\sum_{k=1}^{p} \frac{|x'_k|^2}{d_k(|x'_k|)} + |x''|^\lambda\right]\right\}$$

in $[0, T] \times R^n$ *provided*

$$|x''| \leqq e(R), \text{ where } R^2 = \sum_{k=1}^{p} \frac{|x'_k|^2}{d_k(|x'_k|)} ,$$

where C, β *are positive constants. Then* $u(t, x) \geqq 0$ *in* $[0, T] \times R^n$.

Similar results hold under the assumptions of Theorems 2-5.

We shall prove in this section that, under further restrictions on L, if $Lu = 0$ in $0 < t \leqq T$, $u(0, x) = 0$ on R^n, $u(t, x) \geqq 0$ in $0 \leqq t \leqq T$, then $u(t, x) = 0$ in $0 \leqq t \leqq T$. For such results for nondegenerate parabolic equations, see Friedman [3], Aronson and Besala [1] and the references given there.

We shall take L to have the form

$$(4.1) \qquad \begin{aligned} Lu \equiv \sum_{i,j=1}^{m} a_{ij}(t, x, y)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} a_i(t, x, y)\frac{\partial u}{\partial x_i} \\ + \sum_{i=1}^{k} x_i \frac{\partial u}{\partial y_i} + c(t, x, y)u - \frac{\partial u}{\partial t} \end{aligned}$$

where $k \leqq m$ and the matrix (a_{ij}) is uniformly positive definite. Such operators are called *ultraparabolic* operators. If the coefficients are sufficiently regular, then (see Weber [13], Ilin [7], Sonin [11]) there exists a fundamental solution $\Gamma(t, x, y; \tau, \xi, \eta)$ for the Cauchy problem

$$Lu = 0 \text{ in } 0 < t \leqq T , \tag{4.2}$$

$$u(0, x, y) = \phi(x, y) \text{ for } (x, y) \in R^{m+k} . \tag{4.3}$$

Thus, if ϕ is continuous and bounded, then a solution of (4.2), (4.3) is given by

$$u(t, x, y) = \int_{R^k}\int_{R^m} \Gamma(t, x, y; 0, \xi, \eta)\phi(\xi, \eta)d\xi d\eta .$$

Set

$$\Pi = \{(t, x, y); 0 \leqq t \leqq T, x \in R^m, y \in R^k\} .$$

We shall assume:

(P) There exists a positive constant α_0 such that

$$\sum_{i,j=1}^{m} a_{ij}(t, x, y)\xi_i\xi_j \geqq \alpha_0 |\xi|^2 \text{ for all } \xi \in R^m, \ (t, x, y) \in \Pi .$$

(Q) The functions a_{ij} and their first three derivatives are continuous and bounded in Π; the functions a_i and their first two derivatives are continuous and bounded in Π; the function c and its first derivatives are continuous and bounded in Π.

Under these assumptions the above mentioned fundamental solution exists and, for any $K > 0$, $c > 0$,

$$\begin{aligned} &|\Gamma(t, x, y; \tau, \xi, \eta)| + \sum_{i=1}^{m}\left|\frac{\partial}{\partial x_i}\Gamma(t, x, y; \tau, \xi, \eta)\right| \\ &\leqq M \exp\{-\mu |x|^2 - \mu |x' + y|^2\} \qquad (M > 0, \ \mu > 0) \end{aligned} \tag{4.4}$$

where $x' = (x_1, \cdots, x_k)$, provided $|\xi| + |\eta| \leqq K$, $t - \tau \geqq c$, where M, μ depend on K, c. [Sharper estimates are valid (see [7]) but will not be needed here.] Furthermore, for the adjoint Cauchy problem

$$L^*v(\tau, \xi, \eta) = 0 \text{ in } 0 \leqq \tau < T ,$$
$$v(T, \xi, \eta) = \psi(\xi, \eta) \text{ for } (\xi, \eta) \in R^{m+k}$$

there also exists a fundamental solution $\Gamma^*(\tau, \xi, \eta; \bar{\tau}, \bar{\xi}, \bar{\eta})$ $(\tau < \bar{\tau})$. Using Green's identity as in [3; p. 29] and employing (4.4) and the analogous estimates for Γ^*, one deduces that

$$\Gamma(t, x, y; \tau, \xi, \eta) = \Gamma^*(\tau, \xi, \eta; t, x, y) .$$

Hence,

$$L^*\Gamma(t, x, y; \tau, \xi, \eta) = 0 \text{ for each fixed } (t, x, y) . \tag{4.5}$$

Note that $\Gamma \geqq 0$; the proof is as in the nondegenerate parabolic case [3; p. 45].

THEOREM 6. *Let (P), (Q) hold and let $u(t, x)$ be a regular solution of* (4.2) *satisfying*

$$u(t, x, y) \geqq -C \exp\{\beta[|x|^2 + |y|^2]\} \qquad (C > 0,\ \beta > 0) \tag{4.6}$$

provided $|y| \leqq |x|$. If $u(0, x, y) = 0$ on R^{m+k} then $u(t, x, y) \equiv 0$ in Π.

Proof. In view of Lemma 1 it suffices to prove the theorem in case $u(t, x, y) \geqq 0$ in Π.

If $k < m$, then introduce

$$\tilde{\Pi} = \{(t, x, y, y_{k+1}, \cdots, y_m); 0 < t < T,\ (x, y, y_{k+1}, \cdots, y_m) \in R^{2m}\} ,$$

$$\tilde{u}(t, x, y, y_{k+1}, \cdots, y_m) = u(t, x, y) ,$$

$$\tilde{L}v \equiv Lv + \sum_{i=k+1}^{m} x_i \frac{\partial v}{\partial y_i} .$$

It is clear that $\tilde{u} \geqq 0$ and $\tilde{L}\tilde{u} = 0$ in $\tilde{\Pi}$. Also, $\tilde{u} = 0$ if $t = 0$. Therefore, if we prove the theorem in case $k = m$ then $\tilde{u} \equiv 0$ in $\tilde{\Pi}$ and, consequently, $u \equiv 0$ in Π. Thus, it suffices to prove the original theorem in case $u \geqq 0$ in Π and $k = m$.

For any $R > 0$,

$$u(t, x, y) \geqq \int_{|\xi|^2+|\eta|^2<R^2} \Gamma(t, x, y; \tau, \xi, \eta) u(\tau, \xi, \eta) d\xi d\eta . \tag{4.7}$$

Indeed, denoting the right hand side by $v(t, x, y)$, we have: $u - v \geqq 0$ on $t = \tau$ and

$$\liminf_{|x|+|y|\to\infty} (u - v) \geqq 0.$$

Hence by Theorem 9 in [3; p. 43] (which holds also for degenerate parabolic operators, since the proof requires only the weak form of the maximum principle), $u - v \geqq 0$ in Π.

Integrating both sides of (4.7) with respect to τ, $0 \leqq \tau \leqq t^*$, for some $t^* \in (0, t)$, and taking $R \to \infty$, we obtain

$$\int_0^{t^*} \int_{R^m} \int_{R^k} \Gamma(t, x, y; \tau, \xi, \eta) u(\tau, \xi, \eta) d\xi d\eta d\tau \leqq t^* u(t, x, y) . \tag{4.8}$$

If $\gamma > c(t, x, y)$ then

$$e^{\gamma t} \leqq \int_{R^m}\int_{R^k} \Gamma(t, x, y; s, \xi, \eta) e^{\gamma s} d\xi d\eta . \tag{4.9}$$

Indeed, denoting right hand side by $w(t, x, y)$, we have: $L(e^{\gamma t} - w) = Le^{\gamma t} \leqq 0$, $e^{\gamma t} - w = 0$ on $t = s$, $e^{\gamma t} - w$ is bounded in the strip $[s, t]$. Now apply Theorem 9 in [3; p. 43]. From (4.9) with $s = 0$ we conclude that, for any $(\bar{t}, \bar{x}, \bar{y})$,

$$\Gamma(\bar{t}, \bar{x}, \bar{y}; 0, \xi^0, \eta^0) > 0 \text{ for some } (\xi^0, \eta^0) .$$

Hence, for some $\rho > 0$, $\alpha > 0$,

$$\Gamma(\bar{t}, \bar{x}, \bar{y}; \tau, \xi, \eta) \geqq \alpha > 0 \text{ if } 0 \leqq \tau \leqq t^*,\ |\xi - \xi^0|^2 + |\eta - \eta^0|^2 \leqq \rho^2 , \tag{4.10}$$

provided t^* is sufficiently small.

LEMMA 2. *Under the assumptions of Theorem* 6, *for any* $(\bar{t}, \bar{x}, \bar{y})$ *and* t^*, ρ, ξ^0, η^0 *as in* (4.10), *and for any* $0 < \hat{t} < t^*$, $\hat{t}$ *sufficiently small,*

$$\Gamma(\bar{t}, \bar{x}, \bar{y}; \tau, \xi, \eta) \geqq c \exp\{-\lambda[|\xi - \xi^0|^2 + |\eta - \eta^0|^2]\} \tag{4.11}$$

provided $|\xi - \xi^0|^2 + |\eta - \eta^0|^2 \geqq \rho^2$, $0 \leqq \tau \leqq \hat{t}$, *where* c, λ *are positive constants.*

A similar result holds for nondegenerate parabolic equations; see [3], [1]. The proof given below employs a comparison argument as in [1].

Proof. Take for simplicity $\xi^0 = 0$, $\eta^0 = 0$. Consider the function

$$V(\tau, \xi, \eta) = \exp\left\{-\frac{\lambda |\xi|^2}{\sigma - \tau} - \frac{\lambda}{(\sigma - \tau)^3} |\mu\eta + (\sigma - \tau)\xi|^2\right\}$$

for $0 < \tau < \sigma$, $|\xi|^2 + |\eta|^2 \geqq \rho^2$, where $\sigma \in (0, t^*)$. If σ is sufficiently small and $|\mu| > 1$, then

$$\frac{|\xi|^2}{\sigma - \tau} + \frac{|\mu\eta + (\sigma - \tau)\xi|^2}{(\sigma - \tau)^3} \geqq 1 . \tag{4.12}$$

Writing

$$L^* w = \sum_{i,j=1}^m a_{ij} \frac{\partial^2 w}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^m \tilde{a}_i \frac{\partial w}{\partial \xi_i} - \sum_{i=1}^m \xi_i \frac{\partial w}{\partial \eta_i} + \tilde{c} w + \frac{\partial w}{\partial \tau}$$

and setting

$$\zeta = \frac{\mu\eta + (\sigma - \tau)\xi}{\sigma - \tau}, \qquad \tilde{a} = (\tilde{a}_1, \cdots, \tilde{a}_m),$$

one easily verifies that

$$\frac{L^*V}{V} \geqq \frac{4\lambda^2\alpha_0}{(\sigma-\tau)^2}|\xi + \zeta|^2 - (\Sigma a_{ii})\frac{4\lambda}{\sigma-\tau} - \frac{2\lambda}{\sigma-\tau}\tilde{a}\cdot(\xi+\zeta)$$
$$+ \frac{2\lambda\mu}{(\sigma-\tau)^2}\xi\cdot\zeta - \frac{\lambda|\xi|^2}{(\sigma-\tau)^2} - \frac{3\lambda|\zeta|^2}{(\sigma-\tau)^2} + \frac{2\lambda}{(\sigma-\tau)^2}\xi\cdot\zeta + \tilde{c}.$$

Writing $|\xi + \zeta|^2 = |\xi|^2 + |\zeta|^2 + 2\xi\cdot\zeta$ and taking $\mu = -(4\alpha_0\lambda + 1)$, the terms involving $\xi\cdot\zeta$ disappear from the right hand side. Taking λ sufficiently large and using (4.12), we then conclude that

$$L^*V > 0 \text{ if } 0 \leqq \tau < \sigma,\ |\xi|^2 + |\eta|^2 \geqq \rho^2. \tag{4.13}$$

We shall compare V with

$$w(\tau, \xi, \eta) = \Gamma(\bar{t}, \bar{x}, \bar{y}; \tau, \xi, \eta)$$

in the region $0 < \tau < \sigma$, $|\xi|^2 + |\eta|^2 \geqq \rho^2$. By (4.5), (4.13), $L^*(w - \alpha V) < 0$ in this region, where α is any positive number. Taking α as in (4.10) we have $w - \alpha V \geqq 0$ if $0 \leqq \tau \leqq \sigma$, $|\xi|^2 + |\eta|^2 = \rho^2$. Since also $w - \alpha V = w \geqq 0$ if $\tau = \sigma$, $|\xi|^2 + |\eta|^2 \geqq \rho^2$, we can apply Theorem 9 of [3; p. 43] to conclude that $w - \alpha V \geqq 0$ in the region $0 < \tau < \sigma$, $|\xi|^2 + |\eta|^2 \geqq \rho^2$. This yields the assertion of the lemma (with a different λ) for any $0 < \hat{t} < \sigma$.

Substituting (4.11) into (4.8) we obtain

$$\int_0^{\hat{t}}\int_{R^m}\int_{R^m} u(\tau, \xi, \eta)\exp\{-\mu[|\xi|^2 + |\eta|^2]\}d\xi d\eta d\tau \leqq C < \infty \tag{4.14}$$

where μ, C are positive constants.

We shall deduce from (4.14) that $u \equiv 0$, employing an argument similar to that used in Lemma 5 of [8]. Let

$$Z(t, x, y) = (\hat{t} - t)\exp\left\{-\frac{\varepsilon|x|^2 + \varepsilon|y|^2}{1 + Q(\hat{t} - t)}\right\}.$$

For any $\varepsilon > 0$,

$$L^*Z < 0 \text{ if } 0 < t < \hat{t},\ x \in R^m,\ y \in R^m$$

provided $\hat{t}$ is sufficiently small and Q is sufficiently large. Let $\zeta(x, y)$ be a C^∞ function satisfying $\zeta(x, y) = 1$ if $|x|^2 + |y|^2 < R^2$, $\zeta(x, y) = 0$ if $|x|^2 + |y|^2 > R^2 + 1$, $0 \leqq \zeta(x, y) \leqq 1$ elsewhere. We can choose ζ so that its first derivatives are bounded by a constant independent of R. Let

$$\hat{Z}(t, x, y) = \zeta(x, y)Z(t, x, y) .$$

Then, by Green's identity,

$$\int_0^{\hat{t}} \int_{R^{2m}} (L^*\hat{Z}\cdot u - \hat{Z}Lu)dxdydt = 0 .$$

We have used here the facts that $\hat{Z}(\hat{t}, x, y) = 0$, $u(0, x, y) = 0$. Since $Lu = 0$, we get

$$\int_0^{\hat{t}} \int_{|x|^2+|y|^2<R^2} L^*Z\cdot udxdydt = -\int_0^{\hat{t}} \int_{R^2<|x|^2+|y|^2<R^2+1} L^*\hat{Z}\cdot udxdydt .$$

Taking $\varepsilon > \mu$ and using (4.14) one easily concludes that the right hand side converges to 0 if $R \to \infty$. Since $u \geqq 0$, $L^*Z < 0$, it follows that $u(t, x, y) \equiv 0$ in the strip $0 \leqq t \leqq \hat{t}$. Now proceed step by step to show that $u \equiv 0$ in the strip $0 \leqq t \leqq T$.

REMARK. Denote by $\tilde{L}u$ the operator obtained from Lu in (4.1) upon replacing

$$\sum_{i=1}^k x_i \frac{\partial u}{\partial y_i} \text{ by } \sum_{i=1}^k b_i(t, x', y)\frac{\partial u}{\partial y_i}, \quad x' = (x_1, \cdots, x_k) .$$

Suppose the transformation

$$\bar{x}_i = b_i(t, x', y) \qquad (i = 1, \cdots, k) \tag{4.15}$$

is a diffeomorphism from R^k onto R^k and that the first four derivatives of this mapping and of its inverse are bounded. Then, by the change of variables (4.15) we obtain an operator of the form (4.1) to which Theorem 6 can be applied. Consequently, Theorem 6 extends also to the operator $\tilde{L}$.

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