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ON THE COUNTABLE UNION OF CELLULAR DECOMPOSITIONS OF n-MANIFOLDS

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Suppose that $G_1, G_2 \cdots$ are cellular upper semicontinuous decompositions of an n-manifold with boundary $M (n \neq 4)$ such that for $i = 1, 2, \cdots, M/G_i$ is homeomorphic to M. Let G be the decomposition of M obtained from the decomposition of G_i in the following manner. A set g belongs to G if and only if g is a nondegenerate element of some G_i or g is a point in $M - (\bigcup_{i=1}^{\infty} H_{G_i}^*)$. It will be shown that if the various decompositions fit together in a "continuous" manner and if G is an upper semicontinuous decomposition of M, then M/G is homeomorphic to M.

Our principal result thus extends previous results obtained by the author ([6], [7]) and Lamoreaux [4], by removing the 0-dimensionality restriction in [6] or, alternatively, by eliminating the finiteness condition in [7]. Furthermore, with the aid of recent work of Siebenmann [5], generalizations to n-manifolds ($n \neq 4$) may be made. As was observed in [7], some conditions must be imposed on the manner in which the decompositions are pieced together. The example described by Bing in [2] demonstrates that the continuity condition to be described below is a necessary one.

Notation and terminology. Suppose G is an upper semicontinuous decomposition of a topological space, X. Then X/G will denote the associated decomposition space, P will denote the natural projection map from X onto X/G, and H_G will denote the collection of nondegenerate elements of G. If U is an open subset of X, then U is said to be *saturated* (with respect to G) in case $U = P^{-1}[P[U]]$. If $\mathscr{U}$ is a covering of a subset of X, then $P[\mathscr{U}] = \{P[U]: U \in \mathscr{U}\}$.

The statement that M is an *n-manifold with boundary* means that M is a separable metric space such that each point of M has a neighborhood which is an n-cell. If A is a subset of M, then A is *cellular* in M if there exists a sequence $C_1, C_2, \cdots$ of n-cells in M such that (1) for each positive integer i, $C_{i+1} \subset$ Interior C_i, and (2) $\bigcap_{i=1}^{\infty} C_i = A$. If M is an n-manifold with boundary, the statement that G is *cellular decomposition* of M means that G is an upper semicontinuous decomposition of M and each nondegenerate element of G is a cellular subset of M.

If M is a metric space, A a subset of M, then $S_\varepsilon(A)$ denotes the ε-neighborhood of A and Cl A denotes the closure of A in M. If K

is a collection of subsets of M, then $K^* = \bigcup\{k: k \in K\}$. The word *map* will always be used to indicate a continuous function. If $\mathscr{U}$ is a collection of subsets of M and $A \subset M$, then

$$\mathrm{St}\,(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U}: A \cap U \neq \emptyset\}\,.$$

The main result. The principal theorem will be proved by means of repeated applications of the Lemma which appears below. We say that a cellular decomposition G of a manifold M satisfies condition S if for each saturated open cover $\mathscr{U}$ of H_G^*, there exists a closed map h from M onto M such that (1) $G = \{h^{-1}(x): x \in M\}$, (2) if $x \in M - \mathscr{U}^*$, then $h(x) = x$, and (3) for each $g \in G$ and $g \subset \mathscr{U}^*$, there exists a $U \in \mathscr{U}$ such that $g \cup h(g) \subset U$.

LEMMA 1. *Suppose G is a cellular decomposition of an n-manifold with boundary $M(n \neq 4)$. Then M/G is homeomorphic to M if and only if G satisfies condition S.*

Proof. Clearly if G satisfies condition S, then M/G is homeomorphic to M. Suppose now that M/G is homeomorphic to M and that $\mathscr{U}$ is a saturated open cover of H_G^*. Without loss of generality we may assume that $\mathscr{U}$ is locally finite. Suppose $x \in \mathscr{U}^*$ and $U_1, \cdots, U_n$ are those sets in $\mathscr{U}$ which contain x. Set

$$\varepsilon_x = \max\{d(P(x), M/G - P[U_1]), \cdots, d(P(x), M/G - P[U_n])\}$$

and define $f_1(x) = \varepsilon_x/2$. Then f_1 is a lower semicontinuous function from $\mathscr{U}^*$ into $(0, \infty)$, and, hence, there exists a continuous map f_2 from $\mathscr{U}^*$ into $(0, \infty)$ such that $0 < f_2 < f_1$. For $x \in \mathscr{U}^*$, define $f_3(x)$ to be $d(P(x), M/G - P[\mathscr{U}^*])$, and finally define $f(x)$ to be min $\{f_2(x), f_3(x)\}$. Siebenman's projection approximation theorem [5] may be applied to find a homeomorphism k from $\mathscr{U}^*$ onto $P[\mathscr{U}^*]$ such that $d(P(x), k(x)) < f(x)$ for each $x \in \mathscr{U}^*$. Then $h = k^{-1}P$ is the desired map. To see this we need only check that for $g \in G$ and $g \subset \mathscr{U}^*$, there is a $U \in \mathscr{U}$ such that $h(g) \cup g \subset U$. Let $y = k^{-1}P(g)$. By our construction there exists a $U \in \mathscr{U}$ such that both $P(y)$ and $k(y)$ belong to $P[U]$. But $k(y) = P(g)$; therefore, y and g belong to U, which completes the proof.

Suppose M is a metric space and K is a collection of mutually disjoint subsets of M. If $g \in K$, then K is said to be continuous at g in case for each positive number ε, there exists an open subset V of M containing g such that if $g' \in K$ and $g' \cap V \neq \emptyset$, then $g \subset S_\varepsilon(g')$ and $g' \subset S_\varepsilon(g)$.

THEOREM 1. *Suppose $G_1, G_2, \cdots$ are cellular decompositions of an n-manifold with boundary $M(n \neq 4)$ such that*

(1) *If* $g \in H_{G_i}$ *and* $g \cap H^*_{G_j} \neq \emptyset$, *then* $g \in H_{G_j}$.

(2) *For each* $k = 1, 2, \cdots$, *if* $g \in H_{G_k}$, *then* $\{H_{G_i}: i \neq k\} \cup \{g\}$ *is continuous at* g.

(3) *For* $i = 1, 2, \cdots, M/G_i$ *is homeomorphic to* M.

(4) $G = \{g: g \in \bigcup_{i=1}^{\infty} H_{G_i}$ *or* g *is a point of* $M - (\bigcup_{i=1}^{\infty} H^*_{G_i})\}$ *is an upper semicontinuous decomposition of* M.

Then M/G *is homeomorphic to* M.

Proof. We show that G satisfies condition S. Let $\mathscr{W}$ be a saturated open cover of H^*_G. The required function h will be defined as a limit of a sequence of closed, onto maps which are obtained in the following steps.

Step 1. Let $K_1 = \{p \in M$: there exists a sequence of nondegenerate elements, each from a different H_{G_i}, which converges to $p\}$. Note that K_1 is a closed subset of M. We construct a saturated (with respect to G) open refinement of $\mathscr{W}$ which covers H^*_G and misses K_1. For each $g \in H_G$, let U_g be saturated open set with compact closure such that

(1) If $\varepsilon_g = \min\{\operatorname{diam} g, 1/2\, d(g, K_1), 1\}$, then $U_g \subset S_{\varepsilon_g}(g)$.

(2) If $g_i \in H_{G_i}$ and $g_j \in H_{G_j}$ $(i \neq j)$ and g_i and g_j are contained in U_g, then $1/2 \operatorname{diam} g_i < \operatorname{diam} g_j < 3/2 \operatorname{diam} g_i$.

(3) U_g is contained in some $W \in \mathscr{W}$ which contains g.

Parts (1) and (2) are possible because of the continuity condition imposed on the decompositions. Define $\mathscr{U}'_1 = \{U_g: g \in H_G\}$. Let $\mathscr{U}_1$ be a saturated open locally finite star refinement of $\mathscr{U}'_1$ and $\mathscr{V}_1 = \{U \in \mathscr{U}_1: U \cap H^*_{G_1} \neq \emptyset\}$. Observe that it follows from (1) that if $p \in K_1$, then $p \notin \mathscr{U}_1^*$. Furthermore, from (1) and (2) we have that if $p \in K_1$ and $\{x_i\}$ is a sequence of points in $\mathscr{U}_1^*$ which converge to p, then the sequence $\{\operatorname{St}(x_i, \mathscr{U}_1)\}$ also converges to p.

By Lemma 1, there exists a closed map h_1 from M onto M such that

(1) $G_1 = \{h_1^{-1}(x): x \in M\}$.

(2) If $x \in M - \mathscr{V}_1^*$, then $h_1(x) = x$.

(3) If $g \in G_1$ and $g \subset \mathscr{U}_1^*$, then there exists a set of $U \in \mathscr{U}_1$ such that $g \cup h_1(g) \subset U$.

In addition, since $\mathscr{U}_1$ is saturated with to respect to G, part (3) holds for all $g \in G$ which are contained in $\mathscr{U}_1^*$.

Step 2. The decomposition $G'_2 = \{h_1(g): g \in G_2\}$ is clearly cellular and upper semicontinuous. Let P' be the projection map from M onto M/G'_2 and P the projection map from M onto $M/(G_1 \cup G_2)$. Then $P'h_1P^{-1}$ is readily seen to be a homeomorphism from $M/(G_1 \cup G_2)$ onto

M/G_2'. But it was shown in [7] that $M/(G_1 \cup G_2)$ is homeomorphic to M (using Siebenman's generalization [5] of Armentrout's "projection approximation" theorem [1], the results of [7] may be extended to n-manifolds for $n \neq 4$).

Let $K_2 = \{p \in M$: there exists a sequence of nondegenerate elements, each from a different $H_{h_1[G_i]}$, which converges to $p\}$. We construct a saturated (with respect to $h_1[G]$) open refinement of $h_1[\mathscr{U}_1]$ which covers $H_{h_1[G]}$ and misses K_2. Suppose $g' = h_1(g)$ where $g \in H_G - H_{G_1}$. Choose $U_{g'}$ to be saturated (with respect to $h_1[G]$) open set such that

(1) If $\varepsilon_{g'} = \min\{\operatorname{diam} g', 1/2\, d(g', K_2), 1/2\}$, then $U_{g'} \subset S_{\varepsilon_{g'}}(g')$.

(2) If $g_i \in H_{h_1[G_i]}$ and $g_j \in H_{h_1[G_j]} (i \neq j)$ and g_i and g_j are contained in $U_{g'}$, then $1/2 \operatorname{diam} g_i < \operatorname{diam} g_j < 3/2 \operatorname{diam} g_i$.

(3) $h_1^{-1}(U_{g'}) \subset S_{1/4}(g)$.

(4) If $W = \bigcap\{U\colon U \in h_1[\mathscr{U}_1]$ and $h_1(g) \subset U\}$, then $U_{g'} \subset W$.

(5) If $V \in \mathscr{U}_1$ and $g \cup h_1(g) \subset V$, then $U_{g'} \subset V$.

(6) $U_{g'} \cap \operatorname{Cl}(h_1[H_{G_1}^*]) = \varnothing$.

Let $\mathscr{U}_2' = \{U_{g'}\colon g' \in H_{h_1[G]}\}$ and let $\mathscr{U}_2$ be a saturated open locally finite star refinement of $\mathscr{U}_2'$ covering $H_{h_1[G]}^*$. Let

$$\mathscr{V}_2 = \{U \in \mathscr{U}_2\colon U \cap H_{h_1[G_2]}^* \neq \varnothing\} .$$

Note that $h_1^{-1}(\mathscr{U}_1^*) \subset S_{1/2}(H_G^*)$ and $h_1^{-1}(\mathscr{V}_2^*) \subset S_{1/2}(H_{G_2}^*)$.

By Lemma 1, there is a closed map h_2 from M onto M such that

(1) $G_2' = \{h_2^{-1}(x)\colon x \in M\}$.

(2) If $x \in M - \mathscr{V}_2^*$, then $h_2(x) = x$.

(3) For each $g' \in G_2'$ contained in $\mathscr{U}_2^*$, there exists a $U \in \mathscr{U}_2$ such that $h_2(g') \cup g' \subset U$.

Claim. For each $g \in G$ contained in $\mathscr{U}_1^*$, there exists a $W \in \mathscr{U}_1$ such that $g \cup h_2h_1(g) \subset W$.

Proof of Claim. Suppose $g \in G$ and $g \subset \mathscr{U}_1^*$. Then there exists $U \in \mathscr{U}_1$ such that $h_1(g) \cup g \subset U$. If $g \in H_{G_1}$ or if $h_1(g)$ is not contained in $\mathscr{V}_2^*$, then $h_2h_1(g) = h_1(g)$, and we are done. Suppose then that $g \notin H_{G_1}$ and $h_1(g) \cap \mathscr{V}_2^* \neq \varnothing$. Since $\mathscr{U}_2'$ is a refinement of $h_1[\mathscr{U}_1]$ and $\mathscr{U}_2$ is a locally finite star refinement of $\mathscr{U}_2'$, we may find $U_2 \in \mathscr{U}_2$ and $U_{g'} \in \mathscr{U}_2'$, where $h_1(g) = g'$, such that $h_1(g) \subset U_2 \subset \operatorname{St}(U_2, \mathscr{U}_2) \subset U_{g'}$. We first show that there exists a $V \in \mathscr{U}_1$ such that $U_{g'} \subset V$. Of course, $h_1(g) = g' \subset U_{g'}$. Let $V_1, V_2, \cdots, V_n$ be those members of $\mathscr{U}_1$ which contain g. Then by our construction of $\mathscr{U}_2'$,

$$U_{g'} \subset h_1(V_1) \cap \cdots \cap h_1(V_n) .$$

Since $h_1(g) \subset U_{g'}$, it follows that $g \subset V_1 \cap \cdots \cap V_n$. But for at least one $i = 1, 2, \cdots$, or n, $h_1(g) \cap g \cup V_i$. Therefore, by (5) in our construction of $\mathscr{U}_2'$, it must be the case that $U_{g'}$ is contained in V_i.

We need only observe now that if $Z \in \mathscr{U}_2$ and $h_1(g) \subset Z$, then $Z \subset V_i$. This is clear since $Z \subset \mathrm{St}(U_2, \mathscr{U}_2) \subset U_{g'} \subset V_i$. Hence, we have that $\mathrm{St}(h_1(g), \mathscr{U}_2)$ is contained in V_i and since

$$h_2h_1(g) \subset \mathrm{St}(h_1(g), \mathscr{U}_2) ,$$

the proof of the claim is complete.

We continue inductively. Assume now that covers $\mathscr{U}_1', \cdots, \mathscr{U}_n'$, $\mathscr{U}_1, \cdots, \mathscr{U}_n, \mathscr{V}_1, \cdots, \mathscr{V}_n$ have been defined so that the conditions listed below are satisfied. We denote $h_kh_{k-1} \cdots h_1$ by $\hat{h}_k$, and $h_0 = \hat{h}_0 =$ identity. For $i = 1, 2, \cdots, n$, let $K_i = \{p \in M$: there exists a sequence of nondegenerate elements converging to p where each element is a member of a different $H_{\hat{h}_{i-1}[G_j]}\}$.

(1) $\mathscr{U}_i' = \{U_{g'}: g' \in H_{\hat{h}_{i-1}[G]}\}$ is a collection of saturated (with respect to $\hat{h}_{i-1}[G]$) open sets which refines $\hat{h}_{i-1}[\mathscr{U}_{i-1}]$ and misses K_i. For each g', $U_{g'}$ is chosen to contain g' such that

(a) If $\varepsilon_{g'} = \min\{\operatorname{diam} g', 1/2\, d(g', K_i)1/i\}$, then $U_{g'} \subset S_{\varepsilon_{g'}}(g')$.

(b) If $g_j \in H_{\hat{h}_{i-1}[G_j]}$ and $g_k \in H_{\hat{h}_{i-1}[G_k]}$ $(j \neq k)$ and g_j and g_k are contained in $U_{g'}$, then $1/2 \operatorname{diam} g_j < \operatorname{diam} g_k < 3/2 \operatorname{diam} g_j$.

(2) $\mathscr{U}_i$ is a saturated open locally finite star refinement of $\mathscr{U}_i'$ and $\mathscr{V}_i = \{U \in \mathscr{U}_i: U \cap H^*_{\hat{h}_{i-1}[G_{i-1}]} \neq \varnothing\}$.

(3) For $i = 1, 2, \cdots, n$ and $1 \leqq j \leqq i-1$,

$$h_j^{-1} \cdots h_{i-2}^{-1}h_{i-1}^{-1}(\mathscr{U}_i^*) \subset S_{1/i}(\hat{h}_{j-1}(H_G^*))$$

and

$$h_j^{-1} \cdots h_{i-2}^{-1}h_{i-1}^{-1}(\mathscr{V}_i^*) \subset S_{1/2}(\hat{h}_{j-1}(H_{G_i}^*)) .$$

(4) For $i = 1, 2, \cdots, n$, h_i is a closed map from M onto M such that if $G_i' = \{\hat{h}_{i-1}(g): g \in G_i\}$ then

(1) $G_i' = \{h_i^{-1}(x): x \in M\}$.

(2) If $x \in M - \mathscr{V}_i^*$, then $h_i(x) = x$.

(3) For each $g' \in G_i'$ which is contained in $\mathscr{U}_i^*$, there exists $U \in \mathscr{U}_i$, such that $h_i(g') \cup g' \subset U$.

(5) For $i = 1, 2, \cdots, n$ and $0 \leqq j \leqq i-1$, if $g \in G$ and $\hat{h}_{i-1}(g)$ is contained in $\mathscr{U}_1^*$, then there exists $U \in \mathscr{U}_{j+1}$ such that $\hat{h}_j(g) \cup \hat{h}_i(g) \subset U$.

(6) $\mathscr{U}_i'^* \cap \mathrm{Cl}(h_{i-1}(H^*_{G_1} \cup \cdots \cup H^*_{G_{i-1}})) = \varnothing$.

Step $n+1$. Let $G_{n+1}' = \{\hat{h}_n(g): g \in G_{n+1}\}$. A proof similar to that employed in Step 2 shows that M/G_{n+1}' is homeomorphic to M. Let $K_{n+1} = \{p \in M$: there exists a sequence of nondegenerate elements converging to p where each element is a member of a different $H_{h_n[G_j]}\}$.

We construct a saturated (with respect to $\hat{h}_n[G]$) open refinement of $h_n[\mathscr{U}_n]$ which covers $H_{\hat{h}_n[G]}$ and misses K_{n+1}. Let $g' = \hat{h}_n(g)$ where $g \in H_G - (H_{G_1} \cup \cdots \cup H_{G_n})$. Choose $U_{g'}$ to be a saturated open set containing g' such that

(1) If $\varepsilon_{g'} = \min\{\operatorname{diam} g', 1/2\, d(g', K_{n+1}), 1/n+1\}$, then $U_{g'} \subset S_{\varepsilon_{g'}}(g')$.

(2) If $g_i \in H_{\hat{h}_n[G_i]}$ and $g_j \in H_{\hat{h}_n[G_j]}$ $(i \neq j)$ and g_i and g_j are contained in $U_{g'}$, then $1/2 \operatorname{diam} g_i < \operatorname{diam} g_j < 3/2 \operatorname{diam} g_i$.

(3) For $i = 1, 2, \cdots, n$, $(h_i h_{i+1} \cdots h_n)^{-1}(U_{g'}) \subset S_{1/2n}(\hat{h}_{i-1}(g))$.

(4) For $i = 1, 2, \cdots, n$, if U^i is the intersection of those sets in $\mathscr{U}_i$ which contain $\hat{h}_{i-1}(g)$, then

$$U_{g'} \subset \hat{h}_n(U^1) \cap h_n h_{n-1} \cdots h_2(U^2) \cap \cdots \cap h_n(U^n) .$$

(5) For $0 \leqq i < n$, if $\hat{h}_i(g) \cup \hat{h}_n(g) \subset U \in \mathscr{U}_n$, then $U_{g'} \subset U$.

(6) $U_{g'} \cap \operatorname{Cl} \hat{h}_n(H^*_{G_1} \cup \cdots \cup H^*_{G_n}) = \varnothing$.

Let $\mathscr{U}'_{n+1} = \{U_{g'}: g' \in H_{G'_{n+1}}\}$, let $\mathscr{U}_{n+1}$ be a saturated open locally finite star refinement of $\mathscr{U}'_{n+1}$, and let $\mathscr{V}_{n+1} = \{U \in \mathscr{U}_{n+1}: U \cap \hat{h}_n[H^*_{G_{n+1}}] \neq \varnothing\}$. By Lemma 1 there exists a closed map h_{n+1} from M onto M such that

(1) $G'_{n+1} = \{h^{-1}_{n+1}(x): x \in M\}$.

(2) If $x \in M - \mathscr{V}^*_{n+1}$, then $h_{n+1}(x) = x$.

(3) For each $g \in G'_{n+1}$ contained in $\mathscr{U}^*_{n+1}$, there exists $U \in \mathscr{U}_{n+1}$ such that $g \cup h_{n+1}(g) \subset U$.

Claim. Suppose $g' = \hat{h}_{n+1}(g)$ is contained in $\mathscr{U}^*_{n+1}$ (g is an element of G). Suppose $0 \leqq i < n+1$. Then there exists $U \in \mathscr{U}_{i+1}$ such that $g' \cup \hat{h}_i(g) \subset U$.

A proof patterned after the proof of the Claim in Step 2 may be used to establish this Claim.

Define $h = \operatorname{Lim} \hat{h}_n$. To see that h is well defined, we observe that for each $x \in M$, there exists an integer N such that for $n > N$,

$$\hat{h}_n(x) = \hat{h}_N(x) = h(x) .$$

This is clearly the case if $x \in H^*_G$, since if N is the first integer such that $x \in H^*_{G_N}$, then $h_N(x)$ does not belong to the succeeding $\mathscr{U}^*_n$, and, hence, is left fixed. If $x \notin \operatorname{Cl} H^*_G$ then choose N such that

$$d(x, \operatorname{Cl} H^*_G) > \frac{1}{N} .$$

Then $h_N(x) \notin \mathscr{U}^*_{N+1}$ (see (3) in the inductive Step $n+1$) and it follows that $h(x) = \hat{h}_n(x)$ for each $n > N$. Finally, consider the case where $x \in (\operatorname{Cl} H^*_G) - H^*_G$. If there exists an open set U such that $U \cap H^*_{G_i} = \varnothing$ for all but a finite number of i, then it again follows from (3) of Step $n+1$ that the required positive integer N exists. On the other hand, if no such U exists, then there is a sequence $\{g_{n_i}\}$ of nondegenerate elements from distinct decompositions G_{i_n} which converges to x. But it was noted in Step 1 that in this case $x \notin \mathscr{U}_1^*$ and thus $h(x) = x$.

We next show that h is continuous. Suppose $\{x_i\}$ is a sequence of points in M converging to a point x. If there exists an open set U containing x such that $U \cap H_{G_i}^* = \emptyset$ for all but at most a finite number of i, then it follows again from (3) of the induction Step $n+1$ that $\{h(x_i)\}$ converges to $h(x)$. If no such U exists, then there are two cases to consider.

Case 1. $x \in (\mathrm{Cl}\, H_G^* - H_G^*)$. Suppose for each i, $x_i \in g_{n_i} \in G_{n_i}$. We may assume that the x_i lie in $\mathscr{U}_1^*$ since if not $h(x_i) = x_i$. But as it was observed in Step 1, since the sequence $\{g_{n_i}\}$ converges to x, we have that the corresponding sequence $\{\mathrm{St}\,(g_{n_i}, \mathscr{U}_1)\}$ also converges to x. It follows from the Claim in Step $n+1$, that $h(x_i) \in \mathrm{St}\,(g_{n_i}, \mathscr{U}_1)$, and, therefore, $\{h(x_i)\}$ converges to $h(x)$.

Case 2. $x \in H_G^*$. Let n be the first integer such that $x \in g_n \in H_{G_n}$. But then $\hat{h}_n(g_n)$ is a point and our construction in the inductive steps reduces this case to Case 1.

That h is onto may be seen by the following argument. Suppose p is a point in M. We assume that $p \in g' \in G$ where $g' \subset \mathscr{U}_1^*$ (if not, $h(p) = p$). For each positive integer i, there exists a point x_i in $\mathscr{U}_1^*$ such that $h_i(x_i) = p$. It follows from the Claim in Step $n+1$ that for each i, $x_i \in \mathrm{St}\,(g', \mathscr{U}_1)$. Since $\mathrm{St}\,(g', \mathscr{U}_1)$ has compact closure (see Step 1), there exists an accumulation point x of the sequence $\{x_i\}$. For simplicity of notation let us assume that $\{x_i\}$ converges to x. We show that $h(x) = p$.

Let $g \in G$ be the member of the decomposition which contains x. Choose N large enough so that $\hat{h}_n(g) = h(g)$ for each $n \geqq N$. First we suppose that there exists a positive integer $K \geqq N$ such that for $n \geqq K$, $S_{1/K}(g) \cap H_{G_n}^* = \emptyset$. Of course, the sequence $\{\hat{h}_K(x_i)\}$ converges to $\hat{h}_K(x)$. But it follows from (3) of Step $n+1$, that for i sufficiently large, we will have $\hat{h}_k(x_i) = \hat{h}_i(x_i) = h(x_i)$. Thus $h(x) = p$, since $\hat{h}_i(x_i) = p$ for all i.

Now suppose that each open set containing x intersects an infinite number of the $H_{G_i}^*$, and, hence, each open set containing $\hat{h}_N(x)$ will also intersect infinitely many of the sets $H_{\hat{h}_N[G_i]}^*$. Thus, $\hat{h}_N(x)$ belongs to K_{n+1} (see Step $n+1$). Since $\{\hat{h}_N(x_i)\}$ converges to $\hat{h}_N(x)$, it follows from conditions (1) and (3) of Step $n+1$ that the sequence

$$\{\mathrm{St}(\hat{h}_N(x_i), \mathscr{U}_N)\}$$

also converges to $\hat{h}_N(x)$.
But the Claim in this step ensures that for $j > N$, $\hat{h}_j(x_i) \cup \hat{h}_N(x_i)$ belongs to $\mathrm{St}\,(\hat{h}_N(x_i), \mathscr{U}_N)$. In particular then for $i > N$,

$$\hat{h}_i(x_i) \cup \hat{h}_N(x_i) \subset \mathrm{St}\,(\hat{h}_N(x_i), \mathscr{U}_N)\ ,$$

and since, $\hat{h}_i(x_i) = p$, it again follows that $h(x) = p$. Thus h is an onto map.

It is easily seen from our construction of h that $G = \{h^{-1}(x): x \in M\}$.

Finally, we must show that h is closed. It suffices to show that if K is a compact subset of M, then $h^{-1}(K)$ is also compact. Since h is onto, for each $x \in K$, there exists a unique element $g_x \in G$ such that $h(g_x) = x$. If $g_x \subset \mathscr{U}_1^*$, let U_{g_x} be a member of $\mathscr{U}_1$ which contains g_x. If g_x is not contained in $\mathscr{U}_1^*$ let U_{g_x} be an open set containing g_x with compact closure. Note that it follows from Step 1 that if g_x is contained in $\mathscr{U}_1^*$, then $\mathrm{St}\,(U_{g_x}, \mathscr{U}_1)$ has compact closure. Since if $g_x \subset \mathscr{U}_1^*$, then $g_x \cup h(g_x) \subset \mathrm{St}\,(U_{g_x}, \mathscr{U}_1)$, and if g_x is not contained in $\mathscr{U}_1^*$, then $h(g_x) = g_x$, the collection $\{U_{g_x}: x \in K\}$ is an open cover of K. Let $U_{g_{x_1}}, \cdots, U_{g_{x_n}}$, be a finite subcover of K, where the first i terms are members of $\mathscr{U}_1$. To finish the proof we need only observe that

$$h^{-1}(K) \subset \mathrm{St}\,(g_{x_1}, \mathscr{U}_1) \cup \cdots \cup \mathrm{St}\,(g_{x_i}, \mathscr{U}_1) \cup U_{g_{x_{i+1}}} \cup \cdots \cup U_{g_{x_n}}$$

and that the right hand set has compact closure. Thus, the conditions of property S have been satisfied, and, hence, M/G is homeomorphic to M.

A decomposition of a metric space is said to be *nondegenerately continuous* if for each $g \in G$, $H_G \cup \{g\}$ is continuous at g.

COROLLARY 1. *Suppose G is a cellular nondegenerately continuous upper semicontinuous decomposition of E^3. Suppose there exists a countable number of planes in E^3, $Q_1, Q_2, \cdots$ such that for each $g \in H_G$, g is contained in at least one of these planes. Then E^3/G is homeomorphic to E^3.*

Proof. For $i = 1, 2, \cdots$, let G_i be the decomposition of E^3 such that $H_{G_i} = \{g \in H_G: g \subset Q_i\}$. Then E^3/G_i is homeomorphic to E^3 [3], and since it is readily verified that $G_1, G_2, \cdots$ satisfy the conditions of Theorem 1, E^3/G is homeomorphic to E^3.

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