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AN ASYMPTOTIC PROPERTY OF SOLUTIONS OF $y''' + py' + qy = 0$

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In this paper, the differential equation

$$y''' + p(x)y' + q(x)y = 0 \tag{1}$$

will be studied subject to the conditions that $p(x) \leqq 0$, $q(x) > 0$, and $p(x)$, $p'(x)$, and $q(x)$ are continuous for $x \in [0, +\infty)$. A solution of (1) will be said to be oscillatory if it changes signs for arbitrarily large values of x. It will be shown that if (1) has an oscillatory solution then every nonoscillatory solution tends to zero as x tends to infinity.

The above result answers a question that was raised in [1].
The following theorem due to Lazer [1] will be basic in our proof.

THEOREM 1. *Suppose $p(x) \leqq 0$ and $q(x) > 0$. A necessary and sufficient condition for (1) to have oscillatory solutions is that for any nontrivial nonoscillatory solution $G(x)$, $G(x)G'(x)G''(x) \neq 0$, $\operatorname{sgn} G(x) = \operatorname{sgn} G''(x) \neq \operatorname{sgn} G'(x)$ for all $x \in [0, +\infty)$, and*

$$\lim_{x\to\infty} G'(x) = \lim_{x\to\infty} G''(x) = 0, \lim_{x\to\infty} G(x) = c \neq \pm\infty .$$

LEMMA 2. *If $G(x)$ is a nonoscillatory solution of (1), where (1) has an oscillatory solution, then*

$$\lim_{x\to\infty} xG'(x) = 0 .$$

Proof. Suppose $G(x) < 0$, $G'(x) > 0$, and $G''(x) < 0$. By Theorem 1, $\int_1^\infty G'(x)dx < \infty$. Let $\varepsilon > 0$. There is an $N > 0$ such that $\int_N^x G'(t)dt < \varepsilon$ for all $x > N$. Thus $\varepsilon > \int_N^x G'(t)dt = G'(\Sigma)[x - N]$ for $N < \Sigma < x$. But $G''(x) < 0$, so $G'(\Sigma)[x - N] \geqq G'(x)[x - N] > G'(x)\cdot x - \varepsilon$ for x large since $G'(x) \to 0$. Thus $2\varepsilon > xG'(x)$ for large x. Hence $\lim_{x\to\infty} xG'(x) = 0$.

LEMMA 3. *If $G(x)$ is as in Lemma 2, then*

$$\left| \int_1^\infty xG''(x)dx \right| < \infty .$$

Proof. Suppose that $G(x) > 0$, $G'(x) < 0$, and $G''(x) > 0$. Integrating by parts, $\int_1^x tG''(t)dt = xG'(x) - G'(1) - G(x) + G(1)$. Thus $\int_1^\infty xG''(x)dx < \infty$ since $\lim_{x\to\infty} xG'(x) = 0$ and $\lim_{x\to\infty} G(x) = K < \infty$.

LEMMA 4. *If $G(x)$ is as in Lemma 2, then*

$$\lim_{x\to\infty} x^2G''(x) = 0 .$$

Proof. Suppose $G(x) > 0, G'(x) < 0, G''(x) > 0$. Since

$$\int_1^\infty xG''(x)dx < \infty ,$$

for $\varepsilon > 0$ there is an $N > 0$ so that for all $x > N$

$$\varepsilon > \int_N^x tG''(t)dt = G''(\Sigma)\int_N^x tdt$$

for some $N < \Sigma < x$.
But since $G'''(x) < 0$ by (1), we have

$$G''(\Sigma)\int_N^x tdt \geqq [G''(x)/2][x^2 - N^2] \geqq [G''(x)/2][x^2] - \varepsilon/2$$

for large x, since $\lim_{x\to\infty} G''(x) = 0$. Thus

$$3\varepsilon > x^2G''(x) \text{ for all large } x .$$

Thus $\lim_{x\to\infty} x^2G''(x) = 0$.

THEOREM 5. *If $G(x) > 0$, $G'(x) < 0$, $G''(x) > 0$ is a solution of* (1) *which has oscillatory solutions then two linearly independent oscillatory solutions of*

$$(2) \qquad y''' + p(x)y' + (p'(x) - q(x))y = 0$$

satisfy the differential equation

$$(3) \qquad (y'/G(x))' + [(G''(x) + p(x)G(x))/G^2(x)]y = 0 .$$

Proof. Let $u(x)$ and $v(x)$ be two solutions of (1) defined by $u(1) = u'(1) = 0$, $u''(1) = 1$, $v(1) = v''(1) = 0$, $v'(1) = 1$. By [1], $u(x)$ and $v(x)$ are linearly independent oscillatory solutions of (1). Let

$$U(x) = u(x)G'(x) - G(x)u'(x)$$
$$V(x) = v(x)G'(x) - G(x)v'(x) .$$

Then $U(x)$ and $V(x)$ are linearly independent oscillatory solutions of (2). Now

$$\begin{vmatrix} V(x) & U(x) \\ V'(x) & U'(x) \end{vmatrix} = G(x)\begin{vmatrix} G(x) & v(x) & u(x) \\ G'(x) & v'(x) & u'(x) \\ G''(x) & v''(x) & u''(x) \end{vmatrix}$$

$$= G(x) \begin{vmatrix} G(1) & 0 & 0 \\ G'(1) & 1 & 0 \\ G''(1) & 0 & 1 \end{vmatrix} = G(1)G(x) .$$

Thus

$$G(1)G'(x) = \begin{vmatrix} V(x) & U(x) \\ V''(x) & U''(x) \end{vmatrix}$$

and

$$G(1)G''(x) = \begin{vmatrix} V'(x) & U'(x) \\ V''(x) & U''(x) \end{vmatrix} + \begin{vmatrix} V(x) & U(x) \\ V'''(x) & U'''(x) \end{vmatrix} .$$

Now $U(x)$ and $V(x)$ are solutions of the differential equation

$$(4) \qquad \begin{vmatrix} V(x) & U(x) & y \\ V'(x) & U'(x) & y' \\ V''(x) & U''(x) & y'' \end{vmatrix} = 0 .$$

But

$$\begin{vmatrix} V(x) & U(x) \\ V'''(x) & U'''(x) \end{vmatrix} = V(x)[- p(x)U'(x) - p'(x)U(x) + q(x)U(x)]$$
$$- U(x)[- p(x)V'(x) - p'(x)V(x) + q(x)V(x)] = - p(x)G(1)G(x) .$$

Thus (4) becomes

$$(5) \quad G(1)G(x)y'' - G(1)G'(x)y' + [G(1)G''(x) + p(x)G(1)G(x)]y = 0$$

or

$$(y'/G(x))' + [(G''(x) + p(x)G(x))/G^2(x)]y = 0 .$$

Our main result now follows.

THEOREM 6. *If $G(x)$ is as in Theorem 5, then* $\lim_{x\to\infty} G(x) = 0$.

Proof. Suppose not. By Theorem 1, $\lim_{x\to\infty} G(x) = K < \infty$. Suppose without loss of generality that $K = 1$. Now for large x, $G(x) < 2$, hence

$$1/G(x) > 1/2 .$$

Also

$$G''(x) \geqq G''(x)/G^2(x) \geqq G''(x)/G^2(x) + p(x)G(x)/G^2(x) .$$

Since (3) is oscillatory, by the Sturm-Picone Theorem [2]

$$(y'/2)' + G''(x)y = 0 \tag{6}$$

is oscillatory. Letting $y = x^{1/2}z$, (6) becomes

$$(xz')' + (2x^2G''(x) - 1/4)x^{-1}z = 0 . \tag{7}$$

But since $\lim_{x\to\infty} x^2G''(x) = 0$, $(2x^2G'' - 1/4)$ is eventually negative and so (7) is clearly nonoscillatory. From this contradiction, we conclude $\lim_{x\to\infty} G(x) = 0$.

References

1. A. C. Lazer, *The behavior of solutions of the differential equation* $y''' + p(x)y' + q(x)y = 0$, Pacific J. Math., **17** (1966), 435-466.
2. Walter Leighton, *Ordinary Differential Equations*, Wadsworth Publishing Company, Belmont, California, 1967.

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