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A TOPOLOGICAL CHARACTERIZATION OF COMPLETE, DISCRETELY VALUED FIELDS

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It is shown that the topology of a topological field F is given by a complete, discrete valuation if and only if F is locally strictly linearly compact. More generally, the topology of a topological division ring K is given by a complete, discrete valuation and K is finite dimensional over its center if and only if K is locally centrally linearly compact, that is, if and only if K contains an open subring B, the open left ideals of which form a fundamental system of neighborhoods of zero, such that B, regarded as a module over its center, is strictly linearly compact.

In [5], Jacobson showed that the topology of an indiscrete, totally disconnected, locally compact division ring is given by a discrete valuation (that is, a valuation whose value group is isomorphic to the cyclic group of integers). Consequently, an indiscrete topological division ring K is locally compact and totally disconnected if and only if its topology is given by a complete, discrete valuation whose residue field is finite [4, Prop. 2, p. 118, Prop. 1, p. 156]. From this, it follows rather readily that the center C of K is indiscrete, that K is finite dimensional over C, and that C is either a finite extension of the p-adic number field for some prime p or the field of formal power series over a finite field [4, Theorem 1, p. 158].

Our purpose here is to generalize Jacobson's theorem by characterizing those topological fields whose topology is given by a complete, discrete valuation, and more generally, those topological division rings K such that K is finite dimensional over its center and the topology of K is given by a complete, discrete valuation.

For this purpose, we assume some familiarity with basic properties of linearly compact and strictly linearly compact modules and rings, as developed in [10] or [3, Exercises 14-22, pp. 108-112]. We recall that a (left) topological A-module E (it is not assumed that E is unitary) is *linearly topologized* if the open submodules of E form a fundamental system of neighborhoods of zero; E is *linearly compact* if E is Hausdorff, linearly topologized, and every filter base of cosets of submodules has an adherent point; E is *strictly linearly compact* if E is linearly compact and every continuous epimorphism from E onto a Hausdorff, linearly topologized A-module is open (equivalently, if E/U is an artinian A-module for every open submodule U of E). A topological ring A is respectively linearly topologized, linearly compact, or strictly linearly compact if the associated left A-module A is.

DEFINITION. A topological ring A is *locally strictly linearly compact* if A has an open subring B that is strictly linearly compact for its induced topology.

To handle the noncommutative case, we need the following definition:

DEFINITION. A topological ring B is *centrally linearly compact* if the open left ideals of B form a fundamental system of neighborhoods of zero and if B, regarded as a module over its center C_B, is a strictly linearly compact C_B-module. A topological ring A is *locally centrally linearly compact* if A contains an open subring that is centrally linearly compact for its induced topology.

Thus a commutative topological ring is (locally) centrally linearly compact if and only if it is (locally) strictly linearly compact. Note that if B is a centrally linearly compact ring, then for any subring B_0 of B that contains the center C_B, B is a strictly linearly compact B_0-module (in particular, B is a strictly linearly compact ring); indeed, since the open left ideals of B form a fundamental system of neighborhoods of zero, B is a linearly topologized B_0-module, and since a B_0-submodule is also a C_B-submodule, every filter base of cosets of B_0-submodules necessarily has an adherent point.

By a topological division ring (field) K we mean a topological ring that is algebraically a division ring (field); that is, we do not assume that $x \mapsto x^{-1}$ is continuous on the set K^* of nonzero elements.

LEMMA 1. *If B is an open, centrally linearly compact subring of an indiscrete topological division ring K, then there is an open, centrally linearly compact subring B_1 of K that contains* 1.

Proof. Let B_1 be the closure of the subring generated by B and 1. The open left ideals of B then form a fundamental system of neighborhoods of zero in B_1; each open left ideal $\mathfrak{a}$ of B is a left ideal of B_1, for as $\mathfrak{a}$ is closed, $\{x \in B_1 : x\mathfrak{a} \subseteq \mathfrak{a}\}$ is a closed subring of B_1 containing B and 1 and hence is all of B_1.

Since B is open, $B \neq (0)$; let b be some nonzero element of B, and let c be its inverse in K. Then, $B_1 = B_1bc \subseteq B_1Bc$, so $B_1 \subseteq Bc$ since, as we saw above, B is a left ideal of B_1. Thus $Bc \supseteq B_1 \supseteq B$, so Bc is a linearly topologized C_B-module, where C_B is the center of B. Hence Bc is a strictly linearly compact C_B-module, as it is the image of the strictly linearly compact C_B-module B under the continuous homomorphism $x \mapsto xc$. Consequently, the closed C_B-submodule

B_1 of Bc is strictly compact; as C_B is contained in the center of B_1, B_1 is *a fortiori* strictly linearly compact over its center.

We recall that an element a of a topological ring is *topologically nilpotent* if $\lim a^n = 0$.

LEMMA 2. *Let K be a Hausdorff topological division ring, let B be an open subring of K that contains 1, and let $\mathfrak{r}$ be the radical of B. If B is strictly linearly compact, then B is a (left) noetherian ring, $B/\mathfrak{r}$ is a division ring, the topology of B is the $\mathfrak{r}$-adic topology, and $\mathfrak{r}$ is the set of all topological nilpotents of B.*

Proof. As B is open and as $y \mapsto yx$ is a homeomorphism for each $x \in K^*$, Bx is open for every $x \in K^*$, and hence every nonzero left ideal of B is open. Let $\mathfrak{s} = \bigcap_{n=1}^{\infty} \mathfrak{r}^n$. Assume that $\mathfrak{s} \neq (0)$. Then $\mathfrak{s}$ is open, so $B/\mathfrak{s}$ is an artinian B-module and hence an artinian ring. Consequently, its radical $\mathfrak{r}/\mathfrak{s}$ is nilpotent, so there exists n such that $\mathfrak{r}^n = \mathfrak{s}$. Hence $(0) \neq \mathfrak{r}^n = \mathfrak{r}^{n+1} = \cdots$, in contradiction to [10, Theorem 9]. Therefore, $\bigcap_{n=1}^{\infty} \mathfrak{r}^n = (0)$.

Since every nonzero left ideal of B is open and hence closed, B is a (left) noetherian ring, $B/\mathfrak{r}$ is an artinian ring, and the topology of B is its $\mathfrak{r}$-adic topology by [13, Theorem 16]. Consequently, every element of $\mathfrak{r}$ is a topological nilpotent. Therefore, as B is complete, B is suitable for building idempotents [11, Lemma 4; 6, Definition 1, p. 53]. Thus every idempotent of $B/\mathfrak{r}$ is the coset of $\mathfrak{r}$ determined by an idempotent of B [6, Proposition 4, p. 54]. But as K is a division ring, B has no idempotents other than 0 and 1. Thus $B/\mathfrak{r}$ is an artinian, semisimple ring whose only idempotents are 0 and 1. By the Wedderburn-Artin theorem, therefore, $B/\mathfrak{r}$ is a division ring. In particular, if $x \notin \mathfrak{r}$, then $x + \mathfrak{r}$ is not a nilpotent of $B/\mathfrak{r}$, so x is not a topological nilpotent.

THEOREM 1. *If K is an indiscrete, Hausdorff topological field, then the topology of K is given by a complete, discrete valuation if and only if K is locally strictly linearly compact.*

Proof. Necessity. It is well known that a complete, semilocal noetherian ring, equipped with its natural $\mathfrak{r}$-adic topology, is strictly linearly compact [cf. 13, Corollary of Lemma 2]. In particular, the valuation ring of a complete discrete valuation is strictly linearly compact.

Sufficiency. By Lemma 1, there is an open, strictly linearly compact subring B of K that contains 1. By Lemma 2, B is a complete, local noetherian domain, and its topology is its natural $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the maximal ideal of B. In particular, B is not a field since B is not discrete. Therefore, as B is open in the topological field K, the topology of K is defined by a complete, discrete valuation [12, Theorem 6].

THEOREM 2. *If K is an indiscrete, Hausdorff topological division ring, then the topology of K is given by a complete, discrete valuation and K is finite-dimensional over its center C if and only if K is locally centrally linearly compact; in this case, C is indiscrete, and hence its topology is given by a complete, discrete valuation.*

Proof. Necessity. As K is finite-dimensional over C, the valuation induced on C by that of K is not the improper valuation; hence as C is closed, the topology of C is given by a complete, discrete valuation v. Let $e_1, \cdots, e_n$ be a basis of K over C such that $e_1 = 1$, and let $e_ie_j = \sum_{k=1}^{n} \alpha_{ijk}e_k$. Let $\lambda \in C$ be such that $v(\lambda) \geqq 0$ and $v(\lambda) \geqq -\min\{v(\alpha_{ijk}): 1 \leqq i, j, k \leqq n\}$. Let $f_1 = 1$ and $f_k = \lambda e_k$ for $2 \leqq k \leqq n$. Let V be the valuation ring of C, and for each $m \geqq 0$ let $V_m = \{x \in V: v(x) \geqq m\}$. Let $B = Vf_1 + \cdots + Vf_n$, and for each $m \geqq 0$ let $\mathfrak{b}_m = V_mf_1 + \cdots + V_mf_n$. Easy calculations establish that B is a ring and that $\mathfrak{b}_m$ is an ideal of B for each $m \geqq 0$. By [2, Theorem 2, p. 18], $F: (\lambda_1, \cdots, \lambda_n) \mapsto \sum_{i=1}^{n} \lambda_i f_i$ is a topological isomorphism from the C-vector space C^n onto the C-vector space K. Hence B is an open subring of K, and $(\mathfrak{b}_m)_{m \geqq 0}$ is a fundamental system of neighborhoods of zero in B, each an ideal of B. We saw earlier that V is strictly linearly compact; hence as $B = F(V^n)$, B is a strictly linearly compact V-module and, *a fortiori*, is a centrally linearly compact ring.

Sufficiency. By Lemma 1, there is an open, centrally linearly compact subring B that contains 1. Let $\mathfrak{r}$ be the radical of B. As the $\mathfrak{r}$-adic topology is the given indiscrete topology of B by Lemma 2, there exists a nonzero $a \in B$ such that $\lim a^n = 0$. Let K_0 be the closed subfield generated by C and a, let $B_0 = K_0 \cap B$, and let $\mathfrak{r}_0$ be the radical of B_0. Since the open left ideals of B form a fundamental system of neighborhoods of zero for B, the open ideals of B_0 form a fundamental system of neighborhoods of zero for B_0. Moreover, the center C_B of B is simply $C \cap B$; indeed, if $c \in C_B$ and if $x \in K$, then $a^nx \in B$ for some n as $\lim a^nx = 0$, whence $(a^nx)c = c(a^nx) = (ca^n)x = (a^nc)x$, so $xc = cx$. Thus $C_B = C \cap B \subseteqq K_0 \cap B = B_0$, so B_0 is a closed C_B-submodule of B and hence is a strictly linearly compact C_B-module. Consequently, B_0 is a strictly linearly compact ring, so by Lemma 2, the topology of B_0 is the $\mathfrak{r}_0$-adic topology, and $\mathfrak{r}$ and $\mathfrak{r}_0$ are respectively the set of topological nilpotents in B and B_0, whence $\mathfrak{r}_0 = \mathfrak{r} \cap B_0$. Hence $\bigcap_{n=1}^{\infty} (\mathfrak{r}_0^n B)^- \subseteqq \bigcap_{n=1}^{\infty} \mathfrak{r}^n = (0)$. As the topology of B_0 is indiscrete, $\mathfrak{r}_0^2 \neq (0)$, so $\mathfrak{r}_0^2 B$ is open as it contains a nonzero left ideal of B. By

[13, Theorem 10], $\mathfrak{r}_0 B$ is a finitely generated B_0-module; let $\mathfrak{r}_0 B = B_0 x_1 + \cdots + B_0 x_m$. Also as B is a strictly linearly compact C_B-module and as $\mathfrak{r}_0 B$ is open, $B/\mathfrak{r}_0 B$ is an artinian C_B-module, hence an artinian B_0-module; now $B/\mathfrak{r}_0 B$ admits the structure of $B_0/\mathfrak{r}_0$-module, and $B_0/\mathfrak{r}_0$ is a field by Lemma 2; consequently $B/\mathfrak{r}_0 B$ is an artinian, therefore, finite-dimensional, and hence noetherian $B_0/\mathfrak{r}_0$-vector space; thus $B/\mathfrak{r}_0 B$ is a noetherian B_0-module. Let $x_{m+1}, \cdots, x_n \in B$ be such that $B = B_0 x_{m+1} + \cdots + B_0 x_n + \mathfrak{r}_0 B$. Then $B = B_0 x_1 + \cdots + B_0 x_n$. Consequently, $x_1, \cdots, x_n$ is a set of generators of the K_0-vector space K, for if $z \in K$, there exists t such that $a^t z \in B$, whence $a^t z = b_1 x_1 + \cdots + b_n x_n$ where $b_i \in B_0$, and thus $z = (a^{-t} b_1) x_1 + \cdots + (a^{-t} b_n) x_n \in K_0 x_1 + \cdots + K_0 x_n$. By [1, Theorem 16], the centralizer K_0' of K_0 has degree $\leqq n$ over C. But $K_0' \supseteqq K_0$ as K_0 is commutative. Moreover, the topology of K_0 is given by a discrete valuation by Theorem 1, as B_0 is an open, strictly linearly compact subring. Therefore, as $[K_0: C] \leqq n$, the valuation induced on C is not the improper valuation; hence the topology of C is given by a complete, discrete valuation. As

$$[K: C] = [K: K_0][K_0: C] \leqq n^2 ,$$

the given topology of K is the only topology for which K is a Hausdorff topological vector space over C [2, Theorem 2, p. 18]; by valuation theory, that topology is given by a complete, discrete valuation.

The idea of using [1, Theorem 16] is suggested by Kaplansky's treatment of locally compact division rings in [8].

Jacobson's theorem concerning totally disconnected locally compact division rings follows at once from Theorem 2. Indeed, if K is an indiscrete, totally disconnected, locally compact division ring, then K contains a compact open subring B [9, Lemma 4]; the open ideals of B form a fundamental system of neighborhoods of zero [7, Lemmas 9 and 10], and therefore the compact ring B is clearly centrally linearly compact; by Theorem 2, K is finite-dimensional over its center, which is indiscrete, and the topology of K is given by a complete, discrete valuation.

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