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EXISTENCE OF DIRICHLET FINITE BIHARMONIC FUNCTIONS ON THE POINCARÉ 3-BALL

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In an earlier study we discussed the existence of quasiharmonic functions, i.e., solutions of $\Delta u = 1$. We showed, in particular, that there exist Dirichlet finite quasiharmonic functions on the Poincaré 3-ball

$$B_\alpha : \{|x| < 1, ds = (1 - |x|^2)^\alpha |dx|\}$$

if and only if $\alpha \in (-3/5, 1)$. We now ask: Is the existence of these functions entailed by that of Dirichlet biharmonic functions? This is known to be the case for dimension 2. We shall show that, perhaps somewhat unexpectedly, it is no longer true for dimension 3.

For preparation we first solve the problem, of significance in its own right, of the existence of Dirichlet finite biharmonic functions. In the notation of No. 1 below, we give the complete characterization

$$B_\alpha \notin O_{H^2D} \Longleftrightarrow \alpha > -\frac{3}{5}\ .$$

The problem also offers considerable technical interest, as the generating harmonic functions can not be presented in a closed form, but only by means of expansions at the regular singular point of the related differential equation. This makes the estimates somewhat delicate. Also, the four cases $\alpha \geqq 1$, $\alpha \in (-3/5, 1)$, $\alpha < -3/5$, and $\alpha = -3/5$ must be treated separately, each with its own approach.

To deduce the above result (Theorem 1), we first expand a harmonic function on B_α in terms of spherical harmonics with respect to our non-Euclidean metric (Theorem 2). As important applications of Theorem 1 to the classification theory we obtain a decomposition of the totality of Riemannian 3-manifolds into three disjoint nonempty subclasses induced by O_{QD} and O_{H^2D} (Theorem 3), and establish the existence of parabolic 3-manifolds which carry H^2D-functions and of hyperbolic 3-manifolds which do not carry H^2D-functions (Theorem 4).

An interesting open problem is whether $B_\alpha \notin O_{H^2D}$ if and only if $\alpha > -3/(N+2)$.

1. A function u is harmonic or biharmonic according as it satisfies $\Delta_\lambda u = 0$ or $\Delta_\lambda^2 u = 0$, where Δ_λ is the Laplace-Beltrami operator $\Delta_\lambda = d\delta + \delta d$ with respect to the metric $ds = \lambda(x)\,|dx|$. Denote by H^2 the family of nonharmonic biharmonic functions, by D the family of

functions f with finite Dirichlet integrals $D(f) = \int df \wedge *df < \infty$, and set $H^2D = H^2 \cap D$. Let O_{H^2D} be the class of Riemannian manifolds which do not carry H^2D-functions. We assert:

THEOREM 1. $B_\alpha \notin O_{H^2D} \Leftrightarrow \alpha > -3/5$.

The proof will be given in Nos. 2–7.

2. We start by expanding a harmonic function on B_α in spherical harmonics. We recall that a function $S_n(\theta^1, \theta^2)$, in polar coordinates (r, θ^1, θ^2), is called a spherical harmonic of degree n if $r^n S_n(\theta^1, \theta^2)$ is harmonic with respect to the Euclidean metric. Every such function is a unique linear combination of $2n + 1$ linearly independent fundamental spherical harmonics S_{nm} of degree n. The class $\{S_{nm}; n = 0, 1, 2, \cdots; m = 1, 2, \cdots, 2n + 1\}$ is not only an orthogonal system with respect to the inner product $(f, g) = \int_\omega fg\, dS$, with ω the 2-sphere and dS the surface element, but also a complete system with respect to the family of L^2-functions. For every harmonic function h in B_α, we have a Fourier expansion

$$h(r, \theta) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} d_{nm}(r) S_{nm}(\theta) \tag{1}$$

with $\theta = (\theta^1, \theta^2)$.

By virtue of

$$\Delta_\lambda(f(r)S_{nm}(\theta)) = -\lambda^{-2}[f''(r) + \left(\frac{2}{r} + \frac{\lambda'}{\lambda}\right)f'(r) - n(n + 1)r^{-2}f(r)]S_{nm}(\theta) ,$$

and $\lambda'\lambda^{-1} = -2\alpha r(1 - r^2)^{-1}$, the function $f(r)S_{nm}(\theta)$ is harmonic on B_α if and only if $f(r)$ satisfies the differential equation

$$\begin{aligned} r^2(1 - r^2)f''(r) + r[2(1 - r^2) - 2\alpha r^2]f'(r) \\ - n(n + 1)(1 - r^2)f(r) = 0 . \end{aligned} \tag{2}$$

We shall denote the solution of equation (2) for each n by $f_n(r)$. Since all coefficients in (2) can be expanded into power series of r, the point 0 is a regular singular point of the equation. Thus there exists at least one solution of (2) in the form

$$f_n(r) = r^{p_n} \sum_{i=0}^{\infty} c_{ni} r^i , \tag{3}$$

$c_{n0} \neq 0$. On substituting in (2) we have

$$(4)\qquad \sum_{i=0}^{\infty}[(p_n+i-1)(p_n+i)+2(p_n+i)-n(n+1)]c_{ni}r^{p_n+i}$$
$$-\sum_{i=2}^{\infty}\{(p_n+i-3)(p_n+i-2)+(2+2\alpha)(p_n+i-2)$$
$$-n(n+1)\}c_{n,i-2}r^{p_n+i}=0\,.$$

To determine p_n we equate to 0 the coefficient of r^{p_n} and obtain the indicial equation

$$(p_n-1)p_n+2p_n-n(n+1)=0$$

which gives $p_n=n$ or $p_n=-(n+1)$. Since $0\in B_\alpha$, p_n can not be negative, and therefore $p_n=n$.

We then equate to 0 the coefficient $2(n+1)c_{n1}$ of r^{p_n+1} and obtain $c_{n1}=0$.

To find c_{ni}, $i\geqq 2$, we equate to 0 the coefficient of r^{p_n+i}:

$$[(p_n+i)(p_n+i+1)-n(n+1)]c_{ni}$$
$$=[(p_n+i-2)(p_n+i-1+2\alpha)-n(n+1)]c_{n,i-2}\,.$$

On letting $p_n=n$ and $c_{n0}=1$ we have

$$(5)\qquad c_{n,2i}=\prod_{j=1}^{i}\frac{(n+2j-2)(n+2j-1+2\alpha)-n(n+1)}{(n+2j)(n+2j+1)-n(n+1)}$$

for $i\geqq 1$, and $c_{n,2i+1}=0$ for $i\geqq 0$.

The limit of $f_n(r)=\sum_{i=0}^{\infty}c_{n,2i}r^{n+2i}$ as $r\to 1$ exists since the $c_{n,2i}$ are of constant sign as soon as i is sufficiently large. Furthermore, this limit can not be zero, for otherwise $\lim_{r\to 1}f_nS_{nm}\equiv 0$, and consequently $f_n\equiv 0$, contrary to $c_{n0}=1$. In a similar fashion we see that $f_n(r)\neq 0$ for $0<r<1$. Hence for arbitrary but fixed r_0, $0<r_0<1$, there exist constants a_{nm} such that $a_{nm}f_n(r_0)S_{nm}=d_{nm}(r_0)S_{nm}$, and

$$(6)\qquad \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}a_{nm}f_n(r)S_{nm}(\theta)$$

is a series of functions harmonic on B_α which converges absolutely and uniformly to $h(r_0,\theta)$ on the 2-sphere of radius r_0. Now let $r_0<r'<1$; then by the same argument there exist constants a'_{nm} such that

$$(7)\qquad \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}a'_{nm}f_n(r)S_{nm}(\theta)$$

converges to h on the ball of radius r'. Hence (6) and (7) are identical on the ball of radius r_0, so that $a_{nm}=a'_{nm}$ for all (n,m).

We have proved:

THEOREM. *Every harmonic function $h(r, \theta^1, \theta^2)$ on the Poincaré ball B_α has the expansion in terms of the fundamental spherical harmonics S_{nm},*

$$h(r, \theta^1, \theta^2) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} a_{nm} \sum_{i=0}^{\infty} c_{n,2i} r^{n+2i} S_{nm}(\theta^1, \theta^2) \ , \tag{8}$$

where the $c_{n,2i}$ are given by (5).

3. After this preparation, we proceed with the proof of Theorem 1. An essential aspect of the proof is that the cases $\alpha \geqq 1, \alpha \in (-3/5, 1), \alpha < -3/5$, and $\alpha = -3/5$ all require a different treatment. We first establish the following crucial estimate:

LEMMA 1. *If $\alpha \geqq 1$, then*

$$f_1(r) = \sum_{i=0}^{\infty} c_{1,2i} r^{1+2i} = O((1-r)^{-2\alpha}) \ \text{as} \ r \longrightarrow 1 \ .$$

Proof. By (5),

$$\begin{aligned} c_{1,2i} &= \prod_{j=1}^{i} \frac{(2j-1)(2j+2\alpha)-2}{(2j+1)(2j+2)-2} \\ &= \prod_{j=1}^{i} \frac{2j+2\alpha-1}{2j} \cdot \frac{2j-1+(2j-3)/(2j+2\alpha-1)}{2j+3} \ . \end{aligned}$$

We claim that

$$c_{1,2i} < \prod_{j=1}^{i} \frac{2j+2\alpha-1}{2j} \tag{9}$$

or, equivalently,

$$4 - \frac{2j-3}{2j+2\alpha-1} > 0 \ . \tag{10}$$

In the case $\alpha \geqq 1$ under consideration this is clearly so for all $j \geqq 1$. Consequently

$$f_1(r) < r + \sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \frac{2j+2\alpha-1}{2j} \right) r^{1+2i} \ .$$

We compare this with the expansion

$$\begin{aligned} r(1-r)^{-2\alpha} &= r + \sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \frac{j+2\alpha-1}{j} \right) r^{1+i} \\ &> r + \sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \frac{2j+2\alpha-1}{2j} \right) r^{1+2i} \end{aligned}$$

and obtain the lemma.

4. We shall make use of Lemma 1 to prove:

LEMMA 2. *$B_\alpha \notin O_{H^2D}$ for $\alpha \geqq 1$.*

Proof. A necessary and sufficient condition for the existence of an H^2D-function u is that the Laplacian $\Delta u = h$ satisfies

$$|(h, \varphi)| \leqq K\sqrt{D(\varphi)} \tag{11}$$

for all $\varphi \in C_0^\infty$ and some constant K independent of φ (Nakai-Sario [5]).

Let $h = f_1(r)S_{11} = f_1(r)\cos\theta^1$, and take any $\varphi \in C_0^\infty(B_\alpha)$. By Lemma 1 and the Fourier expansion

$$\varphi = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} b_{nm}(r)S_{nm}(\theta^1, \theta^2)\ ,$$

we obtain

$$\begin{aligned}|(h, \varphi)| &= \left|\text{const}\int_0^1 b_{11}(r)f_1(r)r^2(1-r^2)^{3\alpha}dr\right| \\ &< \text{const}\int_0^1 |b_{11}(r)|(1-r)^\alpha dr\ .\end{aligned}$$

By Schwarz's inequality,

$$\begin{aligned}|(h, \varphi)|^2 &\leqq \text{const}\int_0^1 (1-r)^\alpha dr \cdot \int_0^1 b_{11}^2(r)(1-r)^\alpha dr \\ &= \text{const}\int_0^1 b_{11}^2(r)(1-r)^\alpha dr\ .\end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned}&D(\varphi) \\ &= \int_{B_\alpha} |\operatorname{grad}\varphi|^2 dV \geqq \text{const}\int_{B_\alpha} r^{-2}(1-r^2)^{-2\alpha}\left(\frac{\partial\varphi}{\partial\theta^1}\right)^2 r^2(1-r^2)^{3\alpha}dr \\ &\geqq \text{const}\int_0^1 b_{11}^2(r)(1-r)^\alpha dr\ .\end{aligned} \tag{13}$$

5. Denote by Q the class of quasiharmonic functions u, characterized by $\Delta_\lambda u = 1$. We recall (Sario-Wang [9]) that $B_\alpha \notin O_{QD}$ if and only if $\alpha \in (-3/5, 1)$. Since $QD \subset H^2D$, we have trivially:

LEMMA 3. *$B_\alpha \notin O_{H^2D}$ if $\alpha \in (-3/5, 1)$.*

6. Next we consider the case $\alpha < -3/5$.

LEMMA 4. *$B_\alpha \in O_{H^2D}$ if $\alpha < -3/5$.*

Proof. Suppose there exists an H^2D-function u on B_α, that is,

$\Delta u = h$ satisfies (11). By Theorem 2, h has the expansion

$$h = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \sum_{i=0}^{\infty} a_{nm} c_{n,2i} r^{n+2i} S_{nm} .$$

If $a_{nm} \neq 0$ for some (n, m), choose for our testing functions φ_t, $0 < t \leqq 1$,

$$\varphi_t(r, \theta) = \rho_t(r) S_{nm}(\theta), \qquad \rho_t(r) = g\left(\frac{1-r}{t}\right),$$

where $g(r)$ is a fixed nonnegative C_0^∞-function with $\operatorname{supp} g \subset (\beta, \gamma)$, $0 < \beta < \gamma < 1$. Since $\lim_{r\to 1} f_n(r) \neq 0$,

$$\int_{1-\gamma t}^{1-\beta t} \rho_t(r) dr = t \int_\beta^\gamma g(r) dr ,$$

and $(1 - r^2)^{3\alpha} > 2^{3\alpha}(1-r)^{3\alpha} \geqq 2^{3\alpha}(\gamma t)^{3\alpha}$ for $\alpha < 0$, we have for sufficiently small t,

$$\begin{aligned} |(h, \varphi_t)| &= \text{const} \left| \int_{1-\gamma t}^{1-\beta t} f_n(r) \rho_t(r) r^2 (1-r^2)^{3\alpha} dr \right| \\ &\geqq \text{const } (1-\gamma)^2 (\gamma t)^{3\alpha} \int_{1-\gamma t}^{1-\beta t} \rho_t(r) dr = \text{const } t^{3\alpha+1} . \end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned} D(\varphi_t) &= \int_{B_\alpha} |\operatorname{grad} \varphi_t|^2 dV \\ &= \int_{1-\gamma t}^{1-\beta t} (1-r^2)^{-2\alpha} (c_1(\rho'(r))^2 + c_2 r^{-2} \rho^2(r)) r^2 (1-r^2)^{3\alpha} dr \\ &< \text{const } (\gamma t)^\alpha \int_{1-\gamma t}^{1-\beta t} (c_1(\rho'(r))^2 + c_2 \rho^2(r)) dr \\ &= d_1 t^{\alpha-1} + d_2 t^{\alpha+1} < d t^{\alpha-1} , \end{aligned} \tag{15}$$

$\alpha < 0$, where d_1, d_2, and d are independent of t. If $\alpha < -3/5$, then (11) is violated as $t \to 0$, a contradiction. Thus $B_\alpha \in O_{H^2D}$ for $\alpha < -3/5$.

7. It remains to consider the case $\alpha = -3/5$.

LEMMA 5. $B_{-3/5} \in O_{H^2D}$.

Proof. We choose a decreasing sequence of real numbers $t_j \in (0, 1]$ tending to 0 such that $1 - \beta t_j < 1 - \gamma t_{j+1}$ and (14) is satisfied for each t_j. Set $q_j = t_j^{-3\alpha-1} j^{-1} \cdot \operatorname{sign}\ (h, \varphi_{t_j})$ and take for the testing functions $\varphi_n = \sum_{j=1}^n q_j \varphi_{t_j}$. We obtain by (14)

$$|(h, \varphi_n)| = \left| \sum_{j=1}^n q_j (h, \varphi_{t_j}) \right| > \text{const} \sum_{j=1}^n j^{-1}$$

and by (15)

$$D(\varphi_n) = \sum_{j=1}^{n} q_j^2 D(\varphi_{t_j}) < \text{const} \sum_{j=1}^{n} j^{-2}(t_j^{-5\alpha-3}) .$$

For $\alpha = -3/5$, we have $D(\varphi_n) < \text{const} \sum_1^n j^{-2}$, which stays bounded as $n \to \infty$ whereas $|(h, \varphi_n)| \to \infty$. Thus (11) is violated, and we conclude that $B_{-3/5} \in O_{H^2D}$.

The proof of Theorem 1 is herewith complete.

8. Since $B_\alpha \notin O_{QD}$ if and only if $\alpha \in (-3/5, 1)$, Theorem 1 has the following applications to the classification of Riemannian manifolds, with $\tilde{O}$ standing for the complement of O:

THEOREM 3. *The totality of Riemannian 3-manifolds has the decomposition*

$$\{R\} = O_{H^2D} \cup (O_{QD} \cap \tilde{O}_{H^2D}) \cup \tilde{O}_{QD}$$

into three disjoint nonempty subclasses.

THEOREM 4. *There exist parabolic Riemannian 3-manifolds which carry H^2D-functions, and hyperbolic Riemannian 3-manifolds which do not carry H^2D-functions.*

For dimension 2, this was shown in Nakai-Sario [5], but for higher dimensions it has been an open problem.

For the proof of Theorem 4, let O_G be the class of parabolic Riemannian manifolds. It was proved in Sario-Wang [9] that $B_\alpha \in O_G$ if and only if $\alpha \geqq 1$. As a consequence,

$$B_\alpha \in O_G \cap \tilde{O}_{H^2D} \Longleftrightarrow \alpha \geqq 1 ,$$

$$B_\alpha \in \tilde{O}_G \cap O_{H^2D} \Longleftrightarrow \alpha \leqq -\frac{3}{5} .$$

We shall return to the classification of higher dimensional Riemannian manifolds in further studies.

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