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RELATIONALLY INDUCED SEMIGROUPS

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This paper gives sufficient conditions, of a relation-theoretic nature, in order that a quotient of the state space of a recursion (or topological machine) be a topological semigroup iseomorphic to the endomorphism semigroup of the recursion, generalizing recent function-theoretic results.

Relations. By a *relation* R from a set A to a set B, we mean that R is a subset of $A \times B$. If A and B are topological spaces, we say that R is *closed* to mean that it is a closed subset of the product space. If R is a relation from A to B and S is a relation from B to C, their *composition* is the relation $S \circ R$ from A to C defined by $(a, c) \in S \circ R$ if and only if there is some $b \in B$ with $(a, b) \in R$ and $(b, c) \in S$. This is contrary to the notation in [1], but agrees with the usual (non-algebraist's) notation for the composition of functions. The *inverse* of a relation R is the relation R^{-1} defined by $(b, a) \in R^{-1}$ if and only if $(a, b) \in R$. A relation from A to A is reflexive if R contains $\Delta_A = \{(a, a): a \in A\}$, *symmetric* if $R^{-1} \subseteqq R$ (whence follows $R^{-1} = R$), and *transitive* if $R \circ R \subseteqq R$. R is an *equivalence relation* if it is reflexive, symmetric, and transitive. For any relation R from A to B and any subsets $A' \subseteqq A$, $B' \subseteqq B$, $A'R$ denotes the set $\{b \in B: (a, b) \in R$ for some $a \in A'\}$; RB' is then defined to be the set $B'R^{-1}$. We write aR rather than $\{a\}R$ and Rb for $R\{b\}$, for simplicity's sake. It is known that if A' is compact and R is closed then $A'R$ is closed; if A, B, and C are all compact Hausdorff spaces and R and S are closed relations from A to B and B to C respectively then $S \circ R$ is also closed. It is also known that if A is compact and R is a closed equivalence on A then the quotient space $A/R = \{aR: a \in A\}$ is compact Hausdorff. See Kelley [3] for topological details.

After Riguet [5, 6], a relation R from A to B is called *difunctional* if $R \circ R^{-1} \circ R \subseteqq R$; we observe that any function is difunctional and any symmetric, transitive relation is difunctional; in elementary geometry, the relation of orthogonality is difunctional, as Riguet noted. We use Riguet's 1950 results freely [6] and note in particular that if R is a difunctional relation from A to B satisfying $A = RB$ and $B = AR$, then $R^{-1} \circ R$ and $R \circ R^{-1}$ are equivalence relations on A and B, respectively, closed if R is closed and A and B are compact Hausdorff. Furthermore, $A/(R^{-1} \circ R) = \{Rb: b \in B\}$ and $B/(R \circ R^{-1}) = \{aR: a \in A\}$. For any difunctional relation R, the *slices* aR and $a'R$ either coincide or are disjoint, a property well-known for equivalence relations; the same property holds for slices Rb, Rb', since R^{-1} is difunctional if and only if R is difunctional. In fact, this property of slices characterizes difunctional relations. Unfortunately, the composition of difunctional relations need not be difunctional.

Recursions. A *recursion* is a triple $(T, X, \cdot)$, where T and X are spaces and $T \times X \xrightarrow{\cdot} X$ is a continuous binary operation, the value $t \cdot x$ of which at the point (t, x) is usually denoted by juxtaposition, unless emphasis seems wise. For $T' \subseteqq T$ and $X' \subseteqq X$, we write $T'X'$ (or occasionally $T' \cdot X'$) to denote the set $\{tx\colon t \in T' \text{ and } x \in X'\}$. We frequently avoid the use of curly brackets, writing Tx for $T\{x\}$ and so forth. In particular, if $R \subseteqq T \times X$ and t, z are elements of T, then $t(zR) = \{t\} \cdot (zR)$, the translate of the slice zR. A recursion is *c.o.d.* if both spaces are compact Hausdorff or both are discrete.

For the sake of completeness we state below an easily established folkloric lemma that A. D. Wallace attribute to G. E. Schweigert [7], and a generalization, the Induced Function Theorem (IFT for short), proved in [1]. The lemma is frequently used in what follows.

LEMMA 0. *If A, B, and C are all compact or all discrete spaces, if $f\colon A \rightarrow B$ and $g\colon A \rightarrow C$ are continuous functions with f surjective and if the condition $f(a) = f(a')$ implies $g(a) = g(a')$ for all a, a' holds then there is a unique continuous function $h\colon B \rightarrow C$, satisfying $h(f(a)) = g(a)$ for all a in A.*

Induced Function Theorem. Let A and B be both compact Hausdorff or both discrete spaces, $R \subseteqq A \times B$ a closed relation from A to B, and E and F closed equivalence relations on A and B, respectively. If $A = RB$ and $R \circ E \circ R^{-1} \subseteqq F$ then there is a unique continuous function h making the following diagram of projection and quotient functions analytic:

$$\begin{array}{ccccc} A & \longleftarrow & R & \longrightarrow & B \\ \downarrow & & & & \downarrow \\ A/E & & \xrightarrow{\quad h \quad} & & B/F \,. \end{array}$$

Furthermore, if in addition to the previous hypothesis $B = AR$ and $R^{-1} \circ F \circ R \subseteqq E$, then h is a homeomorphism.

Results.

THEOREM 1. *Suppose $(T, X, \cdot)$ is a c.o.d. recursion and $R \subseteqq T \times X$ is a closed difunctional relation satisfying, for all t', t'', t and $s \in T$,*

(1) $t'R = t''R \Rightarrow t'(tR) = t''(tR)$

(2) $tR = t'(t''R) \Rightarrow t(sR) = t'(t''(sR))$

(3) *$T = RX$ and $X = TR$*

(4) *for each t, t' in T there is some t'' in T with $t(t'R) = t''R$. Then $X/(R \circ R^{-1})$ is a topological semigroup with multiplication $*$ satisfying $tR^*t'R = t(t'R)$* identically.

Proof. From difunctionality and hypothesis (3), $R^{-1} \circ R$ and $R \circ R^{-1}$ are equivalence relations on T and on X, respectively, and are closed if T and X are compact. The Induced Function Theorem implies that there is a unique homeomorphism h making the following diagram of projection and quotient maps analytic.

$$\begin{array}{ccccc} T & \longleftarrow & R & \longrightarrow & X \\ \downarrow & & & & \downarrow \\ T/(R^{-1} \circ R) & & \longrightarrow & & X/(R \circ R^{-1}) \end{array}$$

In the following diagram,

$$\begin{array}{ccc} T \times X & \xrightarrow{\ \cdot\ } & X \\ {\scriptstyle hp \times q}\downarrow & & \downarrow{\scriptstyle q} \\ X/(R \circ R^{-1}) \times X/(R \circ R^{-1}) & \longrightarrow & X/(R \circ R^{-1}) \end{array}$$

we note that $(t, x) \in R$ iff $tR = q(x)$ and $Rx = p(t)$ iff $h(Rx) = tR$. If (t, x) and (t', x') satisfy $[hp \times q]\ (t, x) = [hp \times q]\ (t', x')$ then $h(p(t)) = h(p(t'))$ and $q(x) = q(x')$, so that $tR = t'R$. If $t'' \in Rx$ then $x \in t''R$, hence $tx \in t(t''R) = t'(t''R)$ by hypothesis (1). We also have $t'x' \in t'(t''R)$ since $Rx = Rx'$. Hypothesis (4) allows us to conclude that $(tx, t'x') \in R \circ R^{-1}$, i.e., $q(tx) = q(t'x')$. Hence Lemma 0 applies to give a unique continuous function* making the diagram analytic. We observe that $tR^*q(x) = tq(x)$ for all $t \in T$ and all $x \in X$. Now* is associative, for if $t, t't'' \in T$, then there is some $s \in T$ such that $t(t'R) = sR$, and hence $(tR^*t'R)^*t''R = t(t'R)^*t''R = sR^*t''R = s(t''R) = t(t'(t''R)) = tR^*t'(t''R) = tR^*(t'R^*t''R)$, using hypothesis (2).

THEOREM 2. *Suppose $(T, X, \cdot)$ is a c.o.d. recursion and $R \subseteq T \times X$ is a closed difunctional relation satisfying*

(1) $T = RX$ and $X = TR$

(2) *the set $Z = \{z \in T : tR = t'R \Rightarrow t(zR) = t'(zR)\}$ is not empty*

(3) *for each $t, t' \in T$ there is some $t'' \in T$ with $t(t'R) = t''R$*

(4) *if $tR = t'(t''R)$ then for any $z \in Z$, $t(zR) = t'(t''(zR))$. Then $\{zR : z \in Z\}$ is a topological semigroup in the quotient topology with multiplication* satisfying $zR^*z'R = z(z'R)$ for all $z, z' \in Z$.*

Proof. For simplicity, let $\bar{Z} = \{zR : z \in Z\}$ be the subspace of the quotient space $A/(R \circ R^{-1})$. We dispose topological considerations first.

One verifies easily that if T and X are compact, then Z is closed, and it follows by standard results that ZR and finally $\bar{Z}$ are compact. Of course, if T and X are discrete, so is $\bar{Z}$.

On the algebraic side, we observe that $Z \cdot ZR \subseteqq ZR$, for if $z, z' \in Z$ and $tR = z(z'R)$, then it will be seen that $t \in Z$ (such t exists by hypothesis (3)). To this end, suppose that $t'R = t''R$, and let $t'(zR) = sR$ to infer that $t'(tR) = t'(z(z'R)) = s(z'R)$, by hypothesis (4). Since $z \in Z$, then $t'(zR) = t''(zR)$ and hence $sR = t''(zR)$; it then follows from hypothesis (4) that $s(z'R) = t''(z(z'R))$, so that $t'(tR) = t''(tR)$, implying that $t \in Z$. Hence $Z \cdot ZR \subseteqq ZR$.

If x and x' are points in ZR satisfying $(x, x') \in R \circ R^{-1}$, then $(zx, zx') \in R \circ R^{-1}$ also, and hence we may infer from Lemma 0 that the function $Z \times \bar{Z} \xrightarrow{*} \bar{Z}$ given by $z^*z'R = z \cdot (z'R)$ is continuous.

Finally, if R' is the relation from Z to $\bar{Z}$ defined by $(z, z'R) \in R'$ if $\{z\} \times z'R \subseteqq R$, then we can easily see that R' is closed and difunctional, so that R' and the compact or discrete recursion $(Z, \bar{Z}, {}^*)$ satisfy the hypothesis of Theorem 1. Theorem 2 now follows.

Representation. Assuming the hypothesis of Theorem 2, let S be the semigroup (with compact open topology) of all continuous functions from the quotient space $A/(R \circ R^{-1})$ into itself, and let *end* denote the subsemigroup of S defined by $f \in$ *end* if $t \cdot f(\bar{x}) = f(t \cdot \bar{x})$ for all $t \in T$ and all $\bar{x}$ in $X/(R \circ R^{-1})$. The function $F\colon T \to S$, given by $F_t(t'R) = t' \cdot (tR)$, is easily seen to be continuous and maps Z into *end*; let F' denote the restriction of F to Z. In a similar way, the map $G\colon Z \to ZR/(R \circ R^{-1})$ given by $G(z) = zR$ is a continuous surjection. Lemma 0 is seen easily to apply, giving a continuous function $H\colon ZR/(R \circ R^{-1}) \to$ *end* satisfying $H \circ G = F$, from which we see that for any $z \in Z$ and any $t \in T$, $[H(zR)]\,(tR) = t(zR)$. Routine computation, using hypothesis (3) and (4), shows that $H(zR^*z'R) = H(z'R) \circ H(zR)$, so that H is an anti-homorphism.

THEOREM 3. *If, in addition to the hypothesis of Theorem 2, for some $z_0 \in Z$ and all $t \in T$ it is the case that $t(z_0R) = tR = z_0(tR)$, then H is an anti-iseomorphism and $ZR/(R \circ R^{-1})$ is a monoid with z_0R its identity; furthermore, the set z_0R is a set of generators for X, i.e., $T(z_0R) = X$.*

Proof. That z_0R generates X is clear from the equations $t(z_0R) = tR$ and $TR = X$. That z_0R is the identity follows from the fact that for any $z \in Z$, $zR^*z_0R = z(z_0R) = zR = z_0(zR) = z_0R^*zR$. If $H(zR) = H(z'R)$ then $zR = z_0(zR) = z_0(z'R) = z'R$, so that H is injective. To see that H is also surjective, let $f \in$ *end*, and suppose $f(z_0R) = t_0R$;

we will see that $t_0 \in Z$. To see this suppose $t'R = t''R$ and compute: $t'(t_0R) = t'f(z_0R) = f(t'(z_0R)) = f(t'R)$; similarly, $t''(t_0R) = f(t''R)$; it follows that $t_0 \in Z$. Now for any $t \in T$, we see that $f(tR) = f(t(z_0R)) = tf(z_0R) = t(t_0R) = [H(t_0R)](tR)$, implying that H is surjective.

REMARKS. Theorem 2 obviously generalizes Theorem 1 and also contains a previous result of the author [4]. When R is a continuous function from T onto X it is a closed, difunctional relation and $R \circ R^{-1} = \Delta_X$, so that $X/(R \circ R^{-1})$ is homeomorphic to X, and the set ZR is just the image of Z; R is surjective just in case $X = TR$. Hence Theorem 1 generalizes the theorem of [7] and Theorem 2, the theorem of [8], which in turn elegantly generalize theorems of Aczel-Wallace, Hosszu, Barnes, Fleck, Weeg, Oehmke *et. al.* (see [8] for references). Other applications will be announced elsewhere. In view of recent results of Fay [2], the present work allows one to induce semigroups "in" the objects of many categories. The details of this extension will be left for another time.

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