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WEAKLY ALMOST PERIODIC HOMEOMORPHISMS OF THE TWO SPHERE

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A self-homeomorphism f of the 2-sphere S^2 is weakly almost periodic (w.a.p.) if the collection of orbit closures forms a continuous decomposition of S^2. It is shown that if f is orientation-preserving, w.a.p. and nonperiodic, then f has exactly two fixed points, and every nondegenerate orbit closure is an homology 1-sphere. There is an example with an orbit closure which is an homology 1-sphere but not a real 1-sphere. If f is orientation-reversing, w.a.p. and has a fixed point, then f is shown to be periodic. The orbit structure of orientation-reversing, w.a.p., nonperiodic homeomorphisms on S^2 is studied.

1. Introduction. Let f be a periodic mapping of the 2-sphere S^2 to itself. Kerékjártó [8] and Eilenberg [3] showed that f is topologically equivalent either to the identity (every point fixed), to a rotation (two fixed points), a reflection (a simple closed curve of fixed points), or to a rotation followed by a reflection (no fixed points). If f satisfies the weaker condition of being almost periodic (equivalent to having equicontinuous iterates), then the fixed point set of f again is either empty or an i-sphere, $0 \leqq i \leqq 2$, [9]. (For related results on almost periodic mappings of subsets of S^2, see Hemmingsen [7].)

In the present paper we study the weakly almost periodic homeomorphisms on S^2, (the collection of orbit closures forms a continuous decomposition of S^2), and show that the set of fixed points is still either empty or an i-sphere, $0 \leqq i \leqq 2$, (Theorem 3 and Corollary 5). Some other results are: if $f: S^2 \rightarrow S^2$ is weakly almost periodic (w.a.p.), orientation-reversing, and has a fixed point, then f is periodic (Theorem 4); if $f: S^2 \rightarrow S^2$ is w.a.p., orientation-preserving, and not periodic, then every nondegenerate orbit closure is an homology 1-sphere (Theorem 5).

A homeomorphism of S^2 to itself which is w.a.p. but not almost periodic is given in [12, Example 1]. This example is not almost periodic since it has an orbit closure which is not locally connected, (see [7, Section 5]). The collection of orbit closures, however, is easily seen to be continuous.

Our main theorems are given in §§ 6 and 7. Section 3 gives a summary of results in the theory of prime ends which we need. Section 4 discusses the fixed point theory used in §§ 5, 6, and 7. (Those familiar with prime ends and local fixed point index may skip

§§ 3 and 4.) Many of our techniques are based on those of Cartwright and Littlewood in [2].

2. Definitions and notation.

If $f: X \to X$ is a homeomorphism and $x \in X$, then the *orbit closure* of x is the closure of the set of iterates $\{f^n(x)\}$, $n = 0, \pm 1, \pm 2, \cdots$, $(f^0 = Id)$.

The original definition of weakly almost periodic was given by Gottschalk in [5]. For compact spaces the original definition is equivalent to requiring that the orbit closures form a continuous decomposition [5, Theorem 5]. The equivalent definition which we shall use in our proofs is: $f: S^2 \to S^2$ is *weakly almost periodic* if (a) the collection of orbit closures is a decomposition of S^2, (if two orbit closures meet, they are equal), and (b) for any closed set B, the union of all orbit closures which intersect B is a closed set, [6, Theorem 4. 24, p. 34].

A point $x \in X$ is a *nonwandering point* if for every neighborhood U of x, there is a nonzero integer n such that $f^n(U) \cap U \neq \phi$. If x is not a nonwandering point it is a *wandering point*. It is easily seen that if $f: S^2 \to S^2$ is w.a.p. then every point is a nonwandering point.

A *domain* is a connected open set. If A is a set $\mathrm{Cl}(A)$ and $\mathrm{Bd}(A)$ denote the closure and boundary, respectively, of A. If U is a domain of S^2 and x is a point in a component R of $S^2 - \mathrm{Cl}(U)$, then $\mathrm{Bd}(R)$ is the *outer boundary of U with respect to x*.

An *homology 1-sphere* K in S^2 is a continuum (closed, connected set) such that $S^2 - K$ has exactly two components.

An open triod is a set homeomorphic to the set of all points (x, y) in the plane such that either $-1 < x < 1$ and $y = 0$, or $x = 0$ and $0 \leqq y < 1$. The points $(-1, 0)$, $(1, 0)$, $(0, 1)$ are called the *feet* of the triod.

If U is a domain then a *crosscut* of U is an open arc in U whose closure is an arc which intersects $\mathrm{Bd}(U)$ in two points. An *endcut* of U is a half-open arc in U whose closure is an arc which intersects $\mathrm{Bd}(U)$ in one point.

3. Prime ends.

In this section we state the results and definitions concerning prime ends which we shall use in §§ 5 and 6. The material in the present section is taken from [2], [11], and [15].

Let U be a simply-connected domain in S^2 with a nondegenerate boundary. A *C-transformation* of U onto the open unit disk D is a homeomorphism $T: U \to D$ such that the image of any crosscut in U is a crosscut in D, and the endpoints of such images of crosscuts of U are dense in the boundary of D. The conformal mapping of U onto D given by the Riemann mapping theorem shows that C-transformations always exist. However, C-transformations may be constructed by topological methods, without using conformal mapping theory, [15, Appendix 2].

Given a homeomorphism f of the closure of U onto itself, and a C-transformation T of U onto D, we have that TfT^{-1}: $D \to D$ is a C-transformation which may be extended to a homeomorphism of the closed unit disk onto itself, [15, (4.10) on page 6, and (A1.7) on page 27].

A collection of crosscuts $Q_1, Q_2, \cdots$ of the simply connected domain U is a *chain* if (a) the arcs $\mathrm{Cl}(Q_1)$, $\mathrm{Cl}(Q_2)$, $\cdots$ are pairwise disjoint, (b) Q_n separates Q_{n-1} from Q_{n+1} in U, (c) there is a point on $\mathrm{Bd}(U)$ whose greatest distance from Q_n approaches 0 as $n \to \infty$. Corresponding to each Q_n there is a domain G_n of $U - Q_n$ containing Q_{n+1}. Note that $G_1 \supset G_2 \supset \cdots$.

If $\{Q_i\}$, $\{R_i\}$ are chains of crosscuts, and $\{G_i\}$, $\{H_i\}$ are their respective corresponding domains, then $\{Q_i\}$, $\{R_i\}$ are *equivalent* chains if for every n there is an m such that $H_m \subset G_n$ and $G_m \subset H_n$. Equivalent chains are said to define the same *prime end*. Thus, a prime end of U is an equivalence class of chains of U.

If $Q_1, Q_2, \cdots$ is a chain of crosscuts in U, then their images $T(Q_1)$, $T(Q_2)$, $\cdots$ under the C-transformation T: $U \to D$ is a chain in D, [15, Appendix 2]. If $\{Q_i\}$ and $\{R_i\}$ are equivalent chains in U, then $\{T(Q_i)\}$, and $\{T(R_i)\}$ are equivalent chains in D, and in fact converge to the same point on the boundary of D, ($\{Q_i\}$ and $\{R_i\}$ may not converge to the same point on $\mathrm{Bd}(U)$). Thus, T sets up a $1-1$ correspondence between prime ends of U and points of the unit circle [11, p. 621].

If f: $\mathrm{Cl}(U) \to \mathrm{Cl}(U)$ is a homeomorphism and E is a prime end of U, then E is *fixed* by f if for some chain $\{Q_i\}$ defining E, we have that $\{Q_i\}$ and $\{f(Q_i)\}$ are equivalent chains. This definition is easily seen to be independent of which defining chain is used. If T: $U \to D$ is a C-transformation, h: $\mathrm{Cl}(D) \to \mathrm{Cl}(D)$ is the extension of TfT^{-1}, and e is the point on $\mathrm{Bd}(D)$ corresponding to the fixed prime end E, then $h(e) = e$. Conversely, every fixed point of h on $\mathrm{Bd}(D)$ corresponds to a fixed prime end of f.

If E is a prime end of U, $\{Q_i\}$ is a defining chain for E, and p is the point on $\mathrm{Bd}(U)$ to which the crosscuts $\{Q_i\}$ converge, then p is a *principal point* of E. (We remark that there exists a U with a prime end E such that every point of $\mathrm{Bd}(U)$ is a principal point of E, [13].)

If A is an endcut in U with an endpoint $s \in \mathrm{Bd}(U)$, then there is a chain $\{Q_i\}$ defining a prime end E such that s is a principal point of E and each crosscut Q_i separates the endpoint of A in U from some (open) subarc of A having s as an endpoint. E is the *prime end determined by A*. If $T: U \to D$ is a C-transformation, and e is the point on $\mathrm{Bd}(D)$ corresponding to E, then $T(A)$ is an endcut in D having e as an endpoint, [15, page 5].

4. Lefschetz number and local fixed point index. In this section we state the results concerning fixed points which we shall use in §§ 5, 6, and 7.

If X is a compact polyhedron and $f: X \to X$ is a map (continuous function), then there is a certain rational number $L(f)$, called the *Lefschetz number* of f, associated with f and X, [14, p. 195]. We shall use the following two facts about $L(f)$.

Fact 1. If X is a two cell, then $L(f) = 1$.

Fact 2. If X is a 2-sphere and f is an orientation-preserving homeomorphism, then $L(f) = 2$.

For proofs of Facts 1 and 2, see [14, p. 196].

If e is the category of compact polyhedra and maps, let $A(e)$ denote the set of pairs (f, U), where $f: X \to X$ is a map in e and U is an open subset of X such that f has no fixed points on the boundary of U. Then there is a function i, the *local fixed point index*, from $A(e)$ into the rationals which satisfies the following axioms:

A1. If (f, U), (g, U) belong to $A(e)$, and $f = g$ on the closure of U, then $i(f, U) = i(g, U)$.

A2. If f_t is a homotopy such that $(f_t, U) \in A(e)$ for each t, $0 \leqq t \leqq 1$, then $i(f_0, U) = i(f_1, U)$.

A3. If $(f, U) \in A(e)$ and U contains mutually disjoint open sets $V_j, j = 1, \cdots, k$, such that f has no fixed points on $U - \bigcup_{j=1}^{k} V_j$, then

$$i(f, U) = \sum_{j=1}^{k} i(f, V_j) .$$

In particular, if f has no fixed points on U, $i(f, U) = 0$.

A4. If $f: X \to X$ belongs to e, then $i(f, X) = L(f)$.

A5. If the maps $f: X \to Y$, $g: Y \to X$ belong to e, and

$$(gf, U) \in A(e) ,$$

then $i(gf, U) = i(fg, g^{-1}(U))$.

For further discussion of the local fixed point index see [4] or [1].

REMARK. If D is the open unit disk, and h is a map of the closure of D to itself with no fixed points on $\mathrm{Bd}(D)$, then $i(h, D) = 1$. For, by Fact 1 and Axiom A4, $1 = \mathrm{L}(h) = i(h, \mathrm{Cl}(D))$. Then, by Axiom A3, $i(h, \mathrm{Cl}(D)) = i(h, D)$.

5. Preliminary lemmas. Our first lemma is based on Lemma 11 of [2].

LEMMA 1. *Suppose $f\colon S^2 \to S^2$ is a homeomorphism, U is a simply connected domain with nondegenerate boundary, $f(U) = U$, and every point of U is a nonwandering point. Suppose also that E is a prime end of U which is fixed by f. Then every principal point of E is a fixed point of f.*

Proof. Let $Q_1, Q_2, \cdots$ be a chain of crosscuts defining E which converge to the principal point p of E.

Case 1. $f(Q_i) \cap Q_i = \phi$ for some i. Let V be the component of $U - Q_i$ containing $Q_{i+1}, Q_{i+2}, \cdots$. E is fixed by f, so $\{Q_j\}$ and $\{f(Q_j)\}$ are equivalent chains, hence $f(V) \cap V \neq \phi$. But then $f(V)$ either contains or is contained in V. Assume $f(V) \subset V$. Let W be the nonempty open set $V - \mathrm{Cl}(f(V))$. Then $f^n(W) \cap W = \phi$ if $n \neq 0$. Thus no point of W is a nonwandering point. This contradiction shows that Case 1 cannot occur.

Case 2. $f(Q_i) \cap Q_i \neq \phi$ for all i, $i = 1, 2, \cdots$. For each i, select a point $x_i \in Q_i$ such that $f(x_i) \in Q_i$. The crosscuts $Q_1, Q_2, \cdots$ converge to the principal point p, hence $\{x_i\} \to p$, hence $\{f(x_i)\} \to f(p)$. But $f(x_i) \in Q_i$, hence $\{f(x_i)\} \to p$. Hence $f(p) = p$ and the proof of Lemma 1 is complete.

LEMMA 2. *Suppose $f\colon S^2 \to S^2$ is a homeomorphism, M is an invariant continuum in S^2 which contains no fixed point of f, and every point of S^2 is a nonwandering point. Then $i(f, U) = 1$ for every component U of $S^2 - M$ which is invariant under f. (See* § 4 *for discussion of the fixed point index $i(f, U)$.)*

Proof. Let U be a component of $S^2 - M$ such that $f(U) = U$. M is connected, hence U is simply connected. Also, $\mathrm{Bd}(U)$ is nondegenerate, since M contains no fixed point of f. Let T be a C-transformation of U onto the open unit disk D. Extend TfT^{-1} to a homeomorphism h of $\mathrm{Cl}(D)$ onto itself. Since $\mathrm{Bd}(U)$ contains no fixed point of f, we see by Lemma 1 that U has no fixed prime ends. Hence h has no fixed points on $\mathrm{Bd}(D)$. Hence $i(h, D) = 1$ by the Remark, § 4.

We would like to conclude from Axiom A5 of § 4 that $i(f, U) = 1$. However, D and U are not compact polyhedra. We overcome this difficulty as follows: let X be an open 2-cell which contains the fixed points of f in U and whose closure is contained in U. Let Y be a closed 2-cell in U containing $\mathrm{Cl}(X) \cup f(\mathrm{Cl}(X))$. Let r_1: $\mathrm{Cl}(D) \to T(Y)$, and r_2: $S^2 \to Y$ be retractions. Since $T(X)$ contains all fixed points of h, we have:

$$\begin{aligned} 1 = i(h, D) &= i(h, T(X)) && \text{by Axiom A3} \\ &= i(Tr_2 f T^{-1} r_1, T(X)) && \text{by A1} \\ &= i(f T^{-1} r_1 T r_2, X) && \text{by A5} \\ &= i(f, X) && \text{by A1} \\ &= i(f, U) && \text{by A3}\,. \end{aligned}$$

The proof of Lemma 2 is complete.

6. Fixed point sets of weakly almost periodic homeomorphisms on S^2.

THEOREM 3. *Suppose f: $S^2 \to S^2$ is a w.a.p. orientation-preserving homeomorphism. Then either f is the identity or f has exactly two fixed points.*

Proof. Let $\mathrm{Fix}(f)$ denote the set of fixed points of f. Assume $\mathrm{Fix}(f) \neq S^2$. Since f is orientation-preserving it is easily seen that f leaves every component of $S^2 - \mathrm{Fix}(f)$ invariant, and so we may select an arc A in one of these components such that $f(A) \cap A \neq \phi$. Denote by M the union of all orbit closures which meet A. M is closed, since f is w.a.p.; M contains no fixed point of f; and M is connected since M is the union of the connected set

$$\bigcup_{n=-\infty}^{\infty} f^n(A)$$

and limit points of this set.

Since M and $\mathrm{Fix}(f)$ are disjoint closed sets, we see that $\mathrm{Fix}(f)$ is contained in a finite number $U_1, \cdots, U_s$ of components of $S^2 - M$. By Axioms A3, A4, and Fact 2 of § 4, we have

$$2 = L(f) = i(f, S^2) = \sum_{j=1}^{s} i(f, U_j)\,.$$

But by Lemma 2, $i(f, U_j) = 1$, $1 \leqq j \leqq s$. Hence $s = 2$.

It remains to show that $\mathrm{Fix}(f) \cap U_j$, $j = 1, 2$, is a single point.

Let U be the component of $U_1 - \mathrm{Fix}(f)$ with $\mathrm{Bd}(U_1) \subset \mathrm{Bd}(U)$. Since $\mathrm{Bd}(U_1)$ and $\mathrm{Fix}(f)$ are disjoint closed sets, we see that $\mathrm{Bd}(U) - \mathrm{Bd}(U_1)$ is a closed nonempty subset of $\mathrm{Fix}(f)$.

Case 1. $\mathrm{Bd}(U) - \mathrm{Bd}(U_1)$ has more than one component. Then by [16, Corollary 3.11, p. 109], there is a simple closed curve J in U which separates $\mathrm{Bd}(U) - \mathrm{Bd}(U_1)$. Let B be an arc with one endpoint on $\mathrm{Bd}(U_1)$, the other on J, and contained in U except for one endpoint. Then $\mathrm{Bd}(U_1) \cup J \cup B$ is connected, and

$$f(\mathrm{Bd}(U_1) \cup J \cup B) \cap (\mathrm{Bd}(U_1) \cup J \cup B) \neq \phi .$$

Thus if we denote by N the union of all orbit closures which intersect $\mathrm{Bd}(U_1) \cup J \cup B$, we see that N is an invariant continuum which contains no fixed point of f (this follows similarly to the case of M above). Let $V_1, \cdots, V_t$ be the (finite) number of components of $S^2 - N$ such that $\mathrm{Fix}(f) \cap V_j \neq \phi$ and $V_j \subset U_1$, $1 \leqq j \leqq t$. By Lemma 2, $i(f, V_j) = 1$, $1 \leqq j \leqq t$. By Axiom A3,

$$1 = i(f, U_1) = \sum_{j=1}^{t} i(f, V_j) = t .$$

But J separates two points of $\mathrm{Fix}(f) \cap U_1$, hence $t > 1$. This contradiction shows that Case 1 cannot occur.

Case 2. $\mathrm{Bd}(U) - \mathrm{Bd}(U_1)$ is connected. The proof will be complete if we show that $\mathrm{Bd}(U) - \mathrm{Bd}(U_1)$ is a single point. We assume that $\mathrm{Bd}(U) - \mathrm{Bd}(U_1)$ is a nondegenerate continuum and derive a contradiction.

Assuming $\mathrm{Bd}(U) - \mathrm{Bd}(U_1)$ is a nondegenerate continuum we establish

Claim 1. There is a simply connected invariant domain C_v containing two endcuts A and B such that the endpoint of B on $\mathrm{Bd}(C_v)$ is not a fixed point of f, and the endpoint of A on $\mathrm{Bd}(C_v)$ is a fixed point of f which is not a limit point of $\mathrm{Bd}(C_v) - \mathrm{Fix}(f)$.

Let Q be a crosscut in U both of whose endpoints lie on

$$\mathrm{Bd}(U) - \mathrm{Bd}(U_1) .$$

Let V be the component of $U - Q$ whose boundary does not intersect $\mathrm{Bd}(U_1)$, [15, (5.3), p. 6]. V is a component of

$$S^2 - ((\mathrm{Bd}(U_1) - \mathrm{Bd}(U)) \cup Q) .$$

Let p be a point of $\text{Bd}(V) - \text{Cl}(Q)$. Note that p is a fixed point of f.

Denote by L the union of all orbit closures which intersect $\text{Cl}(Q)$. L is a continuum. p is not a limit point of L so there is a connected neighborhood 0 of p which misses L. Let A be an endcut of V which is contained in 0. Let C_v be the component of

$$S^2 - ((\text{Bd}(U) - \text{Bd}(U_1)) \cup L)$$

which contains the endcut A. The endpoint of A in $\text{Bd}(C_v)$ has a neighborhood 0 which misses L, hence $0 \cap \text{Bd}(C_v) \subset \text{Fix}(f)$.

Let B' be an endcut of V with one endpoint b in C_v and the other in the crosscut Q. Then the component of $B' \cap C_v$ containing b is the required endcut B.

C_v is simply connected because $(\text{Bd}(U) - \text{Bd}(U_1)) \cup L$ is connected, (see [15, (5.3), p. 6] and [10, Theorem 74, p. 217]).

C_v is invariant because (a) $(\text{Bd}(U)) - \text{Bd}(U_1)) \cup L$ is invariant, (b) $\text{Bd}(C_v)$ contains a continuum of fixed points of f, and (c) f is orientation-preserving, (for further details see proof of Claim 2 below). The proof of Claim 1 is complete.

Claim 2. The prime end E of C_v determined by the endcut A is a fixed prime end of f.

Let $S_1, S_2, \cdots$ be a chain of crosscuts converging to the endpoint s of A and defining the prime end E. Since s is not a limit point of $\text{Bd}(C_v) - \text{Fix}(f)$, we may assume that the endpoints of S_i are fixed points of f for every i, $i = 1, 2, \cdots$. We also may assume that every crosscut S_i intersects A. From the crosscut S_i and the endcut A we may construct an open triod T_i (see § 2 for definition) whose feet are fixed points of f. Since f is orientation-preserving, we see easily that $f(T_i) \cap T_i \neq \phi$. (Hence $f(C_v) \cap C_v \neq \phi$, and since $(\text{Bd}(U) - \text{Bd}(U_1)) \cup L$ is invariant, we have $f(C_v) = C_v$.)

Since $f(T_i) \cap T_i \neq \phi$ for $i = 1, 2, \cdots$, we see that $\{S_i\}$ and $\{f(S_i)\}$ are equivalent chains, hence E is a fixed prime end of f. The proof of Claim 2 is complete.

Let T be a C-transformation of C_v onto the open unit disk D. Extend the homeomorphism $TfT^{-1}\colon D \to D$ to a homeomorphism h of the closed unit disk onto itself. h is orientation-preserving, since f is.

By Claim 2, there is a fixed prime end of C_v; hence h has a fixed point on $\text{Bd}(D)$. But then, since h is orientation-preserving, every point of $\text{Bd}(D)$ is either a fixed point of h or converges to a fixed point under positive iterates of h [2, Lemma 14].

Consider the endcut B of Claim 1. The endpoint of B on $\text{Bd}(C_v)$

is not fixed by f, but this endpoint is a principal point of the prime end F determined by B. Hence, by Lemma 1, F is not a fixed prime end. Hence, if e is the endpoint of $T(B)$ on $\mathrm{Bd}(D)$, e is not a fixed point of h. But then, there is a fixed point m of h on $\mathrm{Bd}(D)$ such that $\{h^n(e)\}_{n=0}^{\infty} \to m$. If M is the prime end of C_v corresponding to the point m, then by Lemma 1, every principal point of M is a fixed point of f.

Let $X_1, X_2, \cdots$ be a chain of crosscuts of C_v defining the prime end M. We claim that for large j, $T(X_j)$ intersects the orbit under h of $T(B)$. To see this we proceed as follows. Let b be the endpoint of B in C_v. Then the orbit closure of b is contained in C_v; therefore, the orbit closure of $T(b)$ under h is contained in D. In particular, m is not a limit point of the orbit of $T(b)$. Hence, for large j, the closure of the crosscut $T(X_j)$ separates m and the orbit of $T(b)$ in $\mathrm{Cl}(D)$. But the other endpoint e of $T(B)$ converges to m under positive iterates of h, so for large j, there is a positive integer n such that $h^n(\mathrm{Cl}(T(B)))$ intersects both components of

$$\mathrm{Cl}(D) - \mathrm{Cl}(T(X_j)) .$$

Hence $h^n(T(B))$ intersects $T(X_j)$, and our claim is established.

Hence, for large j, X_j intersects the orbit under f of $\mathrm{Cl}(B)$.

But the chain $X_1, X_2, \cdots$ of crosscuts converges to a principal point q of the prime end M. But then q is a fixed point of f which is a limit point of the orbit of $\mathrm{Cl}(B)$. Therefore, the union of all orbit closures which intersect $\mathrm{Cl}(B)$ is not a closed set. This contradicts the fact that f is w.a.p.

This final contradiction establishes that $\mathrm{Bd}(U) - \mathrm{Bd}(U_1)$ is a single point. Similarly, $\mathrm{Fix}(f) \cap U_2$ is a single point, and so f has exactly two fixed points. The proof of Theorem 3 is complete.

THEOREM 4. *Suppose $f\colon S^2 \to S^2$ is a w.a.p. orientation-reversing homeomorphism. Then either f has no fixed points, or f is periodic with period* 2.

Proof. Suppose f has a fixed point.

Claim. f has more than two fixed points.

Suppose the claim is not true. Let A be an arc intersecting no fixed point, such that $A \cap f(A) \neq \phi$. Denote by M the union of all orbit closures which intersect A. M is an invariant continuum containing no fixed points of f. Let U be a component of $S^2 - M$ containing a fixed point of f. Then $f(U) = U$ and U is simply connected with a nondegenerate boundary. Let T be a C-transformation of U

onto the open unit disk D. Extend TfT^{-1} to a homeomorphism h of the closed unit disk onto itself. h is orientation-reversing, since f is. But then h must have two fixed points on $\mathrm{Bd}(D)$, [16, Theorem 7.3, p. 264]. These fixed points correspond to fixed prime ends of U. By Lemma 1, the principal points of these prime ends are fixed points of f. This contradicts the assumption that M contains no fixed points of f. The proof of our claim is complete.

But now consider the homeomorphism f^2: $S^2 \to S^2$. f^2 is orientation-preserving, w.a.p. [6, Theorem 4.24, p. 34 and Theorem 2.33, p. 17], and by our claim, has more than two fixed points. Hence, by Theorem 3, $f^2 = Id$. The proof of Theorem 4 is complete.

COROLLARY 5. *Suppose* f: $S^2 \to S^2$ *is a w.a.p. orientation-reversing homeomorphism. Then the set of fixed points of* f *is either empty or is a simple closed curve.*

Proof. Follows from Theorem 4 and [3].

7. Orbit closures of weakly almost periodic homeomorphisms on S^2.

THEOREM 6. *Suppose* f: $S^2 \to S^2$ *is a w.a.p. orientation-preserving homeomorphism which is not periodic. Then every nondegenerate orbit closure is a* 1-*dimensional homology* 1-*sphere.*

Proof. $f \neq Id$ so by Theorem 3, f has exactly two fixed points. Let K be a nondegenerate orbit closure. We show that K separates the fixed points of f. Suppose not. Then there is a simple closed curve J which separates K and the fixed points of f, (connect the fixed points by an arc missing K, then "enlarge" the arc slightly to obtain a disk whose boundary is J). We must have $f(J) \cap J \neq \phi$, since otherwise every point of J would be a wandering point. Denote by M the union of all orbit closures which intersect J. Then M is an invariant continuum which separates K and the fixed points of f. Let U be a component of $S^2 - M$ which intersects K. Since every point of U is a nonwandering point, there is an integer n such that $f^n(U) \cap U \neq \phi$. Since M is invariant, $f^n(U) = U$.

f^n is a w.a.p. orientation-preserving homeomorphism [6, p. 34 and p. 17]. f is not periodic, hence $f^n \neq Id$, hence by Theorem 3, f^n has exactly two fixed points. These fixed points are the original fixed points of f, and so the domain U contains no fixed points of f^n. But by Lemma 2, $i(f^n, U) = 1$. This contradiction shows that the orbit closure K must separate the fixed points of f.

We now show that K is connected. Let V be a component of $S^2 - K$ containing a fixed point of f. Let B be the outer boundary of V with respect to the fixed point of f not in V, (see § 2 for definitions). B is connected, [10, Theorem 25, p. 176]. And V and the fixed points of f are invariant, hence B is invariant. But K is a minimal invariant set, and $B \subset K$, hence $B = K$.

K is one dimensional, since outer boundaries contain no interior points.

Finally, $S^2 - K$ has exactly two components. For, if there were more than two components, then some component U would contain no fixed point of f, and we would arrive at the same contradiction as in proving that K separates the fixed points of f.

Thus K is a 1-dimensional homology 1-sphere and the proof of Theorem 6 is complete.

REMARK. [12, Example 1] is an example of a w.a.p. orientation-preserving homeomorphism with an orbit closure which is an homology 1-sphere but not a real 1-sphere.

THEOREM 7. *Suppose f: $S^2 \to S^2$ is a w.a.p. orientation-reversing homeomorphism which is not periodic. Then, with two exceptions, every orbit closure is the union of two disjoint homology* 1*-spheres. The exceptions are* (a) *a period* 2 *orbit, and* (b) *one orbit closure which is an homology* 1*-sphere (the "axis of reflection").*

Proof. f^2 is a w.a.p., orientation-preserving, nonperiodic homeomorphism. Hence, by Theorems 3 and 6, f^2 has two fixed points, and every nondegenerate orbit closure is an homology 1-sphere. The orbit closure under f of a point x is the union of the orbit closure of x under f^2 and the orbit closure of $f(x)$ under f^2. Thus, the two fixed points of f^2 correspond to a period 2 orbit under f, and every other orbit closure under f is the union of two homology 1-spheres which are either disjoint or equal. Let H denote the collection of orbit closures under f which are homology 1-spheres. We show that H has exactly one element.

Let G be the decomposition space whose points are orbit closures under f^2. Let w: $S^2 \to G$ be the natural decomposition map [16, p. 125]. If K is any nondegenerate orbit closure under f^2, then $w(K)$ is a cut point of G, since K separates S^2, w is an open map, [16, p. 130], and orbit closures are connected. Hence G has exactly two noncut points, (the fixed points of f^2), hence G is an arc, [16, p. 54]. Define a map g: $G \to G$ by $g(w(K)) = w(f(K))$ for all orbit closures K of f^2. It is easily seen that g is a nontrivial period 2 map of the arc G onto itself. Fixed points of G correspond to elements of the set H defined above. But g has exactly one fixed point [16, p. 264]. The proof of Theorem 7 is complete.

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