PACIFIC JOURNAL OF MATHEMATICS
Vol. 48, No. 2, 1973

AN ERGODIC PROPERTY OF LOCALLY COMPACT AMENABLE SEMIGROUPS

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Let $M(S)$ be the Banach algebra of all bounded regular Borel measures on a locally compact semigroup S with variation norm and convolution as multiplication and $M_0(S)$ the probability measures in $M(S)$. We obtain necessary and sufficient conditions for the semigroup S to have the (ergodic) property that for each $\nu \in M(S)$, $|\nu(S)| = \inf\{\|\nu*\mu\|: \mu \in M_0(S)\}$, an extension of a known result for locally compact groups.

1. Notations and terminologies. We shall follow Hewitt and Ross [9] for basic notations and terminologies concerning integration on locally compact spaces. Let S be a locally compact semigroup with jointly continuous multiplication and $M(S)$ the Banach algebra of all bounded regular Borel measures on S with total variation norm and convolution $\mu*\nu$, μ, $\nu \in M(S)$ as multiplication where

$$\int f d\mu*\nu = \iint f(xy)d\mu(x)d\nu(y) = \iint f(xy)d\nu(y)d\mu(x)$$

for $f \in C_0(S)$ the space of all continuous functions on S which vanish at infinity. (See for example [1], [6], or [18].) Let $M_0(S) = \{\mu \in M(S): \mu \geqq 0 \text{ and } \|\mu\| = 1\}$ be the set of all probability measures in $M(S)$. Consider the continuous dual $M(S)^*$ of $M(S)$. Denote by 1 the element in $M(S)^*$ such that $1(\mu) = \int d\mu = \mu(S)$, $\mu \in M(S)$. Clearly $\|1\| = 1$.

2. Convolution of functionals and measures, means. Let $F \in M(S)^*$, $\mu \in M(S)$, we define a linear functional $l_\mu F = \mu \odot F$ on $M(S)$ by $\mu \odot F(\nu) = F(\mu*\nu)$, $\nu \in M(S)$. Clearly $\mu \odot F \in M(S)^*$. In fact $\|\mu \odot F\| \leqq \|\mu\| \cdot \|F\|$. Similarly we define $F \odot \mu = r_\mu F$.

A linear functional $M \in M(S)^{**}$ is called a mean if $M(F) \geqq 0$ if $F \geqq 0$ and $M(1) = 1$. Here $F \geqq 0$ means $F(\mu) \geqq 0$ for all $\mu \geqq 0$ in $M(S)$. An equivalent definition is

$$\inf\{F(\mu): \mu \in M_0(S)\} \leqq M(F) \leqq \sup\{F(\mu): \mu \in M_0(S)\}$$

for any $F \in M(S)^*$.

Consequently $\|M\| = M(1) = 1$ for any mean M on $M(S)^*$. It follows that the set of means is weak* compact and convex. Each probability measure $\mu \in M_0(S)$ is a mean if we put $\mu(F) = F(\mu)$, $F \in$

$M(S)^*$. An application of Hahn-Banach Separation Theorem shows that $M_0(S)$ is weak* dense in the set of means on $M(S)^*$.

A mean M is topological left invariant if $M(\mu \odot F) = M(F) \forall \mu \in M_0(S)$ and $F \in M(S)^*$ (see Greenleaf [7] for the case of locally compact groups).

3. Topological right stationarity and ergodic property. Following Wong [16], we say that S is topological right stationary if for each $F \in M(S)^*$, there is a net $\mu_\alpha \in M_0(S)$ and some scalar β such that $F \odot \mu_\alpha \to \beta \cdot 1$ weak* in $M(S)^*$.

THEOREM 3.1. *Let S be a locally compact semigroup, the following statements are equivalent:*

(1) *S is topological right stationary.*

(2) *For each $\nu \in M(S)$, $|\nu(S)| = \inf\{||\nu * \mu||: \mu \in M_0(S)\}$.*

(3) *There is a net $\mu_\alpha \in M_0(S)$ such that $||\mu * \mu_\alpha - \mu_\alpha|| \to 0$ for any $\mu \in M_0(S)$.*

(4) *$M(S)^*$ has a topological left invariant mean.*

Proof.

(1) implies (2).

Assume that S is topological right stationary, we modify the arguments in Glicksberg [5, Lemma 2.1] to show that S has (ergodic) property (2). Observe that

$$||\nu * \mu|| = |\nu * \mu|(S) \geqq |\nu * \mu(S)| = |\nu(S)\mu(S)| = |\nu(S)|$$

for any $\mu \in M_0(S)$, $\nu \in M(S)$. Hence $|\nu(S)| \leqslant \inf\{||\nu * \mu||: \mu \in M_0(S)\}$. Now let $c = \inf\{||\nu * \mu||: \mu \in M_0(S)\} > 0$. By Hahn-Banach Extension Theorem, there is some $F \in M(S)^*$ such that $||F|| = 1$ and

$$c \leqslant |(F, \sigma)| \text{ for any } \sigma \in C_\nu ,$$

the norm closure of the convex set $\{\nu * \mu: \mu \in M_0(S)\}$ in $M(S)$. Let C_F be the weak* closure of the convex set $\{F \odot \mu: \mu \in M_0(S)\}$ in $M(S)^*$. Since $(F, \sigma * \mu) = (F \odot \mu, \sigma)$, it follows that

$$c \leqslant |(G, \sigma)| \quad \forall \sigma \in C_\nu \quad \text{and} \quad G \in C_F .$$

But S is topological right stationary, there is some β such that $\beta \cdot 1 \in C_F$ (here we depart from Glicksberg's proof in [5, Lemma 2.1], see remarks below). Now $\beta \cdot 1$ is constant on C_ν since

$$\begin{aligned}(\beta \cdot 1, \nu * \mu) &= \beta \cdot (\nu * \mu)(S) = \beta \cdot \nu(S) \cdot \mu(S) \\ &= \beta \cdot \nu(S) = (\beta \cdot 1, \nu)\end{aligned}$$

for any $\mu \in M_0(S)$. Moreover,

$$\begin{aligned} c \leqslant |(\beta\cdot 1, \nu)| &= \inf\{|(\beta\cdot 1, \nu*\mu)|: \mu \in M_0(S)\} \\ &\leqslant |\beta|\cdot\inf\{||\nu*\mu||: \mu \in M_0(S)\} \\ &= |\beta|\cdot c\,. \end{aligned}$$

Consequently $|\beta| = 1$ and $c = |(\beta\cdot 1, \nu)| = |\beta|\cdot|\nu(S)| = |\nu(S)|$.

(2) implies (3).

Except that we work with measures instead of functions this is practically the same as in the locally compact group case (Greenleaf [7, Theorem 3.7.3]). Let $\mu \in M_0(S)$ be fixed. Consider the directed system $J = \{\alpha\}$ where $\alpha = (\mu_1, \mu_2, \cdots, \mu_n; \varepsilon)$, $\mu_i \in M_0(S)$, $\varepsilon > 0$, n finite. $\alpha \geqslant \alpha'$ means $\{\mu_i\} \supset \{\mu'_i\}$ and $\varepsilon \leqslant \varepsilon'$. For each $\alpha \in J$, we always have $(1, \mu_i*\mu - \mu) = 0$ $\forall i = 1, 2, \cdots, n$. By assumption, there exist $\sigma_1, \sigma_2, \cdots, \sigma_n \in M_0(S)$ such that

$$\begin{aligned} &||(\mu_1*\mu - \mu)*\sigma_1|| < \varepsilon \\ &||(\mu_2*\mu - \mu)*\sigma_1*\sigma_2|| < \varepsilon \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

and

$$||(\mu_n*\mu - \mu)*\sigma_1*\sigma_2* \cdots *\sigma_n|| < \varepsilon\,.$$

(Note $(1, \nu) = 0$ implies $(1, \nu*\sigma_k) = 0$.) Put $\sigma_\alpha = \sigma_1*\sigma_2* \cdots *\sigma_n$, then

$$\begin{aligned} &||(\mu_k*\mu - \mu)*\sigma_\alpha|| \\ &\leqslant ||(\mu_k*\mu - \mu)*\sigma_1* \cdots *\sigma_k||\cdot||\sigma_{k+1}* \cdots *\sigma_n|| \\ &= ||(\mu_k*\mu - \mu)*\sigma_1* \cdots *\sigma_k|| < \varepsilon \end{aligned}$$

$\forall k = 1, 2, \cdots, n$. Finally define $\mu_\alpha = \mu*\sigma_\alpha \in M_0(S)$ for $\alpha \in J$. Then $||\nu*\mu_\alpha - \mu_\alpha|| \to 0$ for any $\nu \in M_0(S)$.

(3) implies (4) and (4) implies (1).

These are the same as in the locally compact group case and we omit the details. The reader may consult Greenleaf [7] and Wong [16].

4. **Remarks.** Equivalence of (2) and (4) is an analogue of a result of H. Reiter on ergodic property of locally compact amenable groups (see Greenleaf [7, Theorem 3.7.3 p. 77]). Equivalence of (1) and (4) is an extension in a slightly different form of a result in Wong [16].

In the proof of [7, Theorem 3.7.3], Greenleaf used Rickert's fixed point theorem [7, Theorem 3.3.1]. If we were to employ the same arguments in proving that (1) implies (2), we would have to invoke an analogous fixed point theorem (see Wong [17, Theorem 3.3] which has a natural extension to locally compact semigroups) for the compact convex set C_F (referring to the proof of (1) implies (2)) to produce a fixed point $G \in C_F$ of norm 1 such that $G \odot \mu = G \forall \mu \in M_0(S)$. The question is whether $G = \beta \cdot 1$ for some scalar β? If S is a locally compact group, such a G is necessarily "constant" on $M_a(S) = L_1(S)$ (the absolutely continuous measures) and, hence on $M(S)$. For general S, Greenleaf's proof may not carry over.

Finally, it is easy to see that our definitions are consistent with previous ones given in Greenleaf [7] and Wong [16] for locally compact groups.

5. Continuous functions in $M(S)^*$. Let $CB(S)$ be the space of all bounded continuous on S with supremum norm. If $\mu \in M(S)$, $f \in CB(S)$, we can define $l_\mu f = \mu \odot f$ and $r_\mu f = f \odot \mu$ (both in $CB(S)$ again) by putting

$$\mu \odot f(s) = \int f(ts) d\mu(t)$$

$$f \odot \mu(s) = \int f(st) d\mu(t) , \quad s \in S$$

(see Williamson [15]). Invariant means on $CB(S)$ are defined in the obvious and usual way. Each function $f \in CB(S)$ can be regarded as a functional $Tf \in M(S)^*$ such that

$$Tf(\mu) = \int f d\mu, \ \mu \in M(S) .$$

The map $T: CB(S) \to M(S)^*$ is a linear isometry (into) which commutes with convolution operators l_μ (and also r_μ). Moreover $T \geqslant 0$ and $T(1) = 1$. Therefore, the two definitions of invariant mean on $CB(S)$ and its image under T agree. However, unlike the group case, it is not known if $M(S)^*$ has a topological left invariant mean when $CB(S)$ does.

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Received July 17, 1972 and in revised form March 27, 1973. Research supported by NRC of Canada Research Grant No. A8227.

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