INVARIANT SUBSPACES, SIMILARITY AND ISOMETRIC EQUIVALENCE OF CERTAIN COMMUTING OPERATORS IN L_p

ROBERT E. WATERMAN

This paper is concerned with the problem of finding all closed invariant subspaces of operators of the form $T_f = M_f - JM_{f'}$ and the determination of the similarity relationships between such operators. The operator T_f is defined, for suitable conditions on a complex-valued function f and its derivative f', by $T_f g(x) = f(x)g(x) - \int_0^x f'(t)g(t)dt$ for all g in L_p. The main result asserts that the closed invariant subspaces of T_f are precisely those subspaces that are generated by the eigenfunctions of T_f. Conversely, any operator on L_p whose closed invariant subspaces coincide with those of $M - J$ (i.e., T_f where $f(x) \equiv x$) must be of the form T_g for some function g. The closed invariant subspaces of T_f are cyclic and the generating functions have a rather simple description. The algebra of operators $\mathscr{C}$, generated by $M - J$, is maximal abelian. A corollary is that $\mathscr{C}$ is reflexive. It is shown that T_f and T_g are isometrically equivalent in L_p if and only if $f = g$. Finally conditions are given for the similarity of T_f and T_g.

We give a brief history of the problem of finding all closed invariant subspaces of a given class of operators. The case of cyclic hermitian operators is well-known; an exposition appears in Plesner [8]. The general case is accessible via multiplicity theory (see Plesner, loc. cit.). Beurling [1], in an important paper, found all the closed invariant subspaces of the shift operator on ℓ_2 through a detailed study of certain analytic functions on the unit disc. Donoghue [3] found all closed invariant subspaces of the simple Volterra operator J, defined by $f \rightarrow \int_0^x f$, on $L_2(0, 1)$ and those of a certain weighted shift operator. Kalisch [7] found all closed invariant subspaces of an extensive class of integral operators (including J) on $L_p(0, 1)$ for $1 < p < \infty$. Here we examine the invariant subspace structure of $M - J$ (and more generally, $T_f = M_f - JM_{f'}$). In spite of appearing superficially the same, the problem of finding all the closed invariant subspaces of $M - nJ$ (n an integer > 1) is quite different and considerably more difficult. This problem has been solved by the author (using recent results on L_p approximation by splines) and will appear elsewhere.

1. **Definitions.** We are concerned with bounded operators on

$L_p = L_p(0, 1)$ for $1 < p < \infty$. Define (for suitable functions f and r) the *multiplication operator* M_f, the *Volterra operator* J, and the *substitution operator* S_r respectively by $M_f g = fg$, $Jg = \int_0^x g$ (with respect to Lebesgue measure) and $S_r g = g \circ r$ (composition of g and r). For $f \in L_p$ and $g \in L_q$, where $1/p + 1/q = 1$, we define the *Banach inner product* of f and g by $(f, g) = \int_0^1 fg$. We write $B(L_p)$ for the algebra of bounded operators on L_p and write T^* to denote the *Banach adjoint* of $T \in B(L_p)$; i.e., T^* is the unique member of $B(L_q)$ such that $(Tf, g) = (f, T^*g)$ for all $f \in L_p$ and $g \in L_q$. In particular, we have $J^* = \int_x^1$ and $M_f^* = M_f$. If W is a closed invariant subspace of $T \in B(L_p)$ then we will refer to the restriction of T to W as the *part* of T on W or simply as a part of T. For $a \in [0, 1]$ we will write e_a to denote the characteristic function $\chi_{[a,1]}$ of the interval $[a, 1]$. If $E \subseteq [0, 1]$ is a nonempty closed set we define $W(E)$ to be the closed subspace of L_p generated by the set of functions $\{e_a : a \in E\}$. We will say that a complex-valued function $f : [0, 1] \to C$ is absolutely continuous on $[a, b)$ if f is absolutely continuous on every closed interval contained in $[a, b)$; the same convention will apply to the intervals $(a, b]$ and (a, b).

If f is a complex-valued function whose derivative f' exists a.e. then we will write T_f to denote $M_f - JM_{f'}$. We define $\mathscr{C}$ to be the collection of all operators T_f such (i) $f \in L_\infty$ and f is absolutely continuous on $[0, 1)$, and (ii) the function $k_f(x) \equiv (1 - x)^{q/p} \int_0^x |f'(t)|^q dt$ belongs to L_∞. The fact that $\mathscr{C} \subseteq B(L_p)$ follows from a result in [2]. In fact, $T_f \in \mathscr{C}$ implies that M_f and $JM_{f'}$ each belong to $B(L_p)$. Condition (ii), above, implies that if $T_f \in \mathscr{C}$ then f' belongs to $L_q(0, a)$ $\left(\text{i.e., } \int_0^a |f'|^q < \infty\right)$ for all $a < 1$; we will use this fact often in the sequel. We also note that (i) and (ii) together do not imply that f is absolutely continuous on $[0, 1]$. Indeed, it may happen that $T_f \in \mathscr{C}$ while f fails to be continuous at $x = 1$, as the example $f(x) = \sin(-\log(1 - x))$ shows. We define $\mathscr{A}$ to be the subset of $\mathscr{C}$ consisting of those operators $T_f \in \mathscr{C}$ such that f is 1-1 and absolutely continuous on $[0, 1]$ and $m\{x \in [0, 1] : f'(x) = 0\} = 0$ (where m denotes Lebesgue measure).

Our first lemma details the important algebraic properties of $\mathscr{C}$. Part (ii) establishes the fact that $\mathscr{C}$ is indeed an (abelian) algebra. The formula in part (iii) shows that $P(T_r) = T_{P(r)}$, for any polynomial P and any member T_r of $\mathscr{C}$. This operational calculus is easily extended to more general functions. Part (iv) of the lemma plays a crucial role in the proof of our main lemma (Lemma 4) on the invariant subspaces of T_f. The formula appearing in part (iv) becomes somewhat more transparent if one considers the special case $r(x) \equiv x$.

LEMMA 1. *If T_r and T_s belong to $\mathscr{C}$ then:*

(i) $M_r J - J M_r = J M_{r'} J$;

(ii) T_r *and* T_s *commute and their product is equal to* T_{rs} *which also belongs to* $\mathscr{C}$;

(iii) $T_r^n = M_r^n - n J M_{r'} M_r^{n-1}$ *for all* $n \in \mathbf{Z}^+$;

(iv) *if* $r(0) = 0$ *then* $(T_r^n g, h) = n(r^{n-1}, (M_{r'} J^* M_g - M_g M_{r'} J^*)h)$ *for* $n \in \mathbf{Z}^+$ *and all* $g \in L_p$ *and* $h \in L_q$.

Proof. (i) Let $g \in L_p$. Then since r is absolutely continuous on $[0, x]$ for $x < 1$ we may integrate by parts to compute

$$\int_0^x r(t)g(t)dt + \int_0^x r'(t) \int_0^t g(z)dz dt = r(x) \int_0^x g(t)dt ,$$

which, since $x < 1$ and $g \in L_p$ were arbitrary, demonstrates that $J M_r + J M_{r'} J = M_r J$.

(ii) Using part (i) to evaluate $J M_{r'} J$ we have

$$\begin{aligned}
T_r T_s &= M_{rs} - M_r J M_{s'} - J M_{r'} M_s + (J M_{r'} J) M_{s'} \\
&= M_{rs} - J M_{r'} M_s - J M_r M_{s'} \\
&= M_{rs} - J M_{(rs)'} = T_{rs} .
\end{aligned}$$

Similarly we may show that $T_s T_r = T_{sr}$ and since $T_{rs} = T_{sr}$ we have proved that T_r and T_s commute with product equal to T_{rs}. The fact that $T_{rs} \in \mathscr{C}$ is a routine verification.

(iii) This follows immediately from part (ii).

(iv) Let $T = T_r$. Since $r(0) = 0$ it is easily verified that $M_r e_0 = J M_{r'} e_0$ and thus for $n = 1$ and any $g \in L_p$ and $h \in L_q$ we have

$$\begin{aligned}
(Tg, h) &= (M_r g, h) - (J M_{r'} g, h) \\
&= (M_g J M_{r'} e_0, h) - (J M_{r'} M_g e_0, h) \\
&= (e_0, (M_{r'} J^* M_g - M_g M_{r'} J^*)h) .
\end{aligned}$$

We now sketch the inductive step, where the adjoint of (i) is used to evaluate the term $M_{r'}(J^* M_{r'} J^*)$ that arises in our computation. Let $N = J M_{r'}$. Then

$$\begin{aligned}
(T^{n+1}g, h) &= (T^n g, T^* h) \\
&= n(r^{n-1}, (N^* M_g M_r - N^* M_g N^* - M_g N^* M_r)h) \\
&\quad + n(r^{n-1}, M_g M_{r'}(J^* M_{r'} J^*)h) \\
&= n(r^{n-1}, (N^* M_g M_r - N^* M_g N^* - M_g M_r N^*)h) \\
&= n(Nr^{n-1}, M_g(M_r - N^*)h) - n(r^n, M_g N^* h) \\
&= (n+1)(Nr^n, M_g h) - (n+1)(r^n, M_g N^* h) \\
&= (n+1)(r^n, (M_{r'} J^* M_g - M_g M_{r'} J^*)h) .
\end{aligned}$$

The next lemma provides us with a useful tool for deciding when a densely defined "formal" operator (i.e., one given by a formula, such as $M_f - JM_{f'}$) that is capable of being extended to an "abstract" operator in $B(L_p)$ is actually defined everywhere in L_p by the given formula. The rather straightforward proof, which may be based on a standard theorem in analysis [4, p. 156], is left to the reader.

LEMMA 2. *Let* T *be a function from* L_p *into the set of measurable functions on* $[0, 1]$. *Suppose further that* T *is bounded and linear on a dense subspace* D *of* L_p *and* $T(D) \subseteq L_p$. *Then* T *maps* L_p *into* L_p *and is linear (and bounded) if and only if* $f_0 \in L_p$ *and* $\{f_n: n = 1, 2, \cdots\} \subseteq D$ *and* $\|f_n - f_0\|_p \to 0$ *together imply that there exists a subsequence* $\{f_{n_k}\}$ *such that* $Tf_{n_k} \to Tf_0$ *pointwise a.e. on* $[0, 1]$.

The following elementary fact is used repeatedly in the sequel and so we single it out as a lemma.

LEMMA 3. *Let* E *be a nonempty closed subset of* $[0, 1]$ *and let* $g \in L_p$. *Then* $(J^*g)(x) = 0$ *for all* x *in* E *implies that* $g = 0$ *a.e. on* E.

Proof. Let $h = -J^*g$ and define F to be the set consisting of those points x in E such that x is a limit point of E and $h'(x) = g(x)$. Clearly $m(F) = m(E)$ and so if $m(F) = 0$ the lemma is trivial. If $m(F) > 0$ and x belongs to F let $\{x_n\} \subseteq E$ be a sequence converging to x. Then $h(x) = h(x_n) = 0$ so that

$$g(x) = h'(x) = \lim_{n\to\infty} \frac{h(x) - h(x_n)}{x - x_n} = 0 ,$$

which completes our proof.

2. **Invariant subspaces.** We now turn our attention to the classification of the closed invariant subspaces of operators that belong to the set $\mathscr{A}$. Our Main Lemma, which gives a characterization of the functions that belong to a given closed invariant subspace of an operator in $\mathscr{A}$, is used repeatedly in much of what follows.

LEMMA 4 (MAIN LEMMA). *Let* W *be a closed invariant subspace of* $T_f \in \mathscr{A}$. *Let* $E = \{a \in [0, 1]: e_a \in W\}$ *and* $K = \inf E$. *Then*
  (i) E *is closed and nonempty;*
  (ii) *if* $f_0 \in W$ *then* f_0 *is a constant* k_i *a.e. on each component* (c_i, d_i) *of the open set* $[K, 1] \backslash E = \bigcup_{i=1}^{\infty} (c_i, d_i)$ *and* $f_0 = 0$ *a.e. on* $[0, K]$.

Proof. (i) Since $1 \in E$ we have $E \neq \varnothing$. Letting $x_i \to x$ with $\{x_i\} \subseteq E$ one easily verifies that $\|e_{x_i} - e_x\|_p \to 0$. Thus $x \in E$ and so

E is closed.

(ii) Without loss of generality we may suppose that $f(0) = 0$ since T_f and $T_{(f-f(0))} = T_f - f(0)$ have the same invariant subspaces. If $K = 0$ then trivially $f_0 = 0$ a.e. on $[0, K]$. Thus we will assume that $K > 0$ for the remainder of the proof. Let $a \in U$ where $U = (0, K)$ or $U = (c_i, d_i)$ is a component of $[K, 1] \backslash E$. Then $e_a \notin W$ and so by the Hahn-Banach Theorem there exists a continuous linear functional S on L_p such that $S(W) = 0$ and $S(e_a) \neq 0$. According to the Riesz Representation Theorem there exists an h in L_q such that $Sg = (g, h)$ for all g in L_p. Thus $(T_f^n f_0, h) = 0$ for all $n \in Z^+$ and so Lemma 1 (iv) gives us

$$(1) \qquad 0 = (f^{n-1}, (M_{f'} J^* M_{f_0} - M_{f_0} M_{f'} J^*) h)$$

for all $n \in Z^+$. Since f, by hypothesis, is continuous and 1-1, a theorem due to J. L. Walsh [9, p. 39] implies that polynomials in f are dense in the space of complex-valued continuous functions on $[0, 1]$ and consequently are dense in L_p. Thus (1) and $f' \neq 0$ a.e. (since $T_f \in \mathscr{A}$) together imply that

$$(2) \qquad f_0(x) \int_x^1 h(t)dt = \int_x^1 f_0(t)h(t)dt$$

almost everywhere. Now

$$\int_a^1 h(t)dt = \int_0^1 e_a(t)h(t)dt = (e_a, h) = Se_a \neq 0$$

and thus, since $\int_x^1 h(t)dt$ is continuous, there exists an open interval $I_a \subseteq U$ containing a, such that $\int_x^1 h(t)dt \neq 0$ on $\bar{I}_a$. This fact together with (2) implies that f_0 is absolutely continuous on $\bar{I}_a$. Thus we may differentiate both sides of (2) on I_a to obtain $f_0'(x) \int_x^1 h(t)dt = 0$ a.e. on I_a which in turn implies that $f_0' = 0$ a.e. on I_a. Since f_0 is absolutely continuous on $\bar{I}_a$ we conclude that f_0 is equal to a constant m_a on I_a. Now let $\varepsilon > 0$ be given. Repeating the above argument at each point a in $F_\varepsilon = [\varepsilon, K - \varepsilon]$ (or $F_\varepsilon = [c_i + \varepsilon, d_i - \varepsilon]$ as the case may be), and then using the compactness of F_ε, we obtain a finite covering of F_ε by open intervals $I_1, \cdots, I_n$ such that $f_0 = m_i$ on $I_i (i = 1, \cdots, n)$. It is easy to see that all the m_i's must be equal and so f_0 is a constant k_ε on F_ε. The fact that f_0 is constant on U follows easily from the arbitrariness of $\varepsilon > 0$. Finally we must show that when $U = [0, K]$ we have $f_0 = 0$ on U. If $K > 0$ then e_0 does not belong to W and so there exists a continuous linear functional $S \equiv (\cdot, h)$ on L_p such that $S(W) = 0$ and $(e_0, h) \neq 0$. For all $a \in E$ we have

$$0 = Se_a = (e_a, h) = \int_a^1 h(t)dt = (J^*h)(a)$$

and so by Lemma 3 we have $h = 0$ almost everywhere on E. Thus

$$(3) \qquad 0 = Sf_0 = \int_0^K f_0(t)h(t)dt + \sum_{i=1}^\infty \int_{c_i}^{d_i} f_0(t)h(t)dt \ .$$

Since c_i and d_i belong to E for all i and since we have shown that $f_0 = k_i$ on (c_i, d_i) we have

$$
\begin{aligned}
(4) \qquad \int_{c_i}^{d_i} f_0(t)h(t)dt &= k_i \int_{c_i}^{d_i} h(t)dt \\
&= k_i \left(\int_{c_i}^1 h(t)dt - \int_{d_i}^1 h(t)dt \right) \\
&= k_i(Se_{c_i} - Se_{d_i}) = 0 \ .
\end{aligned}
$$

Thus, since f_0 is a constant k_0 on $[0, K]$, (3) reduces to

$$
\begin{aligned}
0 = Sf_0 = \int_0^K f_0(t)h(t)dt &= k_0 \int_0^K h(t)dt \\
&= k_0(Se_0 - Se_K) = k_0 Se_0
\end{aligned}
$$

(since $K \in E$ implies $Se_K = 0$). But $Se_0 \neq 0$ and so we must have $k_0 = 0$.

LEMMA 5. *The spectrum, point spectrum and continuous spectrum of the operator* $T_f \in \mathscr{A}$ *are respectively the sets* $f([0, 1])$, $f([0, 1))$ *and* $\{f(1)\}$. *The point spectrum is simple with the eigenfunction* e_a *corresponding to the eigenvalue* $f(a)$.

Proof. If $a \in [0, 1)$ then a simple calculation shows that $T_f e_a = f(a)e_a$ and so $f([0, 1)) \subseteqq P\sigma(T_f)$ and $f([0, 1]) \subseteqq \sigma(T_f)$. If $c \in C$ does not belong to $f([0, 1])$ then the function $h = 1/(f - c)$ is absolutely continuous on $[0, 1]$ and it is easy to see that $T_h \in \mathscr{C}$. By Lemma 1 (ii) we have $T_{(f-c)}T_h = T_h T_{(f-c)} = T_{h(f-c)} = I$ and so $T_{(f-c)}$ is boundedly invertible on L_p. Since $T_{(f-c)} = T_f - c$ it follows that c is not in the spectrum of T_f. Thus $\sigma(T_f) = f([0, 1])$. Next suppose $(T_f - f(a))g = 0$ for $g \in L_p$ and $a \in [0, 1]$. Then

$$(1) \qquad (f(x) - f(a))g(x) = \int_0^x f'(t)g(t)dt$$

almost everywhere on $[0, 1]$ which implies that g is absolutely continuous on $[0, a)$ and $(a, 1]$. If we differentiate (1) we find that $g' = 0$ a.e. on $[0, 1]$ and therefore g is equal to a constant k_0 a.e. on $[0, a)$ and $g = k_1$ a.e. on $(a, 1]$. Let $a > 0$. Then for almost every $x \in [0, a]$

$$0 = (T_f - f(a))g(x) = k_0(f(x) - f(a)) - k_0(f(x) - f(0))$$
$$= k_0(f(0) - f(a))$$

and since f is 1-1 we must have $k_0 = 0$. Thus $g = k_1 e_a$ which proves that if $a \in [0, 1)$ the eigenvalue $f(a)$ is simple and if $a = 1$ the operator $T_f - f(1)$ is 1-1. We observe that the range of $T_f - f(1)$ is dense in L_p since, for all $a < 1$, $(T_f - f(1))e_a = (f(a) - f(1))e_a$. But the range of $T_f - f(1)$ cannot be all of L_p since $f(1) \in \sigma(T_f)$. Thus $f(1)$ belongs to $C\sigma(T_f)$. This also shows that $P\sigma(T_f) \subsetneqq f([0, 1))$, which, together with the reverse inclusion obtained earlier gives us the desired equality.

If T_f is an operator in $\mathscr{A}$ and if E is an arbitrary nonempty closed subset of $[0, 1]$ then clearly the closed subspace $W(E)$ is invariant for T_f since $W(E)$ is generated by certain eigenfunctions of T_f. Indeed, $W(E)$ is the closed linear span of the eigenfunctions e_a for a belonging to the set E. Our main result, which we now state and prove, demonstrates that these are the only closed invariant subspaces of T_f.

THEOREM 1. (*Invariant Subspaces of* T_f). *The only closed invariant subspaces of the operator* $T_f \in \mathscr{A}$ *are those generated by the eigenfunctions of* T_f. *Specifically, if* W *is a closed invariant subspace of* T_f *then there exists a unique closed set* $E \subseteq [0, 1]$ *containing* 1 *such that* $W = W(E)$. *Consequently the lattice of closed invariant subspaces of* T_f *is isomorphic to the lattice of closed subsets of* $[0, 1]$ *that contain the point* 1.

Proof. Let W be an arbitrary closed invariant subspace of T_f. Define $E = \{a \in [0, 1]: e_a \in W\}$. By Lemma 4 (i) the set E is closed and nonempty and clearly $W(E) \subseteq W$ since the generators of $W(E)$ all belong to W. For the reverse inclusion let S be an arbitrary continuous linear functional on L_p such that $S(W(E)) = 0$. By the Riesz Representation Theorem there exists an $h \in L_q$ such that $Sg = (g, h)$ for all $g \in L_p$. Since $0 = Se_a = (J^*h)(a)$ for all $a \in E$ it follows from Lemma 3 that $h = 0$ almost everywhere on E. Thus for any $g \in W$ we have

$$(1) \qquad Sg = \int_0^K g(t)h(t)dt + \sum_{n=1}^{\infty} \int_{c_i}^{d_i} g(t)h(t)dt$$

where $K = \inf E$ and the (c_i, d_i)'s are the components of $[K, 1] \backslash E$. By Lemma 4 (ii) we see that $0 = \int_0^K g(t)h(t)dt$ since $g = 0$ almost everywhere on $[0, K]$. For the remaining integrals in (1) we again appeal to Lemma 4 (ii) and use the fact that c_i and d_i belong to E to conclude that

$$\int_{c_i}^{d_i} g(t)h(t)dt = k_i(Se_{c_i} - Se_{d_i}) = 0$$

where $g = k_i$ a.e. on (c_i, d_i). Thus $Sg = 0$ and consequently $S(W) = 0$. The Hahn-Banach Theorem now implies that $W \subseteq W(E)$. The uniqueness of the set E, provided $1 \in E$, is straightforward and so we omit the details.

As an elementary application of Theorem 1 let us find the closed invariant subspaces of the operator $M + J$. Straightforward calculation reveals that the spectrum of $M + J$ is the closed unit interval, with $(0, 1]$ serving as residual spectrum and the point 0 as continuous spectrum. Thus $M + J$ is certainly not similar to $M - J$. Nonetheless we can employ the operator $M - J$ to get at the invariant subspace structure of $M + J$ as follows. Let S denote the substitution operator defined by $Sg(x) = g(1 - x)$. Then $S = S^{-1}$ is an invertible isometry and one can verify directly that $S(M - J)S = I - (M + J^*)$. It follows from Theorem 1 (since $M - J$ belongs to $\mathscr{A}$) that each and every closed invariant subspace of $M + J^*$ is generated by a suitable subset of its eigenfunctions. Finally, by taking adjoints, it is an easy matter to determine the closed invariant subspaces of the operator $M + J = (M + J^*)^*$. In fact, the lattice of closed invariant subspaces of $M + J$ is antiisomorphic to the lattice of all closed subsets of $[0, 1]$ that contain the point 0.

Our next result illustrates the manner in which certain subsets of the complex plane may arise as the (pure) point spectrum of an operator.

COROLLARY. *Let E be a nonempty closed subset of $[0, 1]$ not containing 1 and let T be the part of $T_f \in \mathscr{A}$ on the closed invariant subspace $W(E)$. Then the lattice of closed invariant subspaces of T is isomorphic to the lattice of all closed subsets of E. The spectrum of T is the set $f(E)$ and is purely point.*

Proof. The statement about the lattice of invariant subspaces of T follows easily from Theorem 1 and our hypothesis that $1 \notin E$. The claim regarding the spectrum of T is a slight generalization of a result due to G. K. Kalisch [5].

The result of Kalisch referred to is that for any nonempty compact subset of the real numbers there exists a bounded operator on a separable Hilbert space whose spectrum is purely point and equal to the given set. A similar result can be obtained for the residual

spectrum of an operator by considering suitable parts of the operator $M + J$. For the continuous spectrum of an operator the above result is well-known. We remark that all of this can be carried out for arbitrary nonempty compact subsets of the plane; i.e., given a nonempty compact subset of the plane there exists a bounded operator on a separable Hilbert space whose spectrum is purely point (or purely continuous or purely residual, as may be desired) and is equal to the given set. The case of pure point spectrum is also due to Kalisch.

We now state some algebraic results about the algebra $\mathscr{C}$ starting with the assertion that $\mathscr{C}$ is a maximal abelian algebra.

THEOREM 2. (*Commutant of* $M - J$) $\mathscr{C}$ *is a maximal abelian subalgebra of* $B(L_p)$. *In fact if* T *belongs to* $B(L_p)$ *and commutes with* $M - J$ *then* $T \in \mathscr{C}$: *i.e.,* $\mathscr{C}$ *is the commutant of* $M - J$ *relative to* $B(L_p)$.

Proof. By Lemma 1 (ii) we see that $\mathscr{C}$ is an abelian algebra. If T belongs to $B(L_p)$ and commutes with $M - J$ then for every $a \in [0, 1)$ we have

$$(M - J)Te_a = T(M - J)e_a = aTe_a$$

which, according to Lemma 5, implies that Te_a is some scalar multiple, say $h(a)$, of the function e_a. Thus there exists a function $h: [0, 1) \to C$ such that $Te_a = h(a)e_a$ for all $a \in [0, 1)$. For $x \in [0, 1)$ we may then compute

$$\begin{aligned}
(1 - x)h(x) &= \int_0^1 h(x)e_x(t)e_0(t)dt \\
&= (h(x)e_x, e_0) = (Te_x, e_0) = (e_x, T^*e_0) \\
&= \int_0^1 e_x(t)(T^*e_0)(t)dt = \int_x^1 (T^*e_0)(t)dt \ .
\end{aligned}$$

Thus for all $x < 1$

(1) $$h(x) = \left(\frac{1}{1 - x}\right) \int_x^1 (T^*e_0)(t)dt \ .$$

Since T is in $B(L_p)$, its adjoint T^* belongs to $B(L_q)$ and consequently T^*e_0 belongs to L_q. Therefore, (1) implies that h is absolutely continuous on $[0, 1)$ and h' belongs to $L_q(0, a)$ for all $a < 1$. Since $Te_a = h(a)e_a$ we have the set inclusions $h([0, 1)) \subseteq P\sigma(T) \subseteq \sigma(T)$ and it follows, upon defining $h(1)$ arbitrarily, that $h \in L_\infty$. We now consider $T_h = M_h - JM_{h'}$. Since $h \in L_\infty$ the operator M_h is bounded. The operator $JM_{h'}$ maps L_p into the measurable functions since $h' \in L_q(0, a)$ for all $a < 1$.

Furthermore, on the dense subspace D of L_p generated by $\{e_a: a \in [0, 1)\}$ it is clear that $JM_{h'}$ agrees with the bounded operator $T - M_h - T$. Finally it is easy to verify that if $\{g_n\} \subseteq D$ converges to $g \in L_p$ then there exists a subsequence $\{g_{n_k}\}$ such that $\{JM_{h'}g_{n_k}\}$ converges to $JM_{h'}g$ pointwise almost everywhere. In fact, the sequence $\{g_n\}$ itself has this property, for if $x < 1$ we may use Hölder's inequality to compute

$$|JM_{h'}g_n(x) - JM_{h'}g(x)| \leqq \int_0^x |h'(t)| \, |g_n(t) - g(t)| \, dt$$

$$\leqq \left(\int_0^x |h'(t)|^q dt\right)^{1/q} \left(\int_0^x |g_n(t) - g(t)|^p\right)^{1/p}$$

which implies that $\{JM_{h'}g_n\}$ converges pointwise to $JM_{h'}g$ on $[0, 1)$. Thus all of the hypotheses of Lemma 2 are satisfied by $JM_{h'}$ and so $JM_{h'}$ is bounded on L_p. It follows from [2] that

$$(1 - x)^{q/p} \int_0^x |h'(t)|^q dt \in L_\infty$$

and so T_h belongs to $\mathscr{C}$ and clearly $T_h = T$. Thus $\mathscr{C}$ is the commutant of $M - J$ and consequently $\mathscr{C}$ is maximal abelian.

As an easy corollary of Theorem 2 we show that the algebra $\mathscr{C}$ is reflexive; i.e., we prove that $\mathscr{C} = \mathrm{Alg\ Lat\ }\mathscr{C}$ where $\mathrm{Lat\ }\mathscr{C}$ denotes the family of all closed subspaces of L_p that are left invariant by every member of $\mathscr{C}$, and $\mathrm{Alg\ Lat\ }\mathscr{C}$ denotes the algebra of bounded operators on L_p that leaves invariant every member of $\mathrm{Lat\ }\mathscr{C}$. The inclusion $\mathscr{C} \subseteq \mathrm{Alg\ Lat\ }\mathscr{C}$ is obvious. For the reverse inclusion let T be bounded operator on L_p that leaves every member of $\mathrm{Lat\ }\mathscr{C}$ invariant. Then the functions $\{e_a: a \in [0, 1)\}$ are all eigenfunctions of T and consequently T commutes with $M - J$. By Theorem 2 we conclude that T belongs to $\mathscr{C}$ and so $\mathscr{C}$ is reflexive.

Our next theorem is a partial converse of Theorem 1. We remark that Theorem 1 can be extended to a slightly larger collection of operators of the form T_f than those in the set $\mathscr{A}$ (by dropping the requirement that f be continuous at $x = 1$, but insisting that the closure of the range of f does not separate the plane), in which case our next theorem would be a full converse.

THEOREM 3. (*Partial Converse of Theorem* 1). *If the closed invariant subspaces of $T \in B(L_p)$ are precisely those of the form $W(E)$, where E ranges over all nonempty closed subsets of $[0, 1]$, then $T = T_h \in \mathscr{C}$ and*

(i) *h is $1-1$ on $[0, 1)$;*
(ii) *$m\{x: h'(x) = 0\} = 0$.*

Proof. Our hypothesis implies that T commutes with $M - J$ (since e_a is an eigenfunction of T for all a in $[0, 1)$) and thus by Theorem 2 we have $T = T_h \in \mathscr{C}$. If $0 \leqq a < b < 1$ and $h(a) = h(b)$ then T_h would have an eigenspace of dimension $\geqq 2$ at $h(a)$ which would contradict our hypothesis regarding the nature of the invariant subspaces of T. Indeed, the linearly independent functions e_a and $\chi_{[a,b]}$ would belong to the eigenspace of T at $h(a)$. Thus h must be 1-1 on $[0, 1)$ which proves (i). Let $F = \{x: h'(x) = 0\}$ and suppose that $m(F) > 0$. We may assume, without any loss of generality that $F \subsetneqq [0, a]$ for some $a < 1$. Then clearly the closed subspace $V = \chi_F L_p$, obtained by mapping $g \in L_p$ onto $\chi_F g$, is invariant for T_h and so by our hypothesis we must have $V = W(E)$ for some closed nonempty subset $E \subsetneqq [0, 1]$. But if some $b < 1$ belongs to E then $e_b \in V$ which is impossible since every function in $\chi_F L_p$ vanishes on $[a, 1]$. Thus $E = \{1\}$ which implies that $V = 0$ and contradicts our assumption that $m(F) > 0$. Hence $m(F) = 0$ and so (ii) is proved.

THEOREM 4. *(Cyclicity of Invariant Subspaces of T_f).* *Every closed invariant subspace W of the operator $T_f \in \mathscr{A}$ is cyclic. In fact, if we define $E = \{a: e_a \in W\}$ and $K = \inf E$ then a function $s \in L_p$ is cyclic for W if and only if s has the following properties:* (i) *$s = 0$ almost everywhere on $[0, K]$,* (ii) *s is a constant k_i on each component of $[K, 1] \backslash E$ and* (iii) *s is not a.e. equal to a constant on any open interval containing a point $a \in E \backslash \{0, 1\}$.*

Proof. Let W be an arbitrary closed invariant subspace of T_f so that by Theorem 1 we have $W = W(E)$. Let $s \in L_p$ satisfy (i), (ii), and (iii) and let W_s be the closed invariant subspace of T_f generated by s and T_f. Applying Theorem 1 again gives us $W_s = W(F)$ for some closed set $F = \{a \in [0, 1]: e_a \in W_s\}$. We now proceed to show that $W = W_s$ by showing that $E = F$. We first observe that $0 \in E$ if and only if $s \neq 0$ a.e. in any neighborhood of $x = 0$. Similarly $0 \in F$ if and only if $s \neq 0$ near $x = 0$. Thus $0 \in E$ if and only if $0 \in F$. Also, by definition, $1 \in E \cap F$. Suppose next that $a \notin E$ (with $a \neq 0$ and $a \neq 1$). Then there exists an open interval I containing a, such that $T_f^n s$ is constant on I for all $n \in Z^+$ and it follows that $e_a \notin W_s$. Therefore, $a \notin F$ which proves that $F \subsetneqq E$. For the reverse inclusion let $a \in E$ and $a \notin \{0, 1\}$. If $a \notin F$ then upon applying Lemma 4 (ii) to $W_s = W(F)$ we see that there exists an open interval I containing a such that s is constant on I. But this contradicts (iii) since $a \in E$. Thus we must have $a \in F$ and hence $E \subsetneqq F$. For the converse we note that if s is cyclic for W then, since $W = W(E)$, properties (i) and (ii) must hold for s. It is straightforward to show that if s does not satisfy (iii) then s and T_f generate a subspace $W(F)$ properly

contained in $W(E)$. Thus we have shown that conditions (i), (ii), and (iii) are necessary and sufficient for the cyclicity of s. To complete our proof we will demonstrate the existence of a function satisfying these conditions. Let $\{a_i\}_{i=1}^{\infty}$ be a dense subset of $E = \{e_a : a \in W\}$ and define a function s by

$$s \equiv \sum_{i=1}^{\infty} 2^{-i} e_{a_i} \, .$$

Then clearly s satisfies conditions (i), (ii), and (iii) and is therefore cyclic for W. In case the set E is perfect the function $f\chi_E$ is also cyclic for W. However, if E is not perfect then E contains isolated points; if there exists an isolated point $p \in E \cap (0, 1)$ then it is readily verified that $f\chi_E$ fails condition (iii) in a neighborhood of p and consequently $f\chi_E$ is not cyclic for W.

3. **Similarity and isometric equivalence.** We begin our study of the similarity and isometric equivalence of operators belonging to the set $\mathscr{A}$ with a lemma on the boundedness and isometric behavior of certain substitution operators.

LEMMA 6. *Let $r: [0, 1] \to [0, 1]$ and $s = r^{-1}$ be strictly increasing absolutely continuous functions such that $r(0) = 0$ and $r(1) = 1$. Then (i) $S_r \in B(L_p)$ if and only if $s' \in L_\infty$ (ii) S_r is an isometry of L_p into itself if and only if $S_r = I$.*

Proof. (i) Suppose S_r is bounded. If $x < y$ then

$$|s(y) - s(x)| = \int_0^1 \chi_{[s(x), s(y)]}(t)dt = \int_0^1 (S_r \chi_{[x,y]})(t)dt$$
$$= \|S_r \chi_{[x,y]}\|_p^p \leqq \|S_r\|^p \cdot |y - x| \, .$$

Thus for a.e. x_0 in $[0, 1]$ we have

$$|s'(x_0)| = \lim_{x \to x_0} |s(x) - s(x_0)|/|x - x_0| \leqq \|S_r\|^p$$

which shows that $s' \in L_\infty$. Conversely if $s' \in L_\infty$, and $f \in L_p$ is arbitrary then

$$\|S_r f\|_p^p = \int_0^1 |f(r(t))|^p dt = \int_0^1 |f(t)|^p s'(t)dt \leqq \|s'\|_\infty \|f\|_p^p$$

which shows that S_r maps L_p into itself and is bounded.

(ii) If S_r is an isometry then for all x in $[0, 1]$

$$s(x) = \|S_r \chi_{[0,x]}\|_p^p = \|\chi_{[0,x]}\|_p^p = x \, .$$

Thus $s(x) = x$ which implies that $r(x) = s^{-1}(x) = x$ and $S_r = I$.

Our next lemma provides necessary and sufficient conditions for the absolute continuity of the composition $r = g^{-1} \circ f$ of g^{-1} and f where g and f are 1-1 complex-valued absolutely continuous functions. This result is used in the proofs of Theorems 5, 6, and 7 to establish certain properties of the substitution operator S_r, which arises in our investigation of the similarity and isometric equivalence of T_f and T_g.

LEMMA 7. *Let f and $g: [0, 1] \to C$ be 1-1 absolutely continuous functions such that $f(0) = g(0)$ and $f([0, 1]) = g([0, 1])$. Define $E_0 = \{x : g'(x) = 0\}$. Then $r = g^{-1} \circ f$ is absolutely continuous if and only if for each set $E \subseteqq [0, 1]$ of measure zero we have $m(E_0 \cap r(E)) = 0$.*

Proof. Our hypothesis implies that $r = g^{-1} \circ f$ and $s \equiv r^{-1}$ are strictly increasing continuous functions mapping $[0, 1]$ onto itself. A well known theorem [4, p. 288] states that a real-valued function is absolutely continuous if and only if it is continuous, of bounded variation and carries sets of measure zero into sets of measure zero. Thus the proof of the lemma is reduced to showing that the following two conditions are equivalent:

   (i) $m(E) = 0$ implies $m(E_0 \cap r(E)) = 0$,

   (ii) $m(E) = 0$ implies $m(r(E)) = 0$.

Clearly (ii) implies (i) and so to complete the proof we show that the reverse implication, (i) implies (ii), holds. If h is a complex-valued absolutely continuous function on $[0, 1]$ and if $(a, b) \subseteqq [0, 1]$ let us write $V_{(a,b)}h$ to denote the total variation of h on the interval (a, b). A standard theorem in analysis [4, p. 272] together with the fact that $V_{(a,b)}f = V_{(r(a),r(b))}g$ gives us

$$(1) \qquad \int_E |f'(t)| \, dt = \int_{r(E)} |g'(t)| \, dt$$

whenever E is an interval in $[0, 1]$. It is now straightfoward to verify that (1) holds for every measurable subset E in $[0,1]$. Now suppose that (i) holds and let $E \subseteqq [0, 1]$ be an arbitrary set of measure zero. Then using (1) we obtain

$$
\begin{aligned}
(2) \qquad 0 &= \int_E |f'(t)| \, dt \\
&= \int_{r(E)} |g'(t)| \, dt \\
&= \int_{r(E) \cap E_0} |g'(t)| \, dt + \int_{r(E) \setminus E_0} |g'(t)| \, dt \\
&= \int_{r(E) \setminus E_0} |g'(t)| \, dt
\end{aligned}
$$

where the last equality follows from applying (i) to the set E which

is assumed to have measure zero.   Thus

$$0 = \int_{r(E)\setminus E_0} |g'(t)|\, dt$$

and since $g' \neq 0$ a.e. on E_0^c we conclude that $m(r(E)\setminus E_0) = 0$.   On the other hand, (i) says that $m(r(E) \cap E_0) = 0$ so that

$$m(r(E)) = m(r(E) \cap E_0) + m(r(E)\setminus E_0) = 0$$

which shows that (ii) holds, and completes our proof.

Our next result, which gives necessary conditions for the similarity of two operators in $\mathscr{A}$, is the workhorse for much of what follows.

LEMMA 8. *If T_f and T_g belong to $\mathscr{A}$ and are similar in L_p, say $VT_fV^{-1} = T_g$, then:*

  (i) *the functions $r = g^{-1} \circ f$ and $s = r^{-1}$ are well-defined absolutely continuous strictly monotone increasing functions mapping $[0, 1]$ onto itself;*

  (ii) *there exists a function $h: [0, 1) \to C$, absolutely continuous and nonvanishing on $[0, 1)$, such that $Ve_a = h(a)e_{r(a)}$ for all $a \in [0, 1)$.   In fact, for all $x \in [0, 1)$*

$$h(x) = \left(\frac{1}{1 - r(x)} \right) \int_x^1 (V^*e_0)(t)dt \ .$$

Proof. (i) If T_f and T_g are similar in L_p then T_f and T_g must have the same point spectrum and spectrum so Lemma 5 implies that $f([0, 1)) = g([0, 1))$ and $f([0, 1]) = g([0, 1])$.   Since f and g are 1–1 continuous functions it follows that $f(0) = g(0)$ and $f(1) = g(1)$ and consequently the functions $r = g^{-1} \circ f$ and $s = r^{-1}$ are strictly monotone increasing continuous functions mapping $[0, 1]$ onto itself.   Since f' and g' are nonvanishing a.e. it follows that r' and s' also have this property and so Lemma 7 implies that r and s are absolutely continuous.

  (ii) Let $a \in [0, 1)$.   Then

$$T_g(Ve_a) = VT_fV^{-1}Ve_a = VT_fe_a = f(a)Ve_a$$

which shows that Ve_a is an eigenfunction of T_g corresponding to the eigenvalue $f(a) = g(r(a))$.   By Lemma 5 we know that the point spectrum of T_g is simple and so Ve_a must be some scalar multiple, say $h(a)$, of the eigenfunction $e_{r(a)}$.   Thus we have a function $h: [0, 1) \to C$ such that $Ve_a = h(a)e_{r(a)}$ for all $a \in [0, 1)$.   Also, h is nonvanishing on $[0, 1)$ since V is invertible.   For $a \in [0, 1)$ we compute

$$h(a)(1 - r(a)) = \int_0^1 h(a)e_{r(a)}(t)e_0(t)dt = (Ve_a, e_0)$$

$$= (e_a, V^*e_0) = \int_a^1 (V^*e_0)(t)dt .$$

The above computation shows that h is absolutely continuous on $[0, 1)$.

THEOREM 5. (*Isometric Equivalence of T_f and T_g*). *If T_f and T_g belong to $\mathscr{A}$ then they are isometrically equivalent in L_p if and only if $f = g$.*

Proof. Suppose that V is a boundedly invertible isometry such that $VT_fV^{-1} = T_g$. Then by Lemma 8 the functions $r = g^{-1} \circ f$ and $s = r^{-1}$ are well-defined strictly increasing absolutely continuous maps of $[0, 1]$ onto itself. We claim that for every nonempty closed subset $E \subseteqq [0, 1]$

$$(1) \qquad\qquad V[W(E)] = W(r(E)) .$$

To see this let h denote the function appearing in the statement of part (ii) of Lemma 8. Then $Ve_a = h(a)e_{r(a)}$ for all a in $[0, 1)$ and $h(a) \neq 0$. Since V is invertible we clearly have $V(\text{span } e_a) = \text{span }(Ve_a) = \text{span }(e_{r(a)})$. Equation (1) now follows by taking the closures of the above spans. Next we consider the operator

$$V_r \equiv M_{(r')^{1/p}}S_r \colon L_p \longrightarrow L_p .$$

Using the absolute continuity of r this operator is easily seen to be an isometry since for all $h_1 \in L_p$

$$||V_r h_1||_p^p = \int_0^1 |h_1(r(t))(r'(t))^{1/p}|^p dt$$

$$= \int_0^1 |h_1(t)|^p dt = ||h_1||_p^p .$$

Similarly we see that V_s is an isometry and since

$$V_s = M_{(s'(x))^{1/p}}S_s = S_s M_{(1/r'(x))^{1/p}}$$

it follows that V_r is boundedly invertible on L_p. For $t \in [0, 1]$ let E_t denote the projection of $L_p(0, 1)$ onto $W([t, 1])$ defined by $E_t h_1 = e_t h_1$ for all $h_1 \in L_p$. Then for all $h_1 \in L_p$ and a.e. x in $[0, 1]$ we have

$$(2) \quad \begin{aligned} (V_r^{-1}E_t V_r h_1)(x) &= (S_s M_{(1/r')^{1/p}}E_t M_{(r')^{1/p}}S_r h_1)(x) \\ &= (S_s E_t S_r h_1)(x) = e_t(s(x)) \cdot h_1(x) \\ &= e_{r(t)}(x) \cdot h_1(x) = (E_{r(t)} h_1)(x) \end{aligned}$$

for all t in $[0, 1]$. Since V is an invertible isometry it is clear that

$\|VE_t V^{-1}\| = \|E_t\| = 1$. The operator $VE_t V^{-1}$ is obviously idempotent and by (1)

$$(VE_t V^{-1})L_p(0, 1) = (VE_t)L_p(0, 1) = VW([t, 1])$$
$$= W(r[t, 1]) = W([r(t), 1]) ,$$

so that $VE_t V^{-1}$ has range equal to $W([r(t), 1])$. A result due to G. K. Kalisch [6, p. 94] now allows us to conclude that

$$(4) \qquad\qquad VE_t V^{-1} = E_{r(t)} .$$

Upon combining (3) and (4) we obtain

$$(5) \qquad\qquad V_r VE_t = E_t V_r V$$

for all t in $[0, 1]$. Another result of G. K. Kalisch [6, p. 95] now implies that

$$(6) \qquad\qquad M_k = V_r V$$

for some $k \in L_\infty$. Then for all $h_1 \in L_p$

$$Vh_1 = V_r^{-1} M_k h_1$$
$$= (V_r^{-1} M_k e_0)(S_s h_1)$$
$$= (Ve_0)(S_s h_1) = h(0)S_s h_1$$

which shows that $V = h(0)S_s$. However,

$$|h(0)| \cdot \|e_0\|_p = \|Ve_0\|_p = \|e_0\|_p$$

implies that $|h(0)| = 1$ and so S_s is an isometry. Lemma 6 (ii) now tells us that $S_s = I$ and so $T_f = T_g$. A routine argument now demonstrates that $f = g$.

THEOREM 6. (*Similarity of T_f and T_g*). *If T_f and T_g belong to $\mathscr{A}$ then they are similar in L_p if and only if $f([0, 1)) = g([0, 1))$ and there exists a function h, absolutely continuous on $[0, 1)$, such that the operator V_h defined by $V_h \equiv S_{(f^{-1} \circ g)}(M_h - JM_{h'})$ is boundedly invertible on L_p. In this case the similarity is implemented by V_h and we have $V_h T_f V_h^{-1} = T_g$.*

Proof. (i) Suppose T_f and T_g are similar and let V be a boundedly invertible operator on L_p such that $VT_f V^{-1} = T_g$. By Lemma 8 the functions $r = g^{-1} \circ f$ and $s = r^{-1}$ are strictly monotone and absolutely continuous with $r(0) = 0$ and $r(1) = 1$. Let h denote the function in the statement of Lemma 8 (ii) and define the operator V_h by $V_h = S_s T_h$. An elementary calculation shows that $V_h e_a = Ve_a$ for all $a \in [0, 1]$ and so V_h and V agree on the dense subspace D of L_p consisting

of all linear combinations of the e_a's. Thus V_h extends uniquely to a bounded operator, namely V, defined on all of L_p. In a sense this is what we set out to prove. However, the proof of the corollary of the present theorem will require the somewhat sharper result that V is actually given by the formula for V_h, on *all* of L_p. We will show this ultimately by using Lemma 2, but at this point we do not know that V_h is defined on all of L_p. Indeed, to show this we will need the fact that h' belongs to $L_q(0, a)$ for all $a < 1$; but Lemma 8 (ii) merely implies that $h' \in L_1(0, a)$ for all $a < 1$. We begin by showing that V_h is well-defined on L_∞ and agrees with V there. Let $g_1 \in L_\infty$ be arbitrary. For $a < 1$ Lemma 8 (ii) implies that $h' \in L_1(0, s(a))$ and so, by Hölder's inequality,

$$| S_s J M_{h'} g_1(a) | \leqq \int_0^{s(a)} | h'(t) g_1(t) | dt < \infty .$$

Clearly $S_s M_h g_1(a)$ is finite for a.e. $a \in [0, 1]$ and so $V_h g_1$ is well-defined and is a measurable function which is finite a.e. on $[0, 1]$. Let $\{s_n\} \subseteqq D$ be a sequence of step functions such that $|| s_n - g_1 ||_p \to 0$ and $|| s_n ||_\infty \leqslant || g_1 ||_\infty$ for all n. Since V is bounded we have $V s_n \to V g_1$ in L_p and so there exists a subsequence, which we will also denote as $\{s_n\}$ for simplicity, such that $V s_n \to V g_1$ almost everywhere. Similarly we may assume that $s_n \to g_1$ almost everywhere. Let $a < 1$ be arbitrary. Then $| s_n h' | \leqq || g_1 ||_\infty | h' | \in L_1(0, s(a))$ for all n and $s_n h' \to g_1 h'$ a.e. on $[0, s(a)]$. By Lebesgue's Dominated Convergence Theorem we conclude that

$$\lim_{n \to \infty} \int_0^{s(a)} s_n h' = \int_0^{s(a)} g_1 h' .$$

Thus $S_s J M_{h'} s_n$ converges to $S_s J M_{h'} g_1$ for all $a < 1$. Clearly $S_s M_h s_n \to S_s M_h g_1$ a.e. and therefore $V_h s_n \to V_h g_1$ a.e. on $[0, 1]$. On the other hand, $V_h s_n = V s_n \to V g_1$ a.e. and so it follows that $V_h g_1 = V g_1$. Thus V_h and V agree on L_∞. We are now able to demonstrate that $h' \in L_q(0, a)$ for all $a < 1$. Define the function $\hat{h}$ by $\hat{h}(x) = \bar{h}'(x)/| h'(x) |$ if $h'(x) \neq 0$ and by $\hat{h}(x) = 1$ otherwise. Then $|\hat{h}| = 1$ and $h' \hat{h} = | h' |$ almost everywhere. Let $g_1 \in L_p$ be arbitrary and let $\{s_n\} \subseteqq D$ such that $|| s_n - g_1 ||_p \to 0$ and $s_n \to g_1$ almost everywhere. By choosing a suitable subsequence of $\{s_n\}$ we may also suppose that $V(\hat{h} | s_n |)$ converges a.e. to $V(\hat{h} | g_1 |)$. For all $n \in Z^+$ we have $\hat{h} | s_n | \in L_\infty$ and therefore, since $V = V_h$ on L_∞, the sequence $V_h(\hat{h} | s_n |)$ converges a.e. to $V(\hat{h} | g_1 |) \in L_p$. Clearly $S_s M_h(\hat{h} | s_n |) \to S_s M_h(\hat{h} | g_1 |)$ a.e. and so

$$S_s J M_{h'}(\hat{h} | s_n |) = (S_s M_h - V_h)(\hat{h} | s_n |)$$

converges a.e. to

$$S_s M_h(\hat{h} \mid g_1 \mid) - V(\hat{h} \mid g_1 \mid) \ .$$

This last function is finite a.e. and so for almost every a in $[0, 1]$ we have

$$(1) \qquad \lim_{n \to \infty} \int_0^{s(a)} |h'(t) s_n(t)| \, dt = \lim_{n \to \infty} \int_0^{s(a)} h'(t) \hat{h}(t) |s_n(t)| \, dt$$

$$= \lim_{n \to \infty} [S_s J M_{h'}(\hat{h} \mid s_n \mid)](a) < \infty \ .$$

Since $|h' s_n| \geqq 0$ it follows from (1) that

$$(2) \qquad \qquad \lim_{n \to \infty} \int_0^a |h'(t) s_n(t)| \, dt < \infty$$

for all $a < 1$. Now let $a < 1$ be arbitrary. Then (2) and Fatou's Lemma together imply that

$$\left| \int_0^a h'(t) g_1(t) dt \right| \leqq \int_0^a |h'(t)| \cdot |g_1(t)| \, dt$$

$$\leqq \varliminf \int_0^a |h'(t)| \cdot |s_n(t)| \, dt < \infty \ .$$

Since $g_1 \in L_p$ was arbitrary, the converse of Hölder's inequality now implies that $h' \in L_q(0, a)$. Finally, since $a < 1$ was arbitrary, we have $h' \in L_q(0, a)$ for all $a < 1$. It now follows easily that V_h is defined everywhere on L_p and satisfies all of the hypotheses of Lemma 2. Hence V_h is bounded and, since $V_h = V$, boundedly invertible.

(ii) Finally, we consider the converse. If V_h is boundedly invertible then it is straightforward to verify that $V_h^{-1} = (M_{(1/h)} - J M_{(1/h)'}) S_r$ and that $V_h T_f V_h^{-1} = T_g$, demonstrating the similarity of T_f and T_g.

The following corollary lists necessary conditions, involving *only* the functions f and g, for the similarity of T_f and T_g. Thus, since the auxiliary function h appearing in Theorem 6 is dispensed with, the corollary provides us with the means for the direct verification of the similarity of a given T_f and T_g.

COROLLARY. *If T_f and T_g belong to $\mathscr{A}$ and are similar in L_p then, with $r \equiv g^{-1} \circ f$ and $s = r^{-1}$, we have*
  (i) *$T_{(1-r)} S_r$ and $T_{(1-s)} S_s$ belong to $B(L_p)$;*
  (ii) *$(1-r)(r')^{-1/p}$ and $(1-s)(s')^{-1/p}$ belong to L_∞;*
  (iii) *r' and s' belong to $L_\infty(0, a)$ for all $a < 1$.*

Proof. Let V_h denote the operator in the statement of Theorem 6 so that $V_h^{-1} = T_{(1/h)} S_r$ and is bounded. It is straightforward to verify that $T_{h(1-r)} \in B(L_p)$ and hence

$$T_{(1-r)}S_r = T_{h(1-r)}T_{(1/h)}S_r = T_{h(1-r)}V_h^{-1}$$

belongs to $B(L_p)$. If we interchange the roles of T_f and T_g in Lemma 8 and Theorem 6 then we can repeat the above argument to show that $T_{(1-s)}S_s \in B(L_p)$ which proves (i). The operators $V_r \equiv M_{(r')^{1/p}}S_r$ and $V_s \equiv M_{(s')^{1/p}}S_s$ are boundedly invertible isometries on L_p with $V_r^{-1} = S_s M_{(1/r')^{1/p}}$ and $V_s^{-1} = S_r M_{(1/s')^{1/p}}$. Then by part (i) the operators

$$T_{(1-r)}M_{(1/r')^{1/p}} = T_{(1-r)}S_r S_s M_{(1/r')^{1/p}}$$

and

$$T_{(1-s)}M_{(1/s')^{1/p}} = T_{(1-s)}S_s S_r M_{(1/s')^{1/p}}$$

belong to $B(L_p)$. Since $r' \in L_1$ it is clear that the function

$$k_r(x) \equiv (1-x)^{q/p}\int_0^x |(r'(t))^{1/q}|^q dt$$

belongs to L_∞ which by [2] implies that $JM_{(r')^{1/q}} \in B(L_p)$. Thus the operator

$$M_{(1-r)(r')^{-1/p}} = T_{(1-r)}M_{(r')^{-1/p}} - JM_{(1-r)'}M_{(r')^{-1/p}}$$
$$= T_{(1-r)(r')^{-1/p}} + JM_{(r')^{1/q}}$$

belongs to $B(L_p)$. It follows that $(1-r)(r')^{-1/p} \in L_\infty$. A similar argument shows that $(1-s)(s')^{-1/p} \in L_\infty$ and thus (ii) holds. For $a < 1$ the function $1-s$ is bounded away from zero on the interval $[0, r(a)]$ and so (ii) implies that $(s')^{-1/p} \in L_\infty(0, r(a))$. But, since s' is nonvanishing a.e., $s'(r(x)) = 1/r'(x)$ and consequently $r' \in L_\infty(0, a)$. A similar argument shows that $s' \in L_\infty(0, a)$ for all $a < 1$ and completes the proof of (iii).

We remark that the conditions $r' \in L_\infty$ and $s' \in L_\infty$ are sufficient for the similarity of T_f and T_g. Indeed, one may then take $h \equiv 1$ in Theorem 6. These conditions are apparently not too far removed from also being necessary for similarity, as part (iii) of the corollary illustrates.

As an application of the preceding corollary we consider the semi-group $\mathscr{T}$ of bounded operators T^a defined, for all $a > 0$, by $T^a \equiv T_{x^a}$. The semi-group property is an immediate consequence of Lemma 1 (ii) which says that $T^a T^b \equiv T_{x^a}T_{x^b} = T_{x^{a+b}} \equiv T^{a+b}$. Lemma 5 and Theorem 1 imply that the spectrum and closed invariant subspaces of every member of $\mathscr{T}$ are identical. However, the following proposition shows that no two distinct members of $\mathscr{T}$ are similar. This provides an interesting contrast to the semi-group $\mathscr{M}$ of multiplication operators of the form $M_{x^a}(a > 0)$. Not only do all the members of $\mathscr{M}$ have

the same spectrum and closed invariant subspaces; any two members of $\mathscr{M}$ are isometrically equivalent in L_p. In fact, the equivalence of M_{x^a} and M_{x^b} may be implemented by

$$(S_s M_{(1/r')^{1/p}}) M_{x^a} (M_{(r')^{1/p}} S_r) = M_{x^b}$$

where $r = x^{a/b}$ and $s = r^{-1}$.

PROPOSITION. *The operators T^a and T^b in $\mathscr{T}$ are similar if and only if $a = b$.*

Proof. Let $a \leqq b$ and suppose T^a and T^b are similar. Let $f = x^a$ and $g = x^b$. Then $r \equiv g^{-1} \circ f = x^{a/b}$ and $r' = (a/b)x^{-1+a/b}$. Part (iii) of the corollary implies that $-1 + a/b \geqq 0$, or equivalently, $a \geqq b$. Thus $a = b$.

Our final theorem is the analog, for certain operators in $\mathscr{A}$, of the well known theorem about the similarity of matrices with 1-dimensional root spaces; viz., two such matrices are similar if and only if their eigenvalues are the same.

THEOREM 7. *If T_f and T_g belong to $\mathscr{A}$ with f' and g' continuous and nonvanishing then T_f and T_g are similar in L_p if and only if their point spectra coincide.*

Proof. If the point spectra of T_f and T_g are equal then by Lemma 5 we have $f([0, 1)) = g([0, 1))$ and since f and g are 1-1 continuous functions it follows that $f(0) = g(0)$ and $f(1) = g(1)$. Thus $r = g^{-1} \circ f$ and $s = r^{-1}$ are well-defined and our present assumptions on f' and g' imply that r and s satisfy the hypothesis of Lemma 7. Hence r and s are absolutely continuous and it is clear that r' and s' belong to L_∞. By Lemma 6 (i) the operators S_r and S_s are bounded and clearly $S_r^{-1} = S_s$. An easy computation now shows that $S_s T_f S_r = T_{f \circ s} = T_g$ which completes our proof.

REFERENCES

1. A. Beurling, *On two problems concerning linear transformations on Hilbert space*, Acta. Math., **89** (1949), 239-255.
2. D. W. Boyd and J. A. Erdos, *Norm inequalities for a class of Volterra operators*, preprint.
3. W. F. Donoghue, *The lattice of invariant subspaces of a completely continuous quasi-nilpotent transformation*, Pacific J. Math., **7** (1957), 1031-1036.
4. E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York, 1965.
5. G. K. Kalisch, *On operators on separable Banach spaces with arbitrary prescribed point spectrum*, to appear in Proc. Amer. Math. Soc.

6. G. K. Kalisch, *On isometric equivalence of certain Volterra operators*, Proc. Amer. Math. Soc., **11**, (1961), 247–252.

7. ———, *On similarity, reducing manifolds, and unitary equivalence of certain Volterra operators*, Ann. Math., **66** (1957), 481–494.

8. A. I. Plesner, *Spectral Theory of Linear Operators*, Vol. II, Ungar, New York. 1969.

9. J. L. Walsh, *Interpolation and Approximation*, AMS Colloquium Series, Vol. XX, third ed., 1690.

UNIVERSITY OF CALIFORNIA, IRVINE
AND
UNIVERSITY OF OREGON