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DECOMPOSITIONS OF E^3 INTO POINTS AND COUNTABLY MANY FLEXIBLE DENDRITES

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Let G be an upper semicontinuous decomposition of E^3 whose only nondegenerate elements are countably many dendrites. It has been asked by Armentrout whether it is sufficient that each dendrite be tame in E^3 in order that the decomposition space $E^3|G$ be homeomorphic to E^3. In Theorem 3 the sufficiency of the tameness condition is shown as well as the sufficiency of the weaker condition that each dendrite be flexible in E^3. Theorem 2 states that if A and B are flexible dendrites in E^3 whose intersection is a point, then $A \cup B$ is a flexible dendrite. This result is used to construct flexible dendrites in E^3 which are not tame.

An upper semicontinuous decomposition G of a topological space X is a collection of disjoint subsets of X such that X is the union of elements of G and such that for every $g \in G$ and for every open set U in X containing g, there is an open set V in X such that $g \subset V \subset U$ and V is the union of elements of G. The decomposition space of X associated with G, denoted X/G, is the set G with the topology defined by the condition that a subset W of G is open in X/G if and only if the union of the elements of W is open in X. A dendrite is a locally connected continuum which contains no simple closed curve. A tree is a finite 1-dimensional simplicial complex whose geometric realization is a dendrite. If M is an n-manifold with or without boundary, Int M denotes the set consisting of all points of M which have a neighborhood homeomorphic to E^n, and Bd M denotes $M - \text{Int}\, M$. If U is a subset of the space X, then Cl U denotes the closure of U in X.

DEFINITION. A dendrite K in E^3 is tame if there is a homeomorphism h of E^3 onto itself such that $h(K)$ is a subset of the xy-plane.

DEFINITION. A dendrite K in E^3 is flexible if given two subcontinua K_1 and K_2 such that $K = K_1 \cup K_2$ and given two open sets U_1 and U_2 in E^3 such that $K_i \subset U_i (i = 1, 2)$, then there is a homeomorphism f of E^3 onto itself such that $f(K) \subset U_2$ and f is the identity on $E^3 - U_1$.

REMARK. Observe that if K is a dendrite in E^3 and if h is a homeomorphism of E^3 onto itself, then K is flexible if and only if $h(K)$ is flexible.

LEMMA 1. *Let D be a disk contained in the xy-plane P of E^3. If U is an open set in E^3 containing D and if g is a homeomorphism of D onto itself which is the identity on Bd D, then there is a homeomorphism f of E^3 onto itself such that f equals g on D and f is the identity on $(E^3 - U) \cup (P - D)$.*

Proof. Let h be a homeomorphism of E^3 onto itself such that $h(D) = \{(x, y, z) \in E^3 : x^2 + y^2 \leqq 1 \text{ and } z = 0\}$. Since $h(U)$ contains $h(D)$, there is a positive number ε such that the suspension S of $h(D)$ with respect to the points $(0, 0, \varepsilon)$ and $(0, 0, -\varepsilon)$ is contained in $h(U)$. Let k be the homeomorphism of E^3 onto itself which equals the suspension of $h \cdot g \cdot h^{-1} \mid h(D)$ on S and which equals the identity elsewhere. Then f equal to $h^{-1} \circ k \circ h$ is the required homeomorphism.

THEOREM 1. *If K is a tame dendrite in E^3, then K is flexible.*

Proof. Since flexibility is invariant under homeomorphisms of E^3 onto itself, we may assume that K is a subset of the xy-plane P in E^3. Let K_1 and K_2 be subcontinua of K such that $K = K_1 \cup K_2$ and let U_1 and U_2 be open sets in E^3 such that $K_i \subset U_i$ $(i = 1, 2)$. Let

$$\varepsilon = \min \{\text{dist}(K_1, E^3 - U_1), \quad \text{dist}(K_2, E^3 - U_2)\}$$

and let T be a triangulation of P of mesh less than ε such that the 0-skeleton on T misses K. Since K does not separate P, there is a polyhedral disk D in P such that $K \subset \text{Int}\, D$, D misses the 0-skeleton of T, and Bd D is in general position with the 1-skeleton of T in P. Hence if s is a closed 2-simplex of T, then the components of $s \cap D$ consist of finitely many disjoint polyhedral disks. Let $\{D_i\}_{i=1}^n$ be the set of disks in P such that for each $i (1 \leqq i \leqq n)$ there is a closed 2-simplex s in T such that D_i is a component of $s \cap D$ and $D_i \cap K \neq \emptyset$. Hence $\{D_i\}_{i=1}^n$ is a set of polyhedral disks in P such that:

(1) $\text{diam}\, D_i < \varepsilon$ $(1 \leqq i \leqq n)$,

(2) if $D_i \cap D_j \neq \emptyset$, then $D_i \cap D_j$ is an arc for $i \neq j$, and

(3) the nerve of $\{D_i\}_{i=1}^n$ is a tree.

By conditions (2) and (3) we have that the union of all elements of $\{D_i\}_{i=1}^n$ which meet K_2 is a disk E and that the union of all elements of $\{D_i\}_{i=1}^n$ which are not subdisks of E consists of disjoint disks $F_1, \cdots, F_m$ such that for each i $(1 \leqq i \leqq m)$ $F_i \cap E = \text{Bd}\, F_i \cap \text{Bd}\, E$ is an arc J_i. It follows that $K \cap \text{Bd}\, F_i \subset J_i$. By condition (1) and our choice of ε we have $E \subset U_2$ and $F_i \subset U_1$ $(1 \leqq i \leqq m)$. Since

$(K \cap \mathrm{Bd}\, F_i) \subset J_i \subset E \subset U_2$, there is a homeomorphism g_i of F_i onto itself which is the identity on $\mathrm{Bd}\, F_i$ such that $g_i(K \cap F_i) \subset U_2$. The homeomorphism g_i is obtained as follows. Choose arcs A_i and B_i in F_i such that:

(a) $A_i \cap \mathrm{Bd}\, F_i = B_i \cap \mathrm{Bd}\, F_i = \mathrm{Bd}\, J_i = \mathrm{Bd}\, A_i = \mathrm{Bd}\, B_i$,

(b) the disk on F_i bounded by $A_i \cup J_i$ contains $K \cap F_i$, and

(c) the disk on F_i bounded by $B_i \cup J_i$ is contained in U_2.

Now let h_i be an embedding of $A_i \cup \mathrm{Bd}\, F_i$ into F_i which is the inclusion on $\mathrm{Bd}\, F_i$ and which takes A_i onto B_i. The homeomorphism g_i is an extension of h_i to all of F_i.

Now let $V_1, \cdots, V_m$ be disjoint open sets in U_1 such that $F_i \subset V_i$ $(1 \leqq i \leqq m)$. By Lemma 1 there is a homeomorphism f_i of E^3 onto itself such that f_i equals g_i on F_i and f_i is the identity on $(E^3 - V_i) \cup (P - F_i)$. If f equals $f_m \circ f_{m-1} \circ \cdots \circ f_1$, then $f(K) \subset U_2$ and f is the identity on $E^3 - U_1$. Hence K is flexible.

LEMMA 2. *Let K be a flexible dendrite in E^3. If $N, C_1, C_2, \cdots, C_n$ are subcontinua of K and $U, V_1, V_2, \cdots, V_n$ are open sets in E^3 such that:*

(1) $K = N \cup (\bigcup_{i=1}^n C_i)$,

(2) $N \subset U$ *and* $C_i \subset V_i$ $(1 \leqq i \leqq n)$, *and*

(3) $V_i \cap V_j = \emptyset$ *for* $i \neq j$,

then there is a homeomorphism f of E^3 onto itself such that $f(K) \subset U$ and f is the identity on $E^3 - (\bigcup_{i=1}^n V_i)$.

The proof of Lemma 2 is omitted as it is obtained directly with an induction argument.

THEOREM 2. *If A and B are flexible dendrites in E^3 such that $A \cap B = \{p\}$, then $A \cup B$ is a flexible dendrite.*

Proof. It is clear that $A \cup B$ is a dendrite. To show that $A \cup B$ is flexible let K_1 and K_2 be subcontinua of $A \cup B$ such that $A \cup B = K_1 \cup K_2$ and let U_1 and U_2 be open sets in E^3 such that $K_i \subset U_i$ $(i = 1, 2)$. We consider separately the cases when $p \notin K_2$ and when $p \in K_2$.

Case 1. If $p \notin K_2$, then $K_2 \subset A$ or $K_2 \subset B$. Let us say that $K_2 \subset A$. Hence $B \subset K_1$. Using the flexibility of A for the subcontinua $K_1 \cap A$ and $K_2 \cap A$ and for the open sets U_1 and U_2, let g be a homeomorphism of E^3 onto itself such that $g(A) \subset U_2$ and g is the identity on $E^3 - U_1$. Here we used the fact that $K_i \cup A$ $(i = 1, 2)$ is a dendrite and thus unicoherent to say that $K_i \cap A$ is a subcontinuum of A. Let N be a subcontinuum of B such that N is a neighborhood of p in B and $N \subset g^{-1}(U_2)$. Let $C_1, \cdots, C_n$ be the components of $\mathrm{Cl}\,(B - N)$, and let $V_1, \cdots, V_n$ be disjoint open sets in $U_1 - A$ such that $C_i \subset V_i$ $(1 \leqq i \leqq n)$. By Lemma 2, for the flexible dendrite B, for the subcontinua $N, C_1, C_2, \cdots, C_n$, and for the open sets $g^{-1}(U_2), V_1, V_2, \cdots, V_n$, there is a homeomorphism h of E^3 onto itself such that $h(B) \subset g^{-1}(U_2)$ and h is the identity on $E^3 - (\bigcup_{i=1}^n V_i)$. If f equals $g \circ h$, then $f(A \cup B) \subset U_2$ and f is the identity on $E^3 - U_1$.

Case 2. If $p \in K_2$, then let N be a subcontinuum of $A \cup B$ such that N is a neighborhood of K_2 in $A \cup B$ and $N \subset U_2$. Let $C_1, \cdots, C_n$ be the components of $\mathrm{Cl}\,((A \cup B) - N)$. We assume that the set $\{C_i\}_{i=1}^n$ is so numbered that for each i $(1 \leqq i \leqq m)$ $C_i \subset A - B$ and for each i $(m + 1 \leqq i \leqq n)$ $C_i \subset B - A$. Let $V_1, \cdots, V_m$ be disjoint open sets in $U_1 - B$ such that $C_i \subset V_i$ $(1 \leqq i \leqq m)$. By Lemma 2 for the flexible dendrite A, for the subcontinua $N \cap A, C_1, C_2, \cdots, C_m$, and for the open sets $U_2, V_1, V_2, \cdots, V_m$, there is a homeomorphism g of E^3 onto itself such that $g(A) \subset U_2$ and g is the identity on $E^3 - (\bigcup_{i=1}^m V_i)$. Let $V_{m+1}, \cdots, V_n$ be disjoint open sets in $U_1 - g(A)$ such that $C_i \subset V_i$ $(m + 1 \leqq i \leqq n)$. By Lemma 2 for the flexible B, for the subcontinua $N \cap B, C_{m+1}, C_{m+2}, \cdots, C_n$, and for the open sets $U_2, V_{m+1}, V_{m+2}, \cdots, V_n$, there is a homeomorphism h of E^3 onto itself such that $h(B) \subset U_2$ and h is the identity on $E^3 - (\bigcup_{i=m+1}^n V_i)$. If f equals $h \circ g$, then $f(A \cup B) \subset U_2$ and f is the identity on $E^3 - U_1$.

As a result of Cases 1 and 2, we conclude that $A \cup B$ is flexible.

REMARK. The union of two tame arcs in E^3 whose intersection is a point need not be a tame dendrite [1, Example 1.4]. Hence there are flexible dendrites in E^3 which are not tame.

LEMMA 3. *If N is a tree, then the vertexes of N can be numbered $v_1, \cdots, v_n$ such that for each i $(1 \leqq i \leqq n - 1)$, there is a unique integer $s(i)$ satisfying $i < s(i) \leqq n$ and there is a 1-simplex between v_i and $v_{s(i)}$.*

Proof. The proof is by induction on the number of vertexes of N. Any numbering works if N has two vertexes. Assume the lemma is true if N has $n - 1$ $(n \geqq 3)$ vertexes, and consider the case when N has n vertexes. Let w be a vertex of N which is the face of exactly one 1-simplex s in N. We form a new tree N' by removing w and the interior of s from N. By the induction hypothesis we can number the vertexes $u_1, \cdots, u_{n-1}$ of N' such that for each i $(1 \leqq i \leqq n - 2)$, there is a unique integer $s(i)$ satisfying $i < s(i) \leqq n - 1$ and there is a 1-simplex between u_i and $u_{s(i)}$. Now in N let $v_1 = w$ and let $v_i = u_{i-1}$ $(2 \leqq i \leqq n)$. This numbering satisfies the condition.

LEMMA 4. *Let A be a dendrite and let ε be a positive real number. Then A is the finite union of continua $A_1, \cdots, A_n$ of diameter less then ε such that for each i $(1 \leqq i \leqq n-1)$, there is a unique integer $s(i)$ satisfying $i < s(i) \leqq n$ and $A_i \cap A_{s(i)} \neq \emptyset$.*

Proof. The dendrite A can be written as the finite union of continua $A_1, \cdots, A_n$ of diameter less than ε such that each pair intersects in at most a point and each triplet has empty intersection [3, p. 302]. It follows that the nerve N of $\{A_i\}_{i=1}^n$ is a tree. Using Lemma 3 we see that the set $\{A_i\}_{i=1}^n$ can be renumbered such that for each i $(1 \leqq i \leqq n-1)$, there is a unique integer $s(i)$ satisfying $i < s(i) \leqq n$ and $A_i \cap A_{s(i)} \neq \emptyset$.

THEOREM 3. *If G is an upper semicontinuous decomposition of E^3 whose only nondegenerate elements are countably many flexible dendrites, then E^3/G is homeomorphic to E^3.*

Proof. Using the technique of Bing as in [2, Theorem 3], it suffices to show that if G is an upper semicontinuous decomposition of E^3, ε is a positive real number, A is an element of G which is a flexible dendrite, and U is an open set containing A, then there is a homeomorphism f of E^3 onto itself such that f is the identity on $E^3 - U$, diam $f(A) < \varepsilon$, and for each element g of G, either diam $f(g) < \varepsilon$ or $f(g) \subset N(g, \varepsilon)$ where $N(g, \varepsilon) = \{x \in E^3 \colon \text{dist}(x, g) < \varepsilon\}$.

By Lemma 4 the dendrite A is the finite union of continua $A(1)_1, \cdots, A(1)_n$ of diameter less than ε such that for each i $(1 \leqq i \leqq n-1)$, there is a unique integer $s(i)$ satisfying $i < s(i) \leqq n$ and $A(1)_i \cap A(1)_{s(i)} \neq \emptyset$. We may assume that $n > 1$, otherwise f equals to the identity on E^3 would be the required homeomorphism. For each i $(1 \leqq i \leqq n)$ let $U(1)_i$ be an open set in E^3 such that $A(1)_i \subset U(1)_i \subset U$, diam $U(1)_i < \varepsilon$, and $\text{Cl}\, U(1)_i \cap \text{Cl}\, U(1)_j = \emptyset$ if and only if $A(1)_i \cap A(1)_j = \emptyset$. Since A is flexible, for the subcontinua $A(1)_1$ and $\bigcup_{i=2}^n A(1)_i$ and for the open sets $U(1)_1$ and $\bigcup_{i=2}^n U(1)_i$, there is a homeomorphism f_1 of E^3 onto itself such that $f_1(A) \subset \bigcup_{i=2}^n U(1)_i$ and f_i is the identity on $E^3 - U(1)_1$. Once given $\{A(j)_i\}_{i=j}^n$, $\{U(j)_i\}_{i=j}^n$, and f_j for fixed j $(1 \leqq j \leqq n-2)$, define for each i $(j+1 \leqq i \leqq n)$

$$A(j+1)_i = \begin{cases} f_j(A(j)_i) = A(j)_i & \text{if } i \neq s(j) \\ f_j(A(j)_j \cup A(j)_{s(j)}) & \text{if } i = s(j) \,. \end{cases}$$

Also for each i $(j+1 \leqq i \leqq n)$, let $U(j+1)_i$ be an open set in E^3 such that:

(1) $A(j+1)_i \subset U(j+1)_i \subset U(j)_i$, and

(2) $\bigcup_{g \in G}\{g_j \colon g_j \text{ meets } U(j+1)_i\} \subset \bigcup_{k=j+1}^n U(j)_k$, where g_j denotes $f_j \circ \cdots \circ f_1(g)$.

Condition (2) can be satisfied since $f_j \circ \cdots \circ f_1(A)$ which equals $\bigcup_{k=j+1}^{n} A(j+1)_k$ is an element of the upper semicontinuous decomposition $G_j = \{f_j \circ \cdots \circ f_1(g) : g \in G\}$ and a subset of the open set $\bigcup_{k=j+1}^{n} U(j)_k$.

Using the flexibility of $f_j \circ \cdots \circ f_1(A)$ for the subcontinua $A(j+1)_{j+1}$ and $\bigcup_{i=j+2}^{n} A(j+1)_i$ and the open sets $U(j+1)_{j+1}$ and $\bigcup_{i=j+2}^{n} U(j+1)_i$ obtain a homeomorphism f_{j+1} of E^3 onto itself such that

$$f_{j+1}(f_j \circ \cdots \circ f_1(A)) \subset \bigcup_{i=j+2}^{n} U(j+1)_i$$

and f_{j+1} is the identity on $E^3 - U(j+1)_{j+1}$. Let f equal $f_{n-1} \circ \cdots \circ f_1$. We wish to show that f is the required homeomorphism.

It is clear that f is the identity on $E^3 - U$ and diam $f(A) < \varepsilon$. Hence we show that if $g \in G$, then diam $f(g) < \varepsilon$ or $f(g) \subset N(g, \varepsilon)$. Since f_1 is the identity on $E^3 - U(1)_1$, f_1 moves no point of E^3 more than diam $U(1)_1 < \varepsilon$. Hence $f_1(g) \subset N(g, \varepsilon)$. Suppose now we have proven for fixed k $(2 \leqq k \leqq n-1)$ that diam $g_{k-1} < \varepsilon$ or $g_{k-1} \subset N(g, \varepsilon)$ where g_{k-1} denotes $f_{k-1} \circ \cdots \circ f_1(g)$. We show that diam $g_k < \varepsilon$ or $g_k \subset N(g, \varepsilon)$. If g_{k-1} does not meet $U(k)_k$, then g_k equals g_{k-1}. Thus diam $g_k < \varepsilon$ or $g_k \subset N(g, \varepsilon)$. If g_{k-1} meets $U(k)_k$, then $g_{k-1} \subset \bigcup_{j=k} U(k-1)_j$ by condition (2). We consider two cases.

Case 1. If $g_{k-1} \subset U(k-1)_k$, then since f_k is the identity on $E^3 - U(k)_k$ and $U(k)_k \subset U(k-1)_k$, we have $g_k \subset U(k-1)_k$. Thus diam $g_k < \varepsilon$.

Case 2. If g_{k-1} meets $(\bigcup_{j=k}^{n} U(k-1)_j) - U(k-1)_k$, then g_{k-1} meets the boundary B of $U(k-1)_k$ as a subset of $\bigcup_{j=k}^{n} U(k-1)_j$. Let $y \in B \cap g_{k-1}$. We wish to show that $y \in g$. For each i $(1 \leqq i \leqq k-1)$, since there is only one integer $s(i)$ such that $i < s(i) \leqq n$ and $A(i)_i \cap A(i)_{s(i)} \neq \varnothing$, either $\mathrm{Cl}\, U(i)_i \cap \mathrm{Cl}\, U(k-1)_k = \varnothing$ or

$$\mathrm{Cl}\, U(i)_i \cap \left(\bigcup_{j=k+1}^{n} \mathrm{Cl}\ U(k-1)_j \right) = \varnothing \ .$$

Hence for each i $(1 \leqq i \leqq k-1)$, $\mathrm{Cl}\ U(i)_i \cap B = \varnothing$, and thus f_i is the identity on B. Hence $y \in g$. We now show that $g_k \subset N(g, \varepsilon)$ by proving if $x \in g_{k-1}$, then dist $(f_k(x), g) < \varepsilon$. If $x \in U(k-1)_k$, then $f_k(x) \in U(k-1)_k$. Hence

$$\mathrm{dist}\,(f_k(x), g) \leqq \mathrm{dist}\,(f_k(x), y) \leqq \mathrm{diam}\,(\mathrm{Cl}\ U(k-1)_k) < \varepsilon \ .$$

If $x \notin U(k-1)_k$, then $f_k(x) = x$, and we must consider the cases when diam $g_{k-1} < \varepsilon$ and when $g_{k-1} \subset N(g, \varepsilon)$ separately. If diam $g_{k-1} < \varepsilon$, then

$$\mathrm{dist}\,(f_k(x), g) \leqq \mathrm{dist}\,(x, y) \leqq \mathrm{diam}\, g_{k-1} < \varepsilon \ .$$

If $g_{k-1} \subset N(g, \varepsilon)$, then

$$\text{dist}\,(f_k(x), g) = \text{dist}\,(x, g) < \varepsilon\ .$$

Hence we have shown that $g_k \subset N(g, \varepsilon)$.

As a result of Cases 1 and 2, we can conclude by induction that if $g \in G$, then diam $f(g) < \varepsilon$ or $f(g) \subset N(g, \varepsilon)$. Thus f is the required homeomorphism.

DEFINITION. A continuum K in E^3 is cellular if there is a sequence of 3-cells $\{C_i\}_{i=1}^{\infty}$ in E^3 such that $K = \bigcap_{i=1}^{\infty} C_i$ and $C_{i+1} \subset \text{Int}\, C_i$ for $i = 1, 2, \cdots$.

COROLLARY. *If K is a flexible dendrite in E^3, then K is cellular.*

Proof. Let G be an upper semicontinuous decomposition of E^3 into continua with only countably many nondegenerate elements. By Theorem 2 of [4] if E^3/G is homeomorphic to E^3, then each element of G is cellular.

REMARK. For an example of a cellular dendrite which is not flexible consider the cellular arc A of Example 1.2 in [**1**]. This arc has only one wild point, an endpoint. To see that this arc is not flexible, consider another arc B in $E^3 - A$ such that A and B are equivalently embedded in E^3 under a space homeomorphism of E^3. Let J be a tame arc in E^3 which joins the locally tame endpoint of A to the locally tame endpoint of B to form an arc $K = A \cup J \cup B$. If A is flexible, then by Theorem 2 the arc K is flexible. Hence K is cellular. However, a cellular arc in E^3 cannot have isolated wild points for its endpoints [**5**, Theorem 10]. Thus A is not flexible.

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