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DECOMPOSITION OF PLANE CONVEX SETS, PART II: SETS ASSOCIATED WITH A WIDTH FUNCTION

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This paper treats several classes K of plane convex bodies such that the sum of any two members of the class is again a member of the class. In each case K is a class of bodies associated with a certain width function. An explicit characterization is provided for the corresponding subclass $I(K)$ consisting of all indecomposable members of K.

This result was proved by the author [4].

The results and methods of that paper are used to characterize $I(\boldsymbol{K})$ when $\boldsymbol{K}$ consists of all plane convex bodies of constant width, also when $\boldsymbol{K}$ consists of all plane convex bodies whose associated width functions are a multiple of a given width function, known as bodies of constant relative width. It is supposed without further mention that all members of $\boldsymbol{K}$ have piecewise continuously second differentiable support functions.

The class $\boldsymbol{K}$ of all plane convex bodies has the property that the sum of any two members of the class is again a member of the class. The indecomposable subclass $I(\boldsymbol{K})$ is the class of all triangles and line segments.

The reader is referred to [4] for the definitions used in this paper, which are not repeated, for the sake of brevity. For clarity it is, however, pointed out that as in [4], a *body* is a compact set.

This paper shows that, relative to the class $\boldsymbol{K}$ of all plane convex bodies of constant width, the $\boldsymbol{K}$-indecomposable bodies are those whose boundaries are composed solely of circular arcs and have a corner point opposite every arc, or equivalently, are composed solely of circular arcs with radius equal to the width of the body. A more general problem is also considered; what are the $\boldsymbol{K}$-indecomposable members of the family $\boldsymbol{K}$ of sets with width function a multiple of a given continuously second differentiable width function? The $\boldsymbol{K}$-indecomposable members of $\boldsymbol{K}$ are these bodies which have a corner point in every direction, that is, at least one of the two support lines in every direction goes through a corner point.

Let K be a convex body; by the *width* of K in the direction θ, denoted $w(\theta)$, we mean the distance between the two parallel supporting lines of K in the direction perpendicular to θ.

Let w be a positive function, and $\boldsymbol{K}(w)$ be the associated family of sets whose width functions are a multiple of w. The machinery we will set up requires that the sets to be decomposed, or proved indecomposable, have the origin in their interior.

The following remark is elementary but useful. Its proof is left to the reader.

LEMMA 1. *Let w be a positive function, and define the family of sets $\boldsymbol{K}(w)$ as above. A set B in $\boldsymbol{K}(w)$ is $\boldsymbol{K}$-decomposable if and only if it has a $\boldsymbol{K}$-decomposable translate whose interior contains the origin.*

Let $w(\theta)$ be a width function with continuous second derivative, and $\boldsymbol{K}(w)$ the associated family of sets whose width functions are a multiple of $w(\theta)$. We will determine the $\boldsymbol{K}$-indecomposable members of $\boldsymbol{K}(w)$.

LEMMA 2. *An admissible pair $\{\varphi_1, \varphi_2\}$ is the restriction of the support function of a set with width function $w(\theta)$, if*

$$\varphi_1(-t) + \varphi_2(t) = \sqrt{1 + t^2}\, w(\operatorname{arc\,cot} t) ,$$

for all real t.

Proof. Let K be a convex body in R^2, whose interior contains the origin, with width function $w(\theta)$. Then

$$\varphi_2(t) = \max_{(x,y) \in K} (tx + y) ,$$

and

$$\varphi_1(t) = \max_{(x,y) \in K} (tx - y) ,$$

so

$$\varphi_1(-t) = \max_{(x,y) \in K} (-tx - y) = -\min_{(x,y) \in K} (tx + y) .$$

Clearly the extrema of $(tx + y)$ will occur on the boundary of K. For a specified value of t, the extrema will occur when $(dy/dx) = -t$, or where the one sided derivatives have the property

$$\left(\frac{dy}{dx}\right)_- \leqq -t \leqq \left(\frac{dy}{dx}\right)_+ .$$

When we require that $(dy/dx) = -t$, we are actually looking for support lines for K in the direction θ such that $\cot \theta = -(dy/dx) = t$. There exist exactly two points on K with support lines in this direction. Let us denote by $P_2 = (x_2, y_2)$ the one minimizing $(tx + y)$, and by $P_1 = (x_1, y_1)$ the one maximizing $(tx + y)$. Then

$$\varphi_2(t) = x_1 t + y_1, \qquad \varphi_1(-t) = -x_2 t - y_2 ,$$

where

$$t = \cot\theta, \qquad \cot\varphi = \frac{x_2 - x_1}{y_2 - y_1},$$

and

$$w(\theta) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\cos(\theta - \varphi).$$

Substituting all the above, adding, and simplifying,

$$\varphi_2(t) + \varphi_1(-t) = w(\operatorname{arc\,cot} t)\cdot\sqrt{1 + t^2}.$$

On the other hand, if $\{\varphi_1, \varphi_2\}$ is an admissible pair satisfying

$$\varphi_2(t) + \varphi_1(-t) = h(t)$$

for all real t, let K be a set containing the origin in its interior, whose support function corresponds to $\{\varphi_1, \varphi_2\}$.

For a fixed direction θ, suppose $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are the points where the parallel support lines in direction perpendicular to θ intersect K. Then

$$w(\theta) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\cos(\theta - \varphi), \text{ where } \cot\varphi = \frac{x_2 - x_1}{y_2 - y_1}.$$

But P_1 and P_2 are extrema for $(tx + y)$ on the set K, when $t = \cot\theta$; suppose the maximum occurs at P_1 and the minimum at P_2. Then $\varphi_2(t) = x_1 t + y_1$, and $\varphi_1(-t) = -x_2 t - y_2$. Then

$$h(\cot\theta =)\frac{w(\theta)}{|\sin\theta|}.$$

Therefore,

$$w(\theta) = h(\cot\theta)\cdot|\sin\theta|.$$

LEMMA 3. *An admissible pair $\{\varphi_1, \varphi_2\}$ satisfying $\varphi_1(-t) + \varphi_2(t) = h(t) = \sqrt{1 + t^2}\, w(\operatorname{arc\,cot} t)$, for all real t, where $w(\theta)$ has continuous second derivative, can be nontrivially decomposed into the sum of two admissible pairs $\{\sigma_1, \sigma_2\}$, $\{\psi_1, \psi_2\}$, satisfying $\sigma_1(-t) + \sigma_2(t) = 1/2\, h(t)$, $\psi_1(-t) + \psi_2(t) = 1/2\, h(t)$, if there is an interval $I = (a, b]$ on the real line such that neither $\varphi_2(t)$ nor $\varphi_1(-t)$ is linear on I, or on any proper subinterval of I.*

Proof. Defining $\sigma_i(t) = 1/2[\varphi_i(t) + y_i(t)]$, and $\psi_i(t) = 1/2[\varphi_i(t) - y_i(t)]$, as in Theorem 2, [4], we must have $y_1(-t) + y_2(t) = 0$ for all real t.

Let $h(t) = \sqrt{1 + t^2}\, w(\operatorname{arc\,cot} t)$. Then

$$h''(t) = \frac{w(\operatorname{arc\,cot} t) + w''(\operatorname{arc\,cot} t)}{(1 + t^2)^{3/2}}$$

where $w''(\operatorname{arc\,cot} t)$ denotes

$$\left.\frac{d^2w}{d\theta^2}\right|_{\theta=\operatorname{arc}\cot t} .$$

w and w'' are both defined on $[0, \pi]$ and continuous, so are bounded on $[0, \pi]$. Therefore, $h''(t)$ is bounded by some number M, for all real t. Note that $h''(t)$ is also continuous.

Let $f_1(t) = \varphi_1'(-t)$, $f_2(t) = \varphi_2'(t)$, and let $p_1(t) = \varphi_1''(-t)$, $p_2(t) = \varphi_1''(t)$. Consider a closed subinterval $I_1 = [a_1, b_1]$ of I on which p_i is continuous for $i = 1, 2$. Let $T_i = \{x \in I_1 \mid p_i(t) = 0\}$. T_i is compact for $i = 1, 2$, and $T_i \neq I_1$ as φ_i is not linear on I_1. If $I_1 = T_1 \cup T_2$, the connectedness of I_1 implies that $T_1 \cap T_2 \neq \emptyset$. But this is impossible, since $p_1(t) + p_2(t) = h''(t) \neq 0$. Therefore there exists a point $t_0 \in I_1$ such that $p_i(t_0) \neq 0$ for $i = 1, 2$. Let $p(t) = \min_{i=1,2} p_i(t)$. Then $p(t_0) > 0$, and by piecewise continuity of p_i, there is an interval I_2 such that $t_0 \in I_2 \subset I$ and $t \in I_2 \Rightarrow p(t) > 0$. Therefore, $p = \min_{i=1,2} p_i > 0$ almost everywhere. Therefore, $\int_a^x p(t)dt$ is a strictly increasing function of x, on I_2. As in the theorem cited above, select $x_1, \cdots, x_4$ so that f_0 assumes four different positive values, and define $y_1'(x)$ from $f_0(x)$ rather than $\varphi_1'(x)$, as in that theorem. Letting $y_2'(-t) = -y_1'(t)$, we then define $\{\sigma_1, \sigma_2\}$ and $\{\psi_1, \psi_2\}$. It is clear that $f_0(t) \leqq f_i(t)$, $i = 1, 2$, and

$$0 \leqq f_0(t') - f_0(t) \leqq f_i(t') - f_i(t)$$

for $t < t'$, so y certainly satisfies the necessary conditions for admissibility. Thus, the nontrivial decomposition has been accomplished.

LEMMA 4. *If an admissible pair $\{\varphi_1, \varphi_2\}$ satisfying $\varphi_1(-t) + \varphi_2(t) = h(t)$, for all real t, where $h(t) = \sqrt{1 + t^2}\, w(\operatorname{arc}\cot t)$, and $w(\theta)$ has continuous second derivative is decomposable, there is an interval on which both $\varphi_1(-t)$ and $\varphi_2(t)$ are nonlinear.*

Proof. If $y_i(t)$ is a function such that $\sigma_i(t) = \varphi_i(t) + y_i(t)$ and $\psi_i(t) = \varphi_i(t) - y_i(t)$ are convex and satisfy

$$\sigma_2(t) + \sigma_1(-t) = h(t) ,$$

and

$$\psi_2(t) + \psi_1(-t) = h(t) ,$$

then $y_1(t) + y_2(-t) = 0$. This implies that, if on an interval I, say $\varphi_1(t)$ is linear, then for every $x, x' \in I$ with $x' > x$,

$$| y_1'(x') - y_1'(x) | \leqq \varphi_1'(x') - \varphi_1'(x) = 0 ,$$

and also $|y_2'(-x') - y_2'(-x)| = 0$. Therefore, y_1' is constant on I, and y_2' is constant on $-I$. Suppose on every interval $\varphi_1(-t)$ or $\varphi_2(t)$ is linear. Then y_i' is constant on every interval, and continuous on the line, so it is identically a constant. Therefore, $\sigma_i(t)$ and $\psi_i(t)$ differ from a multiple of $\varphi_i(t)$ at most by a linear function, and are therefore translates of multiples of $\varphi_i(t)$. Therefore, $\{\varphi_1, \varphi_2\}$ is indecomposable.

The following theorem is therefore immediate.

THEOREM 1. *An admissible pair* $\{\varphi_1, \varphi_2\}$ *satisfying* $\varphi_1(-t) + \varphi_2(t) = h(t)$, *for all real* t, *where* $h(t) = \sqrt{1+t^2}\ w(\operatorname{arc\,cot} t)$, $w(\theta)$ *having continuous second derivative, is indecomposable if and only if there is a family* $\{I_\alpha\}$ *of disjoint intervals, such that* $\bigcup_\alpha I_\alpha = R^1$ *and on every* I_α *exactly one of* $\varphi_1(t), \varphi_2(-t)$ *is linear.*

We can now characterize the indecomposable sets with a given width function.

THEOREM 2. *Let* $\boldsymbol{K}(w)$ *be the family of plane convex bodies with width function a multiple of* $w(\theta)$, *where* $w(\theta)$ *has a continuous second derivative. The* $\boldsymbol{K}$*-indecomposable members of* $\boldsymbol{K}$ *are those sets with the property that at least one of the support lines of the set in every direction goes through a corner point.*

Proof. Let K be a body with width function $w(\theta)$, and with corresponding supporting admissible pair $\{\varphi_1, \varphi_2\}$. Let $y = f(x)$ be the function satisfied by the boundary of K in a neighborhood of a corner point (x_0, y_0) of K. Then the one-sided tangents, or extreme rays of the bundle of support lines, will have slopes $f_-'(x_0), f_+'(x_0)$. The support function $\varphi_2(t) = \max_{(x,y)\in K}[xt + y]$ is linear on $[t_1, t_2]$ if and only if for every $t \in [t_1, t_2]$ there is a point (x_0, y_0) such that $f_-'(x_0) + t \leqq 0 \leqq f_+'(x_0) + t$, or $f_-'(x_0) = t_2$, and $f_+'(x_0) = -t_1$. The value of the function $\varphi_2(t)$ on $[t_1, t_2]$ is then $\varphi_2(t) = x_0 t + y_0$. The function $\varphi_1(t) = \max_{(x,y)\in K}[xt - y]$ is, correspondingly, linear on a line segment $[t_1, t_2]$ when there is a point $(x_0, y_0) \in K$ such that $f_-'(x_0) = t_1, f_+'(x_0) = t_2$. Therefore by Theorem 1, K is $\boldsymbol{K}$-indecomposable if and only if at least one of the two support lines in every direction goes through a corner point.

Of special interest is the case where $w(\theta)$ is independent of the direction, i.e., $\boldsymbol{K}(w)$ is the family of bodies of *constant width.*

Applying the results of the preceding discussion we note that the admissible pair $\{\varphi_1, \varphi_2\}$ of support functions corresponding to a body of constant width r with origin in its interior satisfies $\varphi_1(-t) + \varphi_2(t) = r\sqrt{1+t^2}$, for all real t.

Applying the results of Theorem 1 a body of constant width is $\boldsymbol{K}$-indecomposable if and only if at least one of the two support lines in every direction goes through a corner point.

We note that if K, a body of constant width r, has a corner point Q, where the bundle of supporting lines occupies an angle equal to $\pi - \theta$, then opposite Q is an arc $\widehat{pp'}$ with the property that every point on $\widehat{pp'}$ is at distance r from Q, i.e., $\widehat{pp'}$ is a circular arc of radius r and angle θ. Therefore, the boundary of K is entirely composed of circular arcs of radius equal to the width.

THEOREM 3. *Let $\boldsymbol{K}$ be the family of bodies of constant width. The $\boldsymbol{K}$-indecomposable sets are precisely those whose boundary is entirely composed of circular arcs whose radius is equal to the width.*

Since a plane convex body of constant width r, whose boundary is composed entirely of arcs of radius r, must have a corner point opposite every arc, this result is immediate.

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