PACIFIC JOURNAL OF MATHEMATICS
Vol. 48, No. 2, 1973

SHAPE GROUPS AND PRODUCTS

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In recent papers S. Mardešić and J. Segal have used ANR-systems to obtain an alternate approach to Borsuk's shape theory. At the same time, they have extended the theory to include all compact Hausdorff spaces. In this paper some of the results of Mardešić and Segal are used to obtain Borsuk's fundamental groups, and to extend some of Borsuk's results that relate shapes and products to arbitrary products. A result relating (direct) products and shape groups is also obtained.

The reader is referred to Chapter 1 of [11] for all categorical definitions; e.g., inverse system, terminal object, inverse limit.

If $\underline{X} = \{X_a, p_{aa'}, A\}$ and $\underline{Y} = \{Y_b, q_{bb'}, B\}$ are inverse systems in a category $\mathscr{A}$, a morphism of inverse systems $\underline{f}: \underline{X} \to \underline{Y}$ consists of an increasing function $f: B \to A$ and a collection of $\mathscr{A}$-morphisms $f_b: X_{f(b)} \to Y_b$ such that if $b \leqq b'$ then $f_b p_{f(b)f(b')} = q_{bb'} f_{b'}$. If $X_\infty = \varprojlim \underline{X}$ and $Y_\infty = \varprojlim \underline{Y}$ exist and $\underline{f}: \underline{X} \to \underline{Y}$ is a morphism of inverse systems then for each $b \in B$ the composition $f_b p_{f(b)}: X_\infty \to Y_b$ satisfies if $b \leqq b'$ then

$$f_b p_{f(b)} = f_b p_{f(b)f(b')} p_{f(b')} = q_{bb'} f_{b'} p_{f(b')} .$$

By the universal mapping property of Y_∞ there is a unique $\mathscr{A}$-morphism $f_\infty: X_\infty \to Y_\infty$ such that if $b \in B$ then $q_b f_\infty = f_b p_{f(b)}$. The $\mathscr{A}$-morphism f_∞ is said to be *induced* by $\underline{f}$.

Under the usual definitions of composition and identities, there is a category, denoted inv $(\mathscr{A})$, whose objects are inverse systems in $\mathscr{A}$ and whose morphisms are morphisms of inverse systems. The reader will note that this category is not the category used by Mardešić and Segal [6]. It is not difficult to show that if $\underline{f}: \underline{X} \to \underline{Y}$ and $\underline{g}: \underline{Y} \to \underline{Z}$ are morphisms of inverse systems and if X_∞, Y_∞ and $Z_\infty = \varprojlim \underline{Z}$ exist then $(gf)_\infty = g_\infty f_\infty$.

An *ANR-system* is an inverse system $\underline{X} = \{(X_a, x_a), P_{aa'}, A\}$ in the category of pointed topological spaces where A is closure-finite (i.e., for every $a \in A$, the set of predecessors of a is finite) and each X_a is a compact ANR for normal spaces. This definition differs from that usually given in that the authors in [6] required each X_a to be compact ANR for metric spaces. This condition is easily relaxed; as was done in [5]. As in [5], we use notions and results from [6] in this setting without specific citations.

If X is a compact Hausdorff space, $x_0 \in X \subset M$, an *inclusion ANR-system* in M associated with (X, x_0) is an ANR-system $\underline{X} = \{(X_a, x_0), i_{aa'}, A\}$ associated with (X, x_0) where

(1) each X_a is a neighborhood of X in M

(2) $X = \bigcap_{a \in A} X_a$

(3) if $a \leqq a'$ then $i_{aa'}\colon (X_{a'}, x_0) \to (X_a, x_0)$ is an inclusion map.

If $A = N$ (the set of natural numbers) then X is said to be an *inclusion ANR-sequence* and is denoted $\underline{X} = \{(X_k, x_0), i_{kk'}\}$. If X is contained in a parallelotope $I^{\Omega} = \prod_{w \in \Omega} I_{\omega}$, $I_{\omega} = I$, then [5] (X, x_0) has an associated inclusion ANR-system (sequence if Ω is countable).

Another useful category is the category of ANR-systems, developed by Mardešić and Segal in [6]. The objects of this category are ANR-systems $\underline{X} = \{(X_a, x_a), P_{aa'}, A\}$ (recall our definition differs somewhat from that used in [6]). A morphism in this category $\underline{f}\colon \underline{X} \to \underline{Y} = \{(Y_b, y_b), q_{bb'}, B\}$, called a *map of systems*, consists of an increasing function $f\colon B \to A$ and a collection of maps (i.e., continuous functions) $f_b\colon X_{f(b)} \to Y_b$ such that if $b \leqq b'$ then $f_b P_{f(b)f(b')} \cong q_{bb'} f_{b'}$; i.e., the diagram

$$\begin{array}{ccc} X_{f(b')} & \xrightarrow{p} & X_{f(b)} \\ {\scriptstyle f_{b'}}\downarrow & & \downarrow{\scriptstyle f_b} \\ Y_{b'} & \xrightarrow{q} & Y_b \end{array}$$

commutes up to homotopy.

2. The shape groups. In [6], Mardešić and Segal define the concept of homotopy between two maps of systems. To be more precise, two maps of systems $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$ are said to be *homotopic*, written $\underline{f} \cong \underline{g}$, provided that for every $b \in B$ there is an $a \in A$, $a \geqq f(b), g(b)$, such that $f_b p_{f(b)a} \cong g_b p_{g(b)a}$. Noting the similarities between the category inv $(\mathscr{A})$ and the category of ANR-systems one can define a similar relation in inv $(\mathscr{A})$.

Let $\mathscr{A}$ be a category. Two morphisms $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$ of inverse systems in $\mathscr{A}$ are *$\sim$-related* ($\underline{f} \sim \underline{g}$) if for each $b \in B$ there is an index $a \in A$, $a \geqq f(b), g(b)$ such that $f_b p_{f(b)a} = g_b p_{g(b)a}$.

THEOREM 2.1. *The relation $\sim$ is an equivalence relation.*

Proof. The proof is as in Theorem 2 of [6].

THEOREM 2.2. *Let $\underline{f}, \underline{f}'\colon \underline{X} \to \underline{Y}$ and $\underline{g}, \underline{g}'\colon \underline{Y} \to \underline{Z}$. If $f \sim f'$ and $\underline{g} \sim \underline{g}'$ then $\underline{g}\underline{f} \sim \underline{g}'\underline{f}'$.*

Proof. See Theorem 3 of [6].

A morphism $\underline{f}\colon \underline{X} \to \underline{Y}$ is a *$\sim$-equivalence* provided there is a morphism $\underline{g}\colon \underline{Y} \to \underline{X}$ (called the *$\sim$-inverse of $\underline{f}$*) such that $\underline{g}\underline{f} \sim \underline{1}_X$ and $\underline{f}\underline{g} \sim \underline{1}_Y$. In this case, $\underline{X}$ and $\underline{Y}$ are said to be $\sim$-equivalent ($\underline{X} \sim \underline{Y}$).

THEOREM 2.3. *The relation $\sim$ is an equivalence relation of inverse systems in $\mathscr{A}$.*

Proof. See Theorem 4 of [6].

THEOREM 2.4. *If $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$ are $\sim$-related morphisms and X_∞ and Y_∞ both exist then $f_\infty = g_\infty$.*

Proof. By definition, $f_\infty\colon X_\infty \to Y_\infty$ is the unique $\mathscr{A}$-morphism satisfying for all $b \in B$, $q_b f_\infty = f_b p_{f(b)}$. Similarily, $g_\infty\colon X_\infty \to Y_\infty$ is the unique $\mathscr{A}$-morphism satisfying for all $b \in B$, $q_b g_\infty = g_b p_{g(b)}$. Choose $a \geqq f(b)$, $g(b)$ such that $f_b p_{f(b)a} = g_b p_{g(b)a}$. Now $p_{f(b)} = p_{f(b)a} p_a$ and $p_{g(b)} = p_{g(b)a} p_a$ so that

$$q_b g_\infty = g_b p_{g(b)} = g_b p_{g(b)a} p_a = f_b p_{f(b)a} p_a = f_b p_{f(b)} \,.$$

By the uniqueness, $f_\infty = g_\infty$.

COROLLARY 2.5. *If $\underline{X} \sim \underline{Y}$ and X_∞, Y_∞ both exist then X_∞ and Y_∞ are $\mathscr{A}$-equivalent objects.*

Proof. If $\underline{f}\colon \underline{X} \to \underline{Y}$ and $\underline{g}\colon \underline{Y} \to \underline{X}$ are such that $\underline{g}\underline{f} \sim \underline{1}_X$ and $\underline{f}\underline{g} \sim \underline{1}_Y$ then $g_\infty f_\infty = (gf)_\infty = 1_{X_\infty}$ and $f_\infty g_\infty = (fg)_\infty = 1_{Y_\infty}$.

If $X = \{(X_a, x_a), p_{aa'}, A\}$ is an ANR-system, let $\pi_n(\underline{X}) = \{(\pi_n(X_a, x_a), \rho_{aa'}, A\}$ denote the inverse system of groups where $\pi_n(X_a, x_a)$ is the nth homotopy group of (X_a, x_a) and if $a \leqq a'$ then $\rho_{aa'}\colon \pi_n(X_{a'}, x_{a'}) \to \pi_n(X_a, x_a)$ is the homomorphism induced by $p_{aa'}$; i.e., if $[\xi] \in \pi_n(X_{a'}, x_{a'})$ then $\rho_{aa'}[\xi] = [p_{aa'}\xi]$.

If $\underline{f}\colon \underline{X} \to \underline{Y}$ is a map of systems, $\underline{f}$ induces a morphism of inverse systems $\underline{f}_*\colon \pi_n(\underline{X}) \to \pi_n(\underline{Y})$ where $f_* = f\colon B \to A$ and $(f_b)_*\colon \pi_n(X_{f(b)}, x_{f(b)}) \to \pi_n(Y_b, y_b)$ is the homomorphism induced by f_b. This gives a covariant functor π_n between the category of ANR-systems and the category of inverse systems of groups.

THEOREM 2.6. *If $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$ are homotopic maps of systems ($\underline{f} \simeq \underline{g}$) then the induced morphisms $\underline{f}_*, \underline{g}_*\colon \pi_n(\underline{X}) \to \pi_n(\underline{Y})$ are $\sim$-related ($\underline{f}_* \sim \underline{g}_*$).*

Proof. For each $b \in B$, choose $a \in A$ such that $a \geqq f(b), g(b)$ and $f_b p_{f(b)a} \cong g_b p_{g(b)a}$. Then if $[\xi] \in \pi_n(X_a, x_a)$,

$$(f_b)_* \rho_{f(b)a}[\xi] = [f_b p_{f(b)a}\xi] = [g_b p_{g(b)a}\xi] = (g_b)_* \rho_{g(b)a}[\xi] .$$

COROLLARY 2.7. *If* $\underline{X} \cong \underline{Y}$ *then* $\pi_n(\underline{X}) \sim \pi_n(\underline{Y})$.

Mardešić and Segal have shown in [6] that if $\underline{X}$ and $\underline{Y}$ are ANR-systems associated with (X, x_0) and (Y, y_0), respectively, then a map $f: (X, x_0) \to (Y, y_0)$ has an associated map of systems $\underline{f}: \underline{X} \to \underline{Y}$. If $\underline{X}$ and $\underline{X}'$ are ANR-systems associated with (X, x_0) then any map of systems $\underline{i}: \underline{X} \to \underline{X}'$ associated with the identity $1_{X,x_0}: (X, x_0) \to (X, x_0)$ is a homotopy equivalence. By Corollary 2.7 and Corollary 2.5, $i_\infty: \varprojlim \pi_n(\underline{X}) \to \varprojlim \pi_n(\underline{X}')$ is an isomorphism. Suppose $f: (X, x_0) \to (Y, y_0)$ is a map, $\underline{X}, \underline{X}'$ are ANR-systems associated with (X, x_0) and $\underline{Y}, \underline{Y}'$ are ANR-systems associated with (Y, y_0). Let $\underline{i}: \underline{X} \to \underline{X}'$ and $\underline{j}: \underline{Y} \to \underline{Y}'$ be homotopy equivalences associated with $1_{X,x_0}$ and $1_{Y,y_0}$, respectively. Let $\underline{f}: \underline{X} \to \underline{Y}$ and $\underline{f}': \underline{X}' \to \underline{Y}'$ be maps of systems associated with f. It follows [5] that $\underline{j}\underline{f} \cong \underline{f}'\underline{i}: \underline{X} \to \underline{Y}'$. By Corollary 2.7 and Corollary 2.5, one has that $j_\infty f_\infty = f'_\infty i_\infty: \varprojlim \pi_n(\underline{X}) \to \varprojlim \pi_n(\underline{Y}')$.

If (X, x_0) is a pointed compact Hausdorff space and $\underline{X}$ is any ANR-system associated with (X, x_0) then the *nth shape group*[1] of (X, x_0) is given by $\underline{\pi}_n(X, x_0) = \varprojlim \pi_n(\underline{X})$. If $f: (X, x_0) \to (Y, y_0)$ then the homomorphism $f_\infty: \underline{\pi}_n(X, x_0) \to \underline{\pi}_n(Y, y_0)$ is said to be induced by f. It is easy to show that $(1_{X,x_0})_\infty = 1_{\underline{\pi}_n(X,x_0)}$ and $(fg)_\infty = f_\infty g_\infty$. Corollary 2.7 also shows that the nth shape group is a shape invariant. It is shown in §3 that this definition of $\underline{\pi}_n$ extends that given by Borsuk in [1].

THEOREM 2.8. *There is a homomorphism* $p: \pi_n(X, x_0) \to \underline{\pi}_n(X, x_0)$ *such that for all* $a \in A$, $(p_a)_* = \rho_a p$ *where* $(p_a)_*: \pi_n(X, x_0) \to \pi_n(X_a, x_a)$ *is the homomorphism induced by* $p_a: (X, x_0) \to (X_a, x_a)$.

Proof. The collection of maps $p_a: (X, x_0) \to (X_a, x_a)$ induces homomorphisms $(p_a)_*: \pi_n(X, x_0) \to \pi_n(X_a, x_a)$ such that if $a \leqq a'$ then $(p_a)_* = \rho_{aa'}(p_{a'})_*$. By the universal mapping property of $\underline{\pi}_n(X, x_0)$ there is a unique homomorphism $p: \pi_n(X, x_0) \to \underline{\pi}_n(X, x_0)$ such that for all $a \in A$, $(p_a)_* = \rho_a p$.

THEOREM 2.9. *If* $X \in ANR$ *then* $\pi_n(X, x_0) \approx \underline{\pi}_n(X, x_0)$.

Proof. Since $X \in$ ANR, there is a special ANR-system $\underline{X} =$

[1] Note the nth shape group is actually an isomorphism class of groups.

$\{(X, x_0), 1_{X, x_0}\}$ associated with (X, x_0). Then $\pi_n(\underline{X}) = \{\pi_n(X, x_0), 1_{\pi_n(X, x_0)}\}$ has as inverse limit the group $\pi_n(X, x_0)$.

THEOREM 2.10. *If X is a compact Hausdorff space, $x_0 \in X$ and X_0 is the component of X containing x_0 then $\underline{\pi}_n(X, x_0) = \underline{\pi}_n(X_0, x_0)$.*

Proof. Assume $X \subset I^\Omega$ and $\underline{X} = \{(X_a, x_0), i_{aa'}, A\}$ is an inclusion ANR-system associated with (X, x_0). For each $a \in A$ let X_{a0} denote the component of X_a containing x_0. Since a compact ANR is locally contractible, it is locally path connected. It follows that each X_{a0} is a compact path connected ANR.

Claim. $\underline{X}_0 = \{(X_{a0}, x_0), i_{aa'}|_{X_{a'0}}, A\}$ is an inclusion ANR-system associated with (X_0, x_0). It suffices to show that $X_0 = \bigcap_{a \in A} X_{a0}$. Certainly $X_0 \subset \bigcap_{a \in A} X_{a0}$ since X_0 a compact connected subset of I^Ω implies that if N is any neighborhood of X_0 there is a path connected neighborhood U of X_0 such that $U \subset N$. Let $x \in \bigcap_{a \in A} X_{a0} - X_0$. Then $x \in X - X_0$ so let X_1 denote the component of X to which x belongs. Then there are disjoint open sets U_0, U_1 such that $U_i \cap X = X_i$ $(i = 0, 1)$. Since I^Ω is normal, there are open sets V_0, V_1 such that $X_i \subset V_i \subset \bar{V}_i \subset U_i$ $(i = 0, 1)$. Since $V = V_0 \cup V_1 \cup [I^\Omega - (\bar{V}_0 \cup \bar{V}_1)]$ is a neighborhood of X in I^Ω there is an $a \in A$ such that $X_a \subset V$. Now $X_{a0} \subset V_0$ and $x \in V_1$ a contradiction since $V_0 \cap V_1 = \emptyset$. Thus $X_0 = \bigcap_{a \in A} X_{a0}$ and the claim is proven. By a well-known theorem, $\pi_n(X_a, x_0) = \pi_n(X_{a0}, x_0)$ so that $\pi_n(\underline{X}) = \pi_n(\underline{X}_0)$. It follows then that $\underline{\pi}_n(X, x_0) = \underline{\pi}_n(X_0, x_0)$.

If $x_0, x_1 \in X$ and $\omega: I \to X$ is a path in X connecting x_0 and x_1 then for each $a \in A$, ω induces an isomorphism $\omega_a: \pi_n(X_a, x_0) \to \pi_n(X_a, x_1)$. If $a \leqq a'$ then $i_{aa'}\omega_{a'} = \omega_a$ and it is not hard to show that $\underline{\pi}_n(X, x_0) \approx \underline{\pi}_n(X, x_1)$. Thus we have the following theorem.

THEOREM 2.11. *If x_0 and x_1 are in the same path component of X then $\underline{\pi}_a(X, x_0) \approx \underline{\pi}_n(X, x_1)$.*

Question. Is Theorem 2.11 valid if one replaces path component with component? Using Theorem 4.1 of [4], one can easily show the following is true.

THEOREM 2.12. *If X is a movable compact metric space and if x_0 and x_1 are in the same component of X then $\underline{\pi}_n(X, x_0) \approx \underline{\pi}_n(X, x_1)$.*

3. Equivalence of the inverse limit and Borsuk's definition of $\underline{\pi}_n$. Let X be a compact metric space and $x_0 \in X$. Assume that X is embedded in Q (Hilbert cube). Let (S, a) denote the pointed n-dimensional sphere. An *approximative map* of (S, a) toward (X, x_0),

$\underline{\xi} = \{\xi_k, (S, a) \to (X, x_0)\}$ is a sequence of maps $\xi_k: (S, a) \to (Q, x_0)$ such that for every neighborhood (V, x_0) of (X, x_0) in Q there is an index k_0 such that if $k \geqq k_0$ then $\xi_k \cong \xi_{k+1}$ in (V, x_0). Two approximative maps $\underline{\xi}$ and $\underline{\eta} = \{\eta_k, (S, a) \to (X, x_0)\}$ are said to be homotopic, $\underline{\xi} \cong \underline{\eta}$, if for each neighborhood (V, x_0) of (X, x_0) in Q there is an index k_0 such that if $k \geqq k_0$ then $\xi_k \cong \eta_k$ in (V, x_0). The homotopy class of an approximative map $\underline{\xi}$ is denoted by $[\underline{\xi}]$.

If $\xi, \eta: (S, a) \to (Q, x_0)$ are maps, their *join*, $\xi * \eta: (S, a) \to (Q, x_0)$ is defined as follows. Let P and P' be n-dimensional balls on S such that $a \in S - \mathring{P}$, $a \in S - \mathring{P}'$ and $\mathring{P}' \subset S - \mathring{P}$. Let $\alpha, \beta: (S, a) \times I \to (S, a)$ be homotopies such that $\alpha(x, 0) = \beta(x, 0) = x$ for all $x \in S$ and $\alpha(S - \mathring{P}, 1) = a = \beta(S - \mathring{P}', 1)$. Define

$$(\xi * \eta)(x) = \begin{cases} \xi\alpha(x, 1) & \text{if } x \in S - \mathring{P}' \\ \eta\beta(x, 1) & \text{if } x \in S - \mathring{P} . \end{cases}$$

Note: if $[\xi], [\eta] \in \pi_n(X, x_0)$ where $(X, x_0) \subset (Q, x_0)$ then $[\xi]*[\eta] = [\xi * \eta]$ is the group operation in $\pi_n(X, x_0)$. Let $\underline{\pi}_n^Q(X, x_0)$ denote the group of fundamental classes of approximative maps of (S, a) toward (X, x_0) with operation $*$ defined as follows. If $[\underline{\xi}], [\underline{\eta}] \in \underline{\pi}_n^Q(X, x_0)$ then

$$[\underline{\xi}]*[\underline{\eta}] = [\{\xi_k * \eta_k, (S, a) \longrightarrow (X, x_0)\}] .$$

Then [3] $\underline{\pi}_n^Q(X, x_0)$ is the nth fundamental group defined by Borsuk in [1].

THEOREM 3.1. *If $\underline{X} = \{(X_k, x_0), i_{kk'}\}$ is an inclusion ANR-sequence in Q associated with $(X, x_0) \subset (Q, x_0)$ then $\underline{\pi}_n^Q(X, x_0) \approx \varprojlim \pi_n(\underline{X}) = \underline{\pi}_n(X, x_0)$.*

Proof. Let $\lambda_k: \underline{\pi}_n^Q(X, x_0) \to \pi_n(X_k, x_0)$ be given as follows. If $[\underline{\xi}] \in \underline{\pi}_n^Q(X, x_0)$ then since (X_k, x_0) is a neighborhood of (X, x_0) in Q there is an index m_k such that if $m \geqq m_k$ then $\xi_m \cong \xi_{m_k}$ in (X_k, x_0). Define $\lambda_k[\underline{\xi}] = [\xi_{m_k}] \in \pi_n(X_k, x_0)$. If $[\underline{\xi}] = [\underline{\eta}]$ then there is an m_0 such that if $m \geqq m_0$ then $\xi_m \cong \eta_m$ in (X_k, x_0) so that λ_k is a well-defined function. If $[\underline{\xi}], [\underline{\eta}] \in \underline{\pi}_n^Q(X, x_0)$ and m_0 is "large enough" then

$$\begin{aligned} \lambda_k([\underline{\xi}]*[\underline{\eta}]) &= \lambda_k[\{\xi_m * \eta_m, (S, a) \longrightarrow (X, x_0)\}] \\ &= [\xi_{m_0} * \eta_{m_0}] \\ &= [\xi_{m_0}]*[\eta_{m_0}] \\ &= \lambda_k[\underline{\xi}]*\lambda_k[\underline{\eta}] . \end{aligned}$$

Thus each λ_k is a group homomorphism.

Note. If $\lambda_k[\underline{\xi}] = \lambda_k[\underline{\eta}]$ for all k, then $[\underline{\xi}] = [\underline{\eta}]$. Let (V, x_0) be a neighborhood of (X, x_0) in Q. Choose k so that $(X_k, x_0) \subset (V, x_0)$. Then $\lambda_k[\underline{\xi}] = \lambda_k[\underline{\eta}]$ implies there is an m_0 such that if $m \geqq m_0$ then $\xi_m \cong \eta_m$ in $(X_k, x_0) \subset (V, x_0)$.

We will now show that $(\underline{\pi}_n^Q(X, x_0), \{\lambda_k\})$ is a terminal object in the category inv $(\pi_n(\underline{X}))$ [9], from which it will follow by uniqueness of inverse limit that $\underline{\pi}_n^Q(X, x_0) \approx \underline{\pi}_n(X, x_0)$. To show $(\underline{\pi}_n^Q(X, x_0), \{\lambda_k\})$ is in the category inv $(\pi_n(\underline{X}))$, one must show that if $k \leqq k'$ then $\lambda_k = \rho_{kk'}\lambda_{k'}$ where $\rho_{kk'}: \pi_n(X_{k'}, x_0) \rightarrow \pi_n(X_k, x_0)$ is the homomorphism induced by $i_{kk'}$: $(X_{k'}, x_0) \rightarrow (X_k, x_0)$. Choose $m_0 \geqq m_k, m_{k'}$. Then $\lambda_k[\underline{\xi}] = [\xi_{m_0}] = \rho_{kk'}\lambda_k[\underline{\xi}]$.

It remains to show $(\underline{\pi}_n^Q(X, x_0), \{\lambda_k\})$ is a terminal object; i.e., if G is any group and $\sigma_k: G \rightarrow \pi_n(X_k, x_0)$ are group homomorphisms such that if $k \leqq k'$ then $\sigma_k = \rho_{kk'}\sigma_{k'}$, then there is a unique group homomorphism $\sigma: G \rightarrow \underline{\pi}_n^Q(X, x_0)$ such that $\sigma_k = \lambda_k\sigma$ for all k. The uniqueness follows immediately from the above note.

Existence. Let $g \in G$. Define $\sigma(g) = [\{\xi_k, (S, a) \rightarrow (X, x_0)\}]$ where $\xi_k: (S, a) \rightarrow (Q, x_0)$ satisfies $\xi_k \in \sigma_k(g) \in \pi_n(X_k, x_0)$. First, $\{\xi_k, (S, a) \rightarrow (X, x_0)\}$ is an approximative map of (S, a) toward (X, x_0). If (U, x_0) is any neighborhood of (X, x_0) in Q choose k_0 such that $k \geqq k_0$ implies that $(X_k, x_0) \subset (U, x_0)$. Then $\sigma_k(g) = \rho_{kk+1}\sigma_{k+1}(g)$ so that $\xi_k \cong \xi_{k+1}$ in $(X_k, x_0) \subset (U, x_0)$. Next, σ is a well-defined function for if $\underline{\xi} = \{\xi_k, (S, a) \rightarrow (X, x_0)\}$ and $\underline{\xi}' = \{\xi'_k, (S, a) \rightarrow (X, x_0)\}$ are such that $\xi_k, \xi'_k \in \sigma_k(g)$ for each k, then if (U, x_0) is any neighborhood of (X, x_0) in Q choose k_0 such that if $k \geqq k_0$ then $(X_k, x_0) \subset (U, x_0)$. Then $\xi_k \cong \xi'_k$ in (U, x_0) and hence $[\underline{\xi}] = [\underline{\xi}']$. Also, σ is a group homomorphism. Each σ_k is a homomorphism so that $\sigma_k(g_1g_2) = \sigma_k(g_1)*\sigma_k(g_2)$. Thus if $\xi_k \in \sigma_k(g_1)$, $\eta_k \in \sigma_k(g_2)$ then $\xi_k*\eta_k \in \sigma_k(g_1)*\sigma_k(g_2) = \sigma_k(g_1g_2)$. That is, $\sigma(g_1g_2) = [\{\xi_k*\eta_k, (S, a) \rightarrow (X, x_0)\}]$. But $\sigma(g_1)*\sigma(g_2) = [\{\xi_k*\eta_k, (S, a) \rightarrow (X, x_0)\}]$ so that σ is group homomorphism.

Finally, $\sigma_k = \lambda_k\sigma$ for each k. Since $\lambda_k\sigma(g) = [\xi_{m_k}] \in \pi_n(X_k, x_0)$, it suffices to show $\xi_{m_k} \cong \xi_k$ in (X_k, x_0). If $k \geqq m_k$ then by the definition of m_k, $\xi_k \cong \xi_{m_k}$ in (X_k, x_0). If $m_k \geqq k$ then $\sigma_k(g) = \rho_{km_k}\sigma_{m_k}(g)$ so that $\xi_{m_k} \cong \xi_k$ in (X_k, x_0).

This completes the proof of the theorem.

4. The product of a family of inverse systems. Let Ω be an index set. For each $\omega \in \Omega$, let $\underline{X}^\omega = \{X_a^\omega, p_{aa'}^\omega, A^\omega\}$ be an inverse system of topological spaces (a similar construction can be made for groups, R-modules, etc.). Let $\Gamma = \{(F, \sigma): F$ is a finite nonempty subset of Ω and $\sigma: F \rightarrow \bigcup_{\omega \in \Omega} A^\omega$ is a function such that $\sigma(\omega) \in A^\omega$ for all $\omega \in F\}$. Order Γ by $(F, \sigma) \leqq (F', \sigma')$ iff $F \subset F'$ and $\sigma(\omega) \leqq \sigma'(\omega)$ for all $\omega \in F$. For $(F, \sigma) \in \Gamma$ let $X_{(F,\sigma)} = \prod_{\omega \in F} X_{\sigma(\omega)}^\omega$. If $(F, \sigma) \leqq (F', \sigma')$ then let $p_{(F,\sigma)(F',\sigma')}: X_{(F'\sigma')} \rightarrow X_{(F,\sigma)}$ be the composition of the natural projection

$$\eta: \prod_{\omega \in F'} X_{\sigma'(\omega)}^\omega \longrightarrow \prod_{\omega \in F} X_{\sigma'(\omega)}^\omega$$

and the product map $\prod p^{\omega}_{\sigma(\omega)\sigma'(\omega)}: \prod_{\omega \in F} X^{\omega}_{\sigma'(\omega)} \rightarrow \prod_{\omega \in F} X^{\omega}_{\sigma(\omega)}$. It is not difficult to show that $\underline{X} = \{X_{(F,\sigma)}, p_{(F,\sigma)(F',\sigma')}, \Gamma\}$ is an inverse system. The inverse system $\underline{X}$ is called the *product* of the family $\{\underline{X}^{\omega}: \omega \in \Omega\}$ and is denoted $\underline{X} = \prod_{\omega \in \Omega} \underline{X}^{\omega}$. It can be shown that $\prod_{\omega \in \Omega} \underline{X}^{\omega}$ is the categorical product of the family $\{\underline{X}^{\omega}: \omega \in \Omega\}$.

EXAMPLE. If each A^{ω} is a singleton, each $X^{\omega}_a = I_{\omega} = I$ is the unit interval and $\underline{X}^{\omega} = \{I_{\omega}, 1_{\omega}\}$ where $1_{\omega}: I_{\omega} \rightarrow I_{\omega}$ is the identity map then the above construction gives the usual representation of $I^{\Omega} = \prod_{\omega \in \Omega} I_{\omega}$ as the inverse limit of $\{I^{\alpha}, p_{\alpha\alpha'}, F(\Omega)\}$ where $F(\Omega)$ is the set of all nonempty finite subsets of Ω ordered by inclusion and $p_{\alpha\alpha'}: I^{\alpha'} = \prod_{\omega \in \alpha'} I_{\omega} \rightarrow I^{\alpha}$, is the natural projection (see [6]).

THEOREM 4.1. $\varprojlim \prod_{\omega \in \Omega} \underline{X}^{\omega} = \prod_{\omega \in \Omega} \varprojlim \underline{X}^{\omega}$.

Proof. Let $X^{\omega} = \varprojlim \underline{X}^{\omega}$. We show $\prod_{\omega \in \Omega} X^{\omega}$ is a terminal object in the category $\mathrm{inv}\,(\prod_{\omega \in \Omega} \underline{X}^{\omega})$. For $(F, \sigma) \in \Gamma$, let $p_{(F,\sigma)}: \prod_{\omega \in \Omega} X^{\omega} \rightarrow X_{(F,\sigma)}$ be the composition of the natural projection $\eta: \prod_{\omega \in \Omega} X^{\omega} \rightarrow \prod_{\omega \in F} X^{\omega}$, and the product map $\prod p^{\omega}_{\sigma(\omega)}: \prod_{\omega \in F} X^{\omega} \rightarrow \prod_{\omega \in F} X^{\omega}_{\sigma(\omega)}$. It is not hard to show that if $(F, \sigma) \leqq (F', \sigma')$ then $p_{(F,\sigma)(F',\sigma')} p_{(F',\sigma')} = p_{(F,\sigma)}$. Thus $(\prod_{\omega \in \Omega} X^{\omega}, \{p_{(F,\sigma)}\})$ is in the category $\mathrm{inv}\,(\prod \underline{X}^{\omega})$.

It remains to show that $\prod_{\omega \in \Omega} X^{\omega}$ is a terminal object. That is, if Y is any space and $f_{(F,\sigma)}: Y \rightarrow X_{(F,\sigma)}$ is a family of maps such that if $(F, \sigma) \leqq (F', \sigma')$ then $p_{(F,\sigma)(F',\sigma')} f_{(F',\sigma')} = f_{(F,\sigma)}$, then there is a unique map $f: Y \rightarrow \prod_{\omega \in \Omega} X^{\omega}$ such that for all $(F, \sigma) \in \Gamma$, $p_{(F,\sigma)} f = f_{(F,\sigma)}$. If $\omega \in \Omega$ and $a \in A^{\omega}$ let $\sigma_a: \{\omega\} \rightarrow A^{\omega}$ be the function $\sigma_a(\omega) = a$. Then $(\{\omega\}, \sigma_a) \in \Gamma$ and $f^{\omega}_a = f_{(\{\omega\},\sigma_a)}: Y \rightarrow X^{\omega}_a$ is a family of maps such that if $a \leqq a'$ then $(\{\omega\}, \sigma_a) \leqq (\{\omega\}, \sigma_{a'})$, so that $p^{\omega}_{aa'} f^{\omega}_{a'} = f^{\omega}_a$. By the universal mapping property of X^{ω}, there is a unique $f^{\omega}: Y \rightarrow X^{\omega}$ such that $p^{\omega}_a f^{\omega} = f^{\omega}_a$ for all $a \in A^{\omega}$. Let $f: Y \rightarrow \prod_{\omega \in \Omega} X^{\omega}$ be the unique map thus defined. Then f satisfies $p_{(F,\sigma)} f = f_{(F,\sigma)}$. Furthermore, if $g: Y \rightarrow \prod_{\omega \in \Omega} X^{\omega}$ is any map that satisfies $p_{(F,\sigma)} g = f_{(F,\sigma)}$ then $p_{(\{\omega\},\sigma_a)} g = f_{(\{\omega\},\sigma_a)} = f^{\omega}_a$. It follows then that $f = g$.

COROLLARY 4.2. *If $\underline{X}^{\omega} = \{X^{\omega}_a, p^{\omega}_{aa'}, A^{\omega}\}$, $\omega \in \Omega$, is a family of ANR-systems where $\underline{X}^{\omega}$ is associated with X^{ω}, then $\prod_{\omega \in \Omega} \underline{X}^{\omega}$ is an ANR-system associated with $\prod_{\omega \in \Omega} X^{\omega}$.*

Proof. It suffices to note that if each A^{ω} is closure-finite then so is Γ and that the product of a finite number of ANR's is an ANR.

Suppose $\underline{X}^{\omega} = \{X^{\omega}_a, p^{\omega}_{aa'}, A^{\omega}\}$, $\omega \in \Omega$; $\underline{Y}^{\lambda} = \{Y^{\lambda}_b, q^{\lambda}_{bb'}, B^{\lambda}\}$, $\lambda \in \Lambda$, are inverse systems (or ANR-systems) and $\theta: \Lambda \rightarrow \Omega$ is a one-to-one function such that for each $\lambda \in \Lambda$ there is a map $\underline{f}^{\lambda}: \underline{X}^{\theta(\lambda)} \rightarrow \underline{Y}^{\lambda}$. Recall, a map

$\underline{f}^\lambda: \underline{X}^{\theta(\lambda)} \to \underline{Y}^\lambda$ consists of an increasing function $f^\lambda: B^\lambda \to A^{\theta(\lambda)}$ together with a family of maps $f_b^\lambda: X_{f^\lambda(b)}^{\theta(\lambda)} \to Y_b^\lambda$, $b \in B^\lambda$, such that if $b \leqq b'$ then $q_{bb'}^\lambda f_{b'}^\lambda = f_b^\lambda$ (in the ANR-system case, $q_{bb'}^\lambda f_{b'}^\lambda \cong f_b^\lambda$). Define $f: \Gamma_Y \to \Gamma_X$ by $f(F, \sigma) = (\theta(F), f_\sigma)$ where $f_\sigma: \theta(F) \to \bigcup_{\omega \in \Omega} A^\omega$ is given by $f_\sigma(\theta(\lambda)) = f^\lambda(\sigma(\lambda)) \in A^{\theta(\lambda)}$. Then $X_{f(F,\sigma)} = \prod_{\omega \in \theta(F)} X_{f_\sigma(\omega)}^\omega = \prod_{\omega \in F} X_{f^\lambda(\sigma(\lambda))}^{\theta(\lambda)}$ so define $f_{(F,\sigma)}: X_{f(F,\sigma)} \to Y_{(F,\sigma)}$ as the product map $\prod f_{\sigma(\lambda)}^\lambda: \prod_{\lambda \in F} X_{f^\lambda(\sigma(\lambda))}^{\theta(\lambda)} \to \prod_{\lambda \in F} Y_{\sigma(\lambda)}^\lambda$. One then checks that if $(F, \sigma) \leqq (F', \sigma')$ then $f(F, \sigma) \leqq f(F', \sigma')$ and $q_{(F,\sigma)(F',\sigma')} f_{(F',\sigma')} = f_{(F,\sigma)}$ (in the ANR-system case, $q_{(F,\sigma)(F',\sigma')} f_{(F',\sigma')} \cong f_{(F,\sigma)}$). Thus there is a map $\underline{f}: \prod_{\omega \in \Omega} \underline{X}^\omega \to \prod_{\lambda \in \Lambda} \underline{Y}^\lambda$.

If $\underline{Z}^\tau = \{Z_c^\tau, r_{cc'}^\tau, C^\tau\}$, $\tau \in T$, is another family of inverse systems and $\phi: \Omega \to T$ is a one-to-one function such that for all $\omega \in \Omega$ there is a $\underline{g}^\omega: \underline{Z}^{\phi(\omega)} \to \underline{X}^\omega$ then there is a "natural composition" given by $\phi\theta: \Lambda \to T$ and $\underline{f}^\lambda \underline{g}^{\theta(\lambda)}: \underline{Z}^{\phi\theta(\lambda)} \to \underline{Y}^\lambda$. It is left to the reader to verify that the map determined by the composition is the same as the composition of the respective determined maps.

There is a "natural identity", $\theta: \Omega \to \Omega$ the identity function and each $\underline{1}^\omega: \underline{X}^\omega \to \underline{X}_\omega$ the identity map. It is left to the reader to verify that the identity $1: \prod_{\omega \in \Omega} \underline{X}^\omega \to \prod_{\omega \in \Omega} \underline{X}^\omega$ is determined by the natural identity.

We now restrict our attention to the ANR-system case when $\Omega = \Lambda$ and θ is the identity.

THEOREM 4.3. *If $\underline{f}^\omega, \underline{g}^\omega: \underline{X}^\omega \to \underline{Y}^\omega$ are families of maps of systems such that $\underline{f}^\omega \cong \underline{g}^\omega$ for all $\omega \in \Omega$ then $\underline{f} \cong \underline{g}: \prod_{\omega \in \Omega} \underline{X}^\omega \to \prod_{\omega \in \Omega} \underline{Y}^\omega$.*

Proof. For each $b \in B^\omega$ there is an $a_b \in A^\omega$, $a_b \geqq f^\omega(b), g^\omega(b)$ such that $f_b^\omega p_{f^\omega(b)a_b}^\omega \cong g_b^\omega p_{g^\omega(b)a_b}^\omega$. Let $\tau_\omega: B^\omega \to A^\omega$ be an increasing function such that $\tau_\omega(b) \geqq a_b$ for all $b \in B^\omega$. If $(F, \sigma) \in \Gamma_Y$, consider $(F, \tau) \in \Gamma_X$ where $\tau: F \to \bigcup_{\omega \in \Omega} A^\omega$ is given by $\tau(\omega) = \tau_\omega(\sigma(\omega))$. First, $(F, \tau) \geqq f(F, \sigma), g(F, \sigma)$. Since θ is the identity, $f(F, \sigma) = (F, f_\sigma)$ where $f_\sigma(\omega) = f^\omega(\sigma(\omega))$. Then $(F, \tau) \geqq (F, f_\sigma)$ since

$$\tau(\omega) = \tau_\omega(\sigma(\omega)) \geqq a_{\sigma(\omega)} \geqq f^\omega(\sigma(\omega)) = f_\sigma(\omega) .$$

Similarly, $(F, \sigma) \geqq g(F, \sigma)$. Furthermore,

$$f_{\sigma(\omega)}^\omega p_{f^\omega(\sigma(\omega))\tau_\omega(\sigma(\omega))}^\omega \cong g_{\sigma(\omega)}^\omega p_{g^\omega(\sigma(\omega))\tau_\omega(\sigma(\omega))}^\omega$$

implies

$$f_{(F,\sigma)} p_{f(F,\sigma)(F,\tau)} \cong g_{(F,\sigma)} p_{g(F,\sigma)(F,\tau)} .$$

Thus $\underline{f} \cong \underline{g}$.

COROLLARY 4.4. *If $Sh(X^\omega) = Sh(Y^\omega)$ for all $\omega \in \Omega$ then*

$$Sh(\prod_{\omega \in \Omega} X^{\omega}) = Sh(\prod_{\omega \in \Omega} Y^{\omega}) .$$

Corollary 4.4 allows one to define the *product of shapes* as follows: $\prod_{\omega \in \Omega} Sh(X^{\omega}) = Sh(\prod_{\omega \in \Omega} X^{\omega})$.

In [5] Mardešić gives the notion of a shape retraction. For our purposes we use the following definition: if $j: X \to Y$ is an embedding then a map of systems $\underline{r}: \underline{Y} \to \underline{X}$ is a *shape retraction* iff $\underline{r}\underline{j} \cong \underline{1}_{\underline{X}}$ where $\underline{j}: \underline{X} \to \underline{Y}$ is a map of systems associated with j. It is routine to verify this definition is equivalent to the one given by Mardešić. If there is an embedding $j: X \to Y$ and a shape retraction $\underline{r}: \underline{Y} \to \underline{X}$ then X is said to be a *shape retract* of Y.

COROLLARY 4.5. *If $\underline{r}^{\omega}: \underline{Y}^{\omega} \to \underline{X}^{\omega}$ is a shape retraction for all ω then $\underline{r}: \prod_{\omega \in \Omega} \underline{Y}^{\omega} \to \prod_{\omega \in \Omega} \underline{X}^{\omega}$ is also a shape retraction.*

Proof. Let $\underline{Y}^{\omega}, \underline{X}^{\omega}$ be associated with Y^{ω}, X^{ω}, respectively, and $j^{\omega}: X^{\omega} \to Y^{\omega}$ the required embeddings. Let $j: \prod_{\omega \in \Omega} X^{\omega} \to \prod_{\omega \in \Omega} Y^{\omega}$ be the embedding determined by the family $\{j^{\omega}: \omega \in \Omega\}$. It is routine to verify that the map determined by the family $\{\underline{j}^{\omega}: \underline{X}^{\omega} \to \underline{Y}^{\omega}\}$ is associated with j. We have that $\underline{r}^{\omega}\underline{j}^{\omega} \cong \underline{1}_{\omega}$ where $\underline{1}_{\omega}: \underline{X}^{\omega} \to \underline{X}^{\omega}$ is the map associated with the identity. By the above theorem, $\underline{r}\underline{j} \cong \underline{1}_{\Pi_{\omega \in \Omega} \underline{X}^{\omega}}$.

5. Products of ASR and ANSR-sets. In [5] Mardešić gives definitions for absolute shape retract (ASR) and absolute neighborhood shape retracts (ANSR). These correspond to Borsuk's FAR and FANR-sets, respectively, in the metric case. We will use the following characterizations: A compact Hausdorff space X is an ASR (respectively, ANSR) if there is a compact AR (respectively, ANR) Y and an embedding $j: X \to Y$ such that X is a shape retract of Y (see [10] and [5]).

THEOREM 5.1. *If $X = \prod_{\omega \in \Omega} X^{\omega}$ then $X \in ASR$ iff $X^{\omega} \in ASR$ for all $\omega \in \Omega$.*

Proof. If $X \in$ ASR there is a $Y \in$ AR, an embedding $j: X \to Y$ and a shape retraction $\underline{r}: \underline{Y} \to \underline{X}$. Since each natural projection $p_{\omega}: X \to X^{\omega}$ is a retraction, the associated maps of systems $\underline{p}_{\omega}: \underline{X} \to \underline{X}^{\omega}$ are shape retractions. It follows [5] that $\underline{p}_{\omega}\underline{r}: \underline{Y} \to \underline{X}^{\omega}$ is a shape retraction. Thus, each X^{ω} is an ASR.

Conversely, if $X^{\omega} \in$ ASR for all $\omega \in \Omega$, then for each $\omega \in \Omega$ there is an AR-set Y^{ω} such that $\underline{X}^{\omega}$ is a shape retract of $\underline{Y}^{\omega}$. Since the product of any family of AR-sets is an AR-set, we have by Corollary 4.5 that $X \in$ ASR.

THEOREM 5.2. *If* $X = \prod_{\omega \in \Omega} X^\omega$ *then* $X \in ANSR$ *iff* $X^\omega \in ANSR$ *for all* ω *and* $X^\omega \in ASR$ *for all but a finite number of* ω.

Proof. If $X^\omega \in$ ANSR for all ω and $X^\omega \in$ ASR for all but finitely many ω, say $\omega_1, \omega_2, \cdots, \omega_n$, then for all ω there is an ANR-set Y^ω and a shape retraction $\underline{r}^\omega: \underline{Y}^\omega \to \underline{X}^\omega$ such that $Y^\omega \in$ AR if $\omega \neq \omega_k$ ($k = 1, 2, \cdots, n$). Then $\prod_{\omega \in \Omega} Y^\omega \in$ ANR and there is a shape retraction $\underline{r}: \prod_{\omega \in \Omega} \underline{Y}^\omega \to \prod_{\omega \in \Omega} \underline{X}^\omega$ so that $\prod_{\omega \in \Omega} X^\omega \in$ ANSR.

Conversely, if $X \in$ ANSR then as in the proof of Theorem 5.1, each $X^\omega \in$ ANSR. We may assume without loss that $X^\omega \subset I^{\Lambda_\omega} = \prod_{\lambda \in \Lambda_\omega} I^\lambda$ and $X \subset I^\Lambda = \prod_{\omega \in \Omega} I^{\Lambda_\omega}$. By Theorem IV. 2.10 of [10], there is a closed neighborhood W of X in I^Λ and a shape retraction $\underline{r}: \underline{W} \to \underline{X}$. There is a finite subset of Ω, $\{\omega_1, \omega_2, \cdots, \omega_n\}$ and neighborhoods U_i of X^{ω_i} in $I^{\Lambda_{\omega_i}}$ ($i = 1, 2, \cdots, n$) such that

$$X = \prod_{\omega \in \Omega} X^\omega \subset \prod_{i=1}^{n} U_i \times \prod_{\omega \neq \omega_i} I^{\Lambda_\omega} \subset W .$$

Let $i: X \to W$, $j_\omega: X^\omega \to I^{\Lambda_\omega}$ denote the inclusion maps and let $p_\omega: X \to X^\omega$ be the natural projections. Choose inclusion maps $j'_\omega: I^{\Lambda_\omega} \to W$ for $\omega \neq \omega_i$ ($i = 1, 2, \cdots, n$) and $i_\omega: X^\omega \to X$ such that $j'_\omega j_\omega = ii_\omega$ and $p_\omega i_\omega = 1_{X_\omega}$. Then $\underline{ri} \cong \underline{1}_X$ so that for $\omega \neq \omega_i$ ($i = 1, 2, \cdots, n$),

$$\underline{p}_\omega \underline{r} \underline{j}'_\omega \underline{j}_\omega \cong \underline{p}_\omega \underline{r} \underline{i} \underline{i}_\omega \cong \underline{p}_\omega \underline{i}_\omega \cong \underline{1}_X \omega .$$

Hence $\underline{p}_\omega \underline{r} \underline{j}'_\omega : \underline{I}^{\Lambda_\omega} \to \underline{X}^\omega$ is a shape retraction for $\omega \neq \omega_i$ ($i = 1, 2, \cdots, n$). Thus X^ω, $\omega \neq \omega_i$ ($i = 1, 2, \cdots, n$), is an ASR-set.

6. Products and shape groups. An inspection of Theorem 4.1 shows that the proof does not involve the fact that each X^ω is a topological space. It remains valid, for example, whenever the objects are groups. This fact together with the fact that the (usual) homotopy group of a product is the direct product of the (usual) homotopy groups of its factors, [11] Exercise B.5, p. 419, gives the following theorem.

THEOREM 6.1. *If* $(X, x_0) = \prod_{\omega \in \Omega} (X^\omega, x_0^\omega)$ *then* $\underline{\pi}_n(X, x_0) = \prod_{\omega \in \Omega} \underline{\pi}_n \ (X^\omega, x_0^\omega)$.

Proof. For each ω let $\underline{X}^\omega = \{(X_a^\omega, x_a^\omega), p_{aa'}^\omega, A^\omega\}$ be an ANR-system associated with (X^ω, x_0^ω). Then

$$\begin{aligned}\underline{\pi}_n(X, x_0) &= \underline{\pi}_n(\prod_{\omega \in \Omega} (X^\omega, x_0^\omega)) \\ &= \varprojlim \pi_n(\prod_{\omega \in \Omega} \underline{X}^\omega)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\longleftarrow} \pi_n\{X_{(F,\sigma)}, p_{(F,\sigma)(F',\sigma')}, \Gamma\} \\
&= \lim_{\longleftarrow} \{\pi_n(\prod_{\omega \in F} (X^{\omega}_{\sigma(\omega)}, x^{\omega}_{\sigma(\omega)}), \rho_{(F,\sigma)(F',\sigma')}, \Gamma\} \\
&= \lim_{\longleftarrow} \{\prod_{\omega \in F} \pi_n(X^{\omega}_{\sigma(\omega)}, x^{\omega}_{\sigma(\omega)}), \prod_{\omega \in F} \rho_{\sigma(\omega)\sigma'(\omega)}\eta, \Gamma\} \\
&= \lim_{\longleftarrow} \prod_{\omega \in \Omega} \{\pi_n(X^{\omega}_a, x^{\omega}_a), \rho^{\omega}_{aa'}, A^{\omega}\} \\
&= \prod_{\omega \in \Omega} \lim_{\longleftarrow} \{\pi_n(X^{\omega}_a, x^{\omega}_a), \rho^{\omega}_{aa'}, A^{\omega}\} \\
&= \prod_{\omega \in \Omega} \underline{\pi}_n(X^{\omega}, x^{\omega}_0)\ .
\end{aligned}
$$

Note. In recent correspondence M. Moszyńska indicated that she has defined the concept of "limit homotopy groups" which correspond to our definition of the shape groups. Her approach, to appear in [8] and [9], is more categorical than ours. For completeness we have included our definition and proof that the shape are isomorphic to Borsuk's fundamental groups. The approach to the latter (Theorem 3.1) is somewhat different than the approach she used (see [9]).

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Received June 23, 1972 and in revised form August 3, 1972. This paper is taken from the author's Ph. D. dissertation which was done under the direction of Dr. Leonard Rubin at The University of Oklahoma, Norman, Oklahoma.

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