PACIFIC JOURNAL OF MATHEMATICS
Vol. 48, No. 2, 1973

IDEALIZERS AND NONSINGULAR RINGS

K. R. GOODEARL

This paper deals with the relationship between a ring T and the idealizer R of a right ideal M of T. [The ring R is the largest subring of T which contains M as a two-sided ideal.] Assuming M to be a finite intersection of maximal right ideals of T, the properties of T and R are shown to be very similar. The main theorem of the first section shows that under these hypotheses the right global dimensions of T and R almost always coincide. In the second section, where T is assumed to be a nonsingular ring, the major theorem asserts that the singular submodule of every R-module is a direct summand if and only if the corresponding property holds for T-modules.

We assume throughout the paper that all rings are associative with identity, and that all modules are unitary. Unless otherwise noted, all modules are right modules.

1. *Idealizers.* This section is concerned with idealizers in arbitrary rings, and is based on the work of J. C. Robson in [7].

Given a ring T and a right ideal M of T, the *idealizer* of M in T is the set $R = \{t \in T \mid tM \leqq M\}$, which is easily seen to be the largest subring of T which contains M as a two-sided ideal. The aim of this investigation is to discover properties of T which carry over to R (and vice versa).

We shall mainly consider the case when M is a finite intersection of maximal right ideals of T; following [7], we say in this case that M is a *semimaximal* right ideal of T. Equivalently, M is a semimaximal right ideal of T if T/M is a semisimple right T-module, i.e., a module which is a sum of simple submodules. In accordance with this terminology, we use the term "semisimple ring" to refer to a ring which is semisimple as a module over itself, rather than a ring whose Jacobson radical is zero.

The concept of the idealizer of M is of course not needed if M is already a two-sided ideal of T, i.e., if $TM = M$. When M is maximal, the only other possibility is $TM = T$, and in general this condition seems to be required for some proofs. Fortunately, [7, Proposition 1.7] allows us to assume it without loss of generality: Assuming that M is a semimaximal right ideal of T, then there is another semimaximal right ideal M', containing M, such that $TM' = T$ and the idealizers of M and M' coincide.

Thus we assume throughout this section that M is a semimaximal right ideal of T satisfying $TM = T$.

PROPOSITION 1. [Robson] (a) *R/M is a semisimple ring.*
(b) *T/R is a semisimple right R-module.*
(c) *T is a finitely generated projective right R-module.*
(d) *The natural map $T \otimes_R T \to T$ is an isomorphism.*

Proof. (b), (c), and (d) are contained in Corollary 1.5 and Lemma 2.1 of [7], while (a) follows from the observation [7, Proposition 1.1] that R/M is isomorphic to the endomorphism ring of the right T-module T/M.

A simple consequence of (d) is that for any modules A_T and ${}_TB$, the natural map $A \otimes_R B \to A \otimes_T B$ is an isomorphism, from which we infer that the following maps are also isomorphisms: $A \otimes_R T \to A$, $T \otimes_R B \to B$, $A \to A \otimes_R T$, $B \to T \otimes_R B$. Then for any modules A_T and C_T we conclude using the isomorphisms $A \to A \otimes_R T$ and $C \to C \otimes_R T$ that $\mathrm{Hom}_R(A, C) = \mathrm{Hom}_T(A, C)$. Given these observations and the projectivity of T_R, a straightforward induction establishes the following results:

PROPOSITION 2. (a) $\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^T(A, B)$ *for all A_T, ${}_TB$ and all $n > 0$.*
(b) $\mathrm{Ext}_R^n(A, C) \cong \mathrm{Ext}_T^n(A, C)$ *for all A_T, C_T and all $n > 0$.*

These results suggest comparing the global dimensions of R and T, which is done in [7, Theorem 2.9] for the case when T is right noetherian: Provided that $R \neq T$, then

$$\text{r. gl. dim.}(R) = \max\{1, \text{r. gl. dim.}(T)\}.$$

In Theorem 5 we shall remove the noetherian restriction on this theorem, but first two intermediate results are needed.

The key to the next two propositions is a consideration of the module JT/J, where J is a right ideal of R. There is an epimorphism $f: F \to JT/J$ for some direct sum F of copies of T/R, and we see from Proposition 1 that F is a semisimple right R-module, hence *ker f* must be a summand of F. Thus JT/J is isomorphic to a summand of a direct sum of copies of T/R. For the proof of Theorem 10, we must notice that this same conclusion follows when J is an R-submodule of a right T-module.

PROPOSITION 3. *T is a flat left R-module.*

Proof. The natural maps $R \otimes_R T \to T \otimes_R T \to T$ and $T \otimes_R T \to T$ are both isomorphisms; hence $R \otimes_R T \to T \otimes_R T$ is an isomorphism. Inasmuch as T_R is projective, it follows that $\mathrm{Tor}_1^R(T/R, T) = 0$. Now given any right ideal J of R, JT/J is isomorphic to a summand of a direct sum of copies of T/R, from which we infer that $\mathrm{Tor}_1^R(JT/J, T) = 0$. According to Proposition 2 we also have $\mathrm{Tor}_1^R(T/JT, T) = 0$, whence $\mathrm{Tor}_1^R(T/J, T) = 0$. Thus $J \otimes_R T \to T \otimes_R T$ is injective, hence $J \otimes_R T \to R \otimes_R T$ must be injective.

We shall use the notation $pd_R(A)$ to stand for the projective dimension of an R-module A.

PROPOSITION 4. *If J is any right ideal of R, then $pd_R(J) = pd_T(JT)$.*

Proof. Since $_RT$ is flat, the tensor product of T with any projective resolution of J_R yields a projective resolution of $(J \otimes_R T)_T$; thus $pd_T(J \otimes_R T) \leqq pd_R(J)$. The flatness of $_RT$ also implies that $J \otimes_R T \cong JT$; hence we get $pd_T(JT) \leqq pd_R(J)$.

In view of the projectivity of T_R and R_R, $pd_R(T/R) \leqq 1$. Inasmuch as JT/J is isomorphic to a summand of a direct sum of copies of T/R, we obtain $pd_R(JT/J) \leqq 1$. Examining the long exact sequence of Ext, we infer from this that $pd_R(J) \leqq pd_R(JT)$. Recalling again that T_R is projective, we see that any projective resolution of $(JT)_T$ is also a projective resolution of $(JT)_R$, from which we conclude that $pd_R(JT) \leqq pd_T(JT)$. Thus $pd_R(J) \leqq pd_T(JT)$.

[After the preparation of this paper, Professor Robson informed the author that he too had obtained the following theorem, which appears in [8, Theorem 2.8].]

THEOREM 5. *If $R \neq T$, then* $\text{r. gl. dim.}(R) = \max\{1, \text{r.gl. dim.}(T)\}$.

Proof. If $\text{r. gl. dim.}(R) > 0$, then from Proposition 4 we obtain $\text{r. gl. dim.}(R) = 1 + \sup\{pd_R(J) \mid J \leqq R_R\} \leqq 1 + \sup\{pd_T(K) \mid K \leqq T_T\} = \max\{1, \text{r. gl. dim.}(T)\}$. On the other hand, it is immediate from Proposition 2 that $\text{r. gl. dim.}(T) \leqq \text{r. gl. dim.}(R)$. Thus it only remains to prove that $\text{r. gl. dim.}(R) \geqq 1$.

In view of the assumption $R \neq T$, we see that M cannot be a two-sided ideal of T; hence $1 \notin M$ and $M < R$. Inasmuch as $TM = T$, it follows that the map $R \otimes_R (R/M) \to T \otimes_R (R/M)$ is not injective, from which we conclude that $_R(R/M)$ is not flat. Thus $\mathrm{GWD}(R) > 0$; hence $\text{r. gl. dim.}(R) > 0$.

For weak dimension, the proofs of Proposition 4 and Theorem 5 can be used, mutatis mutandis, to prove the following theorem:

THEOREM 6. *If* $R \neq T$, *then* $\mathrm{GWD}(R) = \max\{1, \mathrm{GWD}(T)\}$.

2. Nonsingular rings. In this section we shall assume that T is a nonsingular ring and then investigate the relationship between singular and nonsingular modules over T and R. First we recall the relevant definitions: Letting $\mathscr{S}(T)$ denote the collection of essential right ideals of T, then the *singular submodule* of a right T-module A is the set $Z_T(A) = \{x \in A \mid xI = 0 \text{ for some } I \in \mathscr{S}(T)\}$. We say that A is *singular* [*nonsingular*] provided $Z_T(A) = A$ [$Z_T(A) = 0$]. The singular submodule of T_T is a two-sided ideal of T, called the *right singular ideal* of T and denoted $Z_r(T)$; T is a *right nonsingular ring* if $Z_r(T) = 0$. Analogous definitions and notations hold for R and its modules.

Throughout this section, we assume that T is a right nonsingular ring and that M is an essential right ideal of T, and we investigate the idealizer R of M. For all but the next two propositions, we make the additional assumptions that M is a semimaximal right ideal of T and that $TM = T$.

PROPOSITION 7. (a) $\mathscr{S}(T) = \{K \leqq T_T \mid K \cap R \in \mathscr{S}(R)\}$.
(b) $\mathscr{S}(R) = \{J \leqq R_R \mid JM \in \mathscr{S}(T)\}$.
(c) $Z_T(A) = Z_R(A)$ *for all* A_T.
(d) $Z_r(R) = Z_R(T) = 0$.

Proof. (a) Suppose that $K \in \mathscr{S}(T)$ and $A \leqq R_R$ such that $A \cap (K \cap R) = 0$. Then $AM \cap K = 0$, whence $AM = 0$ [because AM is a right ideal of T and $K \in \mathscr{S}(T)$]. Thus $A \leqq Z_r(T) = 0$ and so $K \cap R \in \mathscr{S}(R)$.

Now let $K \leqq T_T$ and assume that $K \cap R \in \mathscr{S}(R)$. If $A \leqq T_T$ and $A \cap K = 0$, then from $(A \cap R) \cap (K \cap R) = 0$ we obtain $A \cap R = 0$, hence $A \cap M = 0$. Thus $A = 0$ and so $K \in \mathscr{S}(T)$.

(b) If $J \leqq R_R$ and $JM \in \mathscr{S}(T)$, then $JM \in \mathscr{S}(R)$ by (a), whence $J \in \mathscr{S}(R)$.

Now consider any $J \in \mathscr{S}(R)$. Inasmuch as $M \in \mathscr{S}(T)$ and $Z_r(T) = 0$, the left annihilator of M in T is zero. In particular, it follows that every nonzero element of J has a nonzero right multiple in JM. Thus JM is an essential R-submodule of J, hence $JM \in \mathscr{S}(R)$, and then $JM \in \mathscr{S}(T)$ by (a).

(c) follows directly from (a) and (b).

(d) According to (c), $Z_R(T) = Z_r(T) = 0$, and then $Z_r(R) = 0$ also.

Let Q denote the maximal right quotient ring of T. From [3, Theorem 1 + 2, p. 69] we obtain the following information: Q_T is an injective hull for T_T, Q is a von Neumann regular ring, and Q_Q is injective. Note that $T \cap Z_T(Q) = Z_r(T) = 0$, from which we obtain $Z_T(Q) = 0$.

PROPOSITION 8. *Q is also the maximal right quotient ring of R.*

Proof. We first show that Q is a right quotient ring of R, i.e., that Q_R is a rational extension of R_R. (See [3, pp. 58, 64] for the definitions.) Inasmuch as $Z_r(R) = 0$, [3, Proposition 5, p. 59] says that it suffices to prove that Q_R is an essential extension of R_R. Thus consider any $A \leqq Q_R$ such that $A \cap R = 0$. Then $AM \cap M = 0$. Since M is an essential right ideal of T, it must be an essential T-submodule of Q, so that we obtain $AM = 0$ and $A \leqq Z_T(Q) = 0$. Therefore, Q is a right quotient ring of R; hence we may assume that Q is a subring of the maximal right quotient ring P of R. The injectivity of Q_Q implies that $P_Q = Q \oplus B$ for some B. Then from $R \cap B = 0$ we infer that $B = 0$ and $P = Q$.

In view of Proposition 8, we may refer to [3, Theorem 1 + 2, p. 69] again and conclude that Q_R is an injective hull for R_R. Now we obtain from [5, Proposition 1, p. 427] the following alternate description of the singular submodule of a right R-module A: $Z_R(A) = \bigcap \{\ker f \mid f \in \mathrm{Hom}_R(A, Q)\}$. In particular, A is singular if and only if $\mathrm{Hom}_R(A, Q) = 0$, from which we conclude that any extension of a singular module by a singular module is singular.

N.B.—From this point on, the assumption that M is a semimaximal right ideal of T satisfying $TM = T$ will hold.

It follows from Proposition 7 that every nonsingular right T-module is also a nonsingular right R-module. A partial converse is provided in the next proposition: Any nonsingular right R-module can be canonically embedded in a nonsingular right T-module.

PROPOSITION 9. *If A_R is nonsingular, then the natural map $A \to A \otimes_R T$ is injective and $(A \otimes_R T)_T$ is nonsingular.*

Proof. In view of the discussion following Proposition 8, the intersection of the kernels of the homomorphisms from A into Q_R must be zero. Thus we may assume that A is a submodule of some direct product B of copies of Q.

Since Q is a nonsingular right T-module, so is B. We now get a natural map $A \otimes_R T \to B \otimes_R T \to B$, and the composition $A \to A \otimes_R T \to B$

is just the inclusion map, whence $A \to A \otimes_R T$ must be injective. Also, we see from the flatness of ${}_R T$ that $A \otimes_R T \to B \otimes_R T$ is injective. Since $B \otimes_R T \to B$ is an isomorphism, we infer that $A \otimes_R T \cong AT$; hence $(A \otimes_R T)_T$ is nonsingular.

We say that R is a *splitting ring* provided that for any right R-module A, $Z_R(A)$ is a direct summand of A. It is noted in [1, Proposition 1.12] that R is a splitting ring if and only if $\mathrm{Ext}^1_R(A, C) = 0$ for all nonsingular A_R and all singular C_R.

THEOREM 10. *R is a splitting ring if and only if T is a splitting ring.*

Proof. Suppose that R is a splitting ring. Given a nonsingular right T-module A and a singular right T-module C, it follows from Proposition 7 that A_R is nonsingular and C_R is singular. Thus $\mathrm{Ext}^1_R(A, C) = 0$; hence from Proposition 2 we obtain $\mathrm{Ext}^1_T(A, C) = 0$.

Now assume that T is a splitting ring. Given a nonsingular module A_R and a singular module C_R, we must show that $\mathrm{Ext}^1_R(A, C) = 0$. It suffices to prove that $\mathrm{Ext}^1_R(A, C/CM) = 0$ and $\mathrm{Ext}^1_R(A, CM) = 0$. Inasmuch as $M^2 = MTM = MT = M$, we may thus assume without loss of generality that either $CM = 0$ or $CM = C$.

Case I. $CM = 0$. We first show that $\mathrm{Tor}^R_1(A, R/M) = 0$.

According to Proposition 9, we may assume that A is an R-submodule of a nonsingular right T-module B. The natural map $T \otimes_R M \to T \otimes_R R \to T$ is injective because T_R is projective; hence in view of the condition $TM = T$ we see that $T \otimes_R M \to T$ is an isomorphism. Thus $AT \otimes_T T \otimes_R M \to AT \otimes_T T$ is an isomorphism; equivalently, $AT \otimes_R M \to AT$ is an isomorphism.

Inasmuch as the natural map $R \otimes_R M \to T \otimes_R M \to T$ is injective, $R \otimes_R M \to T \otimes_R M$ must be injective. In light of the projectivity of T_R, we obtain from this that $\mathrm{Tor}^R_1(T/R, M) = 0$. Now since AT/A is isomorphic to a summand of a direct sum of copies of T/R, we must have $\mathrm{Tor}^R_1(AT/A, M) = 0$. Therefore, the map $A \otimes_R M \to AT \otimes_R M \to AT$ is injective, hence $A \otimes_R M \to A \otimes_R R$ is injective. Thus $\mathrm{Tor}^R_1(A, R/M) = 0$.

Now consider any short exact sequence $E: 0 \to C \to B \to A \to 0$. Since $\mathrm{Tor}^R_1(A, R/M) = 0$, we obtain another exact sequence $E^*: 0 \to C \to B/BM \to A/AM \to 0$. The sequence E^* splits because R/M is a semisimple ring, hence E splits.

Case II. $CM = C$. Here $C \cong P/J$ for some direct sum P of copies of M and some R-submodule J of P. To prove that $\mathrm{Ext}^1_R(A, C) = 0$, it suffices to show that $\mathrm{Ext}^1_R(A, P/JM) = 0$ and $\mathrm{Ext}^2_R(A, J/JM) = 0$.

Inasmuch as $M \in \mathscr{S}(R)$, J/JM is a singular right R-module. Choosing an exact sequence $0 \to K \to F \to A \to 0$ with F_R free, we have $\mathrm{Ext}_R^2(A, J/JM) \cong \mathrm{Ext}_R^1(K, J/JM)$. Since $Z_r(R) = 0$, F and thus K are nonsingular; hence $\mathrm{Ext}_R^1(K, J/JM) = 0$ by Case I. Therefore, $\mathrm{Ext}_R^2(A, J/JM) = 0$.

All that remains is to show that $\mathrm{Ext}_R^1(A, D) = 0$, where $D = P/JM$. Inasmuch as P is a right T-module and JM is a T-submodule of P, D is a right T-module. Since P/J and J/JM are both singular R-modules, it follows from the discussion after Proposition 8 that D_R must be singular. Thus from Propositions 7 and 9 we obtain that D_T is singular and $(A \otimes_R T)_T$ is nonsingular.

Given any exact sequence $0 \to D \to B \to A \to 0$, we get a commutative diagram with exact rows as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D \otimes_R T & \longrightarrow & B \otimes_R T & \longrightarrow & A \otimes_R T & \longrightarrow & 0\,. \end{array}$$

The bottom row splits because T is a splitting ring; hence the top row splits. Therefore, $\mathrm{Ext}_R^1(A, D) = 0$.

One special case of Theorem 10 has been proved in [4]. The authors start with a left and right principal ideal domain C such that C is a simple ring but not a division ring, and such that every simple right C-module is injective. (Examples of such rings are constructed in [2].) Then they choose a maximal right ideal M of C and prove that the idealizer I of M in C is a splitting ring [Lemma 2].

It is not hard to prove that every singular right C-module is semisimple, and hence that every singular right C-module is injective. (Details may be found in [6, Chapter 3].) Thus C is certainly a splitting ring. The right ideal M is nonzero because C is not a division ring; hence from the simplicity of C we obtain $CM = C$. Also, C is a right Ore domain, from which it follows easily that M is an essential right ideal of C. Thus it now also follows from Theorem 10 that I is a splitting ring.

REFERENCES

1. V. C. Cateforis and F. L. Sandomierski, *The singular submodule splits off*, J. Algebra, **10** (1968), 149-165.

2. J. H. Cozzens, *Homological properties of the ring of differential polynomials*, Bull. Amer. Math. Soc., **76** (1970), 75-79.

3. C. Faith, *Lectures on Injective Modules and Quotient Rings*, Springer-Verlag Lecture Notes No. 49, (1967).

4. J. D. Fuelberth and M. L. Teply, *A splitting ring of global dimension two*, Proc. Amer. Math. Soc., **35** (1972), 317-324.

5. E. R. Gentile, *Singular submodule and injective hull*, Indag. Math., **24** (1962), 426-433.
6. K. R. Goodearl, *Singular torsion and the splitting properties*, Amer. Math. Soc. Memoirs No. 124, (1972).
7. J. C. Robson, *Idealizers and hereditary noetherian prime rings*, J. Algebra, **22** (1972), 45-81.
8. J. C. Robson, *Idealizer rings*, in Ring Theory (R. Gordon, Ed.), New York (1972), Academic Press, 309-317.

Received February 11, 1972 and in revised form March 23, 1973.

UNIVERSITY OF CHICAGO

Current Address: University of Utah, Salt Lake City, Utah 84112