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CAPACITY THEORY IN BANACH SPACES

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In classical potential theory one way of defining capacity of a compact $K \subset R^n$ puts cap K equal to the total mass of μ, where μ is the measure associated with the inferior envelope of the family of nonnegative superharmonic functions majorizing the characteristic function I_K. A second (equivalent) definition puts cap $K = 1/\|\gamma_0\|_e$ where γ_0 is the projection of the null measure onto the set of positive Radon measures γ supported by K, satisfying $\int d\gamma \geqq 1$ and having finite energy: $\|\gamma\|_e = \int U^\gamma d\gamma < +\infty$.

In the axiomatic Hilbert space setting of Dirichlet spaces Beurling and Deny have shown that equivalence of definitions of the two above types leads to a rich capacity theory. In this article all of these results are extended to the family of Banach-Dirichlet (BD) spaces, i.e., uniformly convex Banach spaces of (equivalence classes of) functions satisfying the Dirichlet space axioms. This is accomplished by using a capacity of the first type in the BD space D, and of the second type in the dual space D'.

THEOREM 1. *The two types of capacity are equal.*

THEOREM 2. *Exterior capacity is a true capacity in the sense of Brelot.*

THEOREM 3. *A set E has zero exterior capacity iff E is capacitable and $\mu E = 0$ for all measures μ generating a pure potential $u^\mu \in D'$.*

THEOREM 4. *For every quasi-continuous representative u^* of $u \in D$ and every μ generating a pure potential u^μ, the formula $(u, u^\mu) = \int u^* d\mu$ holds, where $(\cdot, \cdot)$ is the bi-linear form on $D \times D'$.*

The reader will be aided by familiarity with [6]. Some definitions therein will be reiterated in § 2.

1. Preliminary lemmas concerning certain Banach spaces.

Let $\{E_i\}_{i \in I}$ be a nonvoid family of nonvoid subsets of a set A.

DEFINITION 1.1. The family $\{E_i\}_{i \in I}$ is *directed downward by inclusion* if for each pair $i, j \in I$ there exists $k \in I$ with $E_k \subset E_i \cap E_j$.

The family $\{E_i\}$ is also called a filter base.

DEFINITION 1.2. The family $\{E_i\}_{i \in I}$ is *directed upward by containment* if for each pair $i, j \in I$ there exists $k \in I$ with $E_k \supset E_i \cup E_j$.

NOTE. If $\{E_i\}_{i \in I}$ is directed downward by inclusion (upward by containment) and $x_i \in E_i$ for each $i \in I$, then $\{x_i\}_{i \in I}$ is a net in A when I is directed by the rule $i \geqq j$ iff $E_i \subset E_j$ $(E_i \supset E_j)$.

LEMMA 1.3. *In a Banach space B with norm $\|\cdot\|$ let $\{E_i\}_{i \in I}$ be a family of closed convex sets directed downward by inclusion such that the set of numbers $\{\inf\{\|z\| \mid z \in E_i\}\}_{i \in I}$ has a supremum $M < \infty$.*

(i) *If B is reflexive, then $E = \bigcap_{i \in I} E_i \neq \emptyset$ and there exists $z \in E$ with $\|z\| \leqq M$.*

(ii) *If B is uniformly convex and for each $i \in I$, x_i is the unique element of minimum norm of E_i, then the net $\{x_i\}_{i \in I}$ is Cauchy and $x = \lim_{i \in I} x_i$ is the unique element of minimum norm of E.*

Proof. (i) Let $B_M = \{z \in B \mid \|z\| \leqq M\}$. The family $\{E_i \cap B_M\}_{i \in I}$ is directed downward by inclusion. Each $E_i \cap B_M$ is closed and convex, thus weakly closed. Since B is reflexive, B_M is weakly compact. Hence $\bigcap_{i \in I} E_i \cap B_M \neq \emptyset$, i.e., there exists $z \in E$ with $\|z\| \leqq M$.

(ii) Since each E_i is closed and convex, E is also. By uniform convexity there exists a unique $x \in E$ of minimum norm and (i) assures $\|x\| \leqq M$. Moreover, $x \in E_i$ for each $i \in I$ and $M = \sup\{\|x_i\| \mid i \in I\}$ entail $\|x\| \geqq M$. Thus $\|x\| = M$.

The net $\{x_i\}_{i \in I}$ is Cauchy. In fact, it is clear that $\lim_{i \in I} \|x_i\| = M$, i.e., for $\varepsilon > 0$ there exists $i \in I$ such that $j \geqq i$ implies $\|x_j\| > M - \varepsilon/2$. Moreover, for all $j, k \geqq i$,

$$2M \geqq \|x_k\| + \|x_j\| \geqq \|x_k + x_j\| = 2\|(x_k + x_j)/2\| .$$

But $x_k, x_j \in E_i$ so convexity assures $(x_k + x_j)/2 \in E_i$. Since x_i is the unique element of minimum norm in E_i, we have

$$2M \geqq 2\|(x_k + x_j)/2\| \geqq 2\|x_i\| > 2M - \varepsilon .$$

This shows $\lim_{j,k \in I} \|x_k + x_j\| = 2M$. The fact that $\{x_i\}_{i \in I}$ is Cauchy follows directly from the definition of uniform convexity. Put $y = \lim x_i$. Then $\|y\| = \lim \|x_i\| = M$. Since $\{E_i\}_{i \in I}$ is directed downward by inclusion and each E_i is closed, we have $y \in \bigcap_{i \in I} E_i = E$. But x is the unique element of minimum norm in E, so $y = x$.

LEMMA 1.4. *Let B be a uniformly convex Banach space and $\{E_i\}_{i \in I}$ a family of closed convex subsets of B directed upward by containment. Let $K \subset B$ be closed and convex with $K \supset \bigcup_{i \in I} E_i$. Denote by x_i, x the unique elements of minimum norm of E_i, K respectively. If $\|x\| = \inf\{\|x_i\| \mid i \in I\}$, then $\lim_{i \in I} x_i = x$.*

Proof. To see that $\{x_i\}_{i \in I}$ is Cauchy, first observe that $\lim \|x_i\| =$

$\inf\{\|x_i\| \mid i \in I\} = \|x\|$. Given $\varepsilon > 0$ choose $n \in I$ such that $m \geqq n$ implies $\|x\| + \varepsilon/2 \geqq \|x_m\|$. Then for $i, j \geqq n$,

$$\begin{aligned} 2\|x\| + \varepsilon &\geqq \|x_i\| + \|x_j\| \geqq \|x_i + x_j\| \\ &= 2\|(x_i + x_j)/2\| \geqq 2\|x_m\| \end{aligned}$$

for any $m \geqq i, j$ since $m \geqq i, j$ implies $E_m \supset E_i \cup E_j$ and E_m is convex. Thus $\lim_{i,j \in I}\|x_i + x_j\| = 2\|x\|$ and uniform convexity assure $\{x_i\}$ is Cauchy. Put $y = \lim x_i$. As in the proof of Lemma 1.3, $y = x$.

COROLLARY 1.5. *Let B, $\{E_i\}_{i \in I}$, $\{x_i\}_{i \in I}$ be as in Lemma 1.4. Then $H = \overline{\bigcup E_i}$ is the closed convex hull of $\bigcup E_i$, and $\lim x_i = x$ where x denotes the unique element of minimum norm in H.*

Proof. Since each E_i is convex and family is directed upward by containment, $\bigcup E_i$ is convex. Thus $H = \overline{\bigcup E_i}$ is the closed convex hull of $\bigcup E_i$. Given $\varepsilon > 0$ there exists $i \in I$ and $z \in E_i$ with $\|x\| \geqq \|z\| - \varepsilon \geqq \|x_i\| - \varepsilon$, so $\|x\| = \inf\{\|x_i\| \mid i \in I\}$ and Lemma 1.4 applies.

2. Review of definitions and basic facts. Much of the below is expanded upon in [6].

A *normal contraction* $T: R \to R$ of the line verifies $T(0) = 0$ and $|T(x) - T(y)| \leqq |x - y|$. A *duality map* $S: N \to N'$ of a smooth normed linear space N to its dual is the unique map satisfying $\|S(u)\| = \|u\|$ and $|(u, S(u))| = \|u\|^2$. Also, for nonzero $u \in N$

$$(x, S(u)) = \|u\| \cdot \lim_{t \to 0} \frac{\|u + tx\| - \|u\|}{t}$$

for all $x \in N$. Let X denote a locally compact Hausdorff space, $\mathscr{C} = \mathscr{C}(X)$ the continuous real valued functions φ on X with support $\mathscr{S}(\varphi)$ compact supplied with the inductive limit topology, ξ a positive Radon measure on X. Let $F = F(X, \xi)$ denote a Banach space with norm $\|\cdot\|$ of equivalence classes of real valued, locally ξ-integrable functions on X. As with L^p spaces, we assume each equivalence class contains all functions which are equal ξ-a.e. to a given representative of that class. (A departure from this convention is suggested in § 10, where "refinements" of classes are considered.)

The three *Dirichlet axioms* are

(a) For any compact $K \subset X$ there exists a constant $A(K) \geqq 0$ such that for $u \in F$

$$\int_K |u|\, d\xi \leqq A(K)\,\|u\|\,.$$

(b) The measure ξ is everywhere dense in X, and $F \cap \mathscr{C}$ is dense in F and in $\mathscr{C}$.

(c) For any normal contraction T and $u \in F$ we have the composition $Tu \in F$ and $\|Tu\| \leqq \|u\|$.

A *Banach-Dirichlet* (BD) space is a Banach space $D = D(X, \xi)$ of equivalence classes of real valued locally ξ-integrable functions which satisfies the three Dirichlet axioms. Several examples of BD spaces are given in [6]. *Pure potentials* are elements of the positive dual cone F'^{+} where the natural order is assumed on F. If F is uniformly convex and satisfies axioms (a) and (c), then $S(u) \in F'^{+}$ implies $u \geqq 0$ a.e. ξ. If $f \in D'^{+}$ where D is a BD space, there exists a unique Radon measure $\mu \geqq 0$ such that

$$(1) \qquad (\varphi, f) = \int \varphi d\mu \quad \text{for all} \quad \varphi \in D \cap \mathscr{C} .$$

The *measure associated with* f is μ and μ *generates* f. Write $f = u^{\mu}$, or in case $\mu = g \cdot \xi$, write $f = u^{g}$. A *potential* f satisfies (1) where μ need not be positive.

3. Capacity and dual capacity of open sets.

Throughout the remainder of this article it is assumed that $F(X, \xi)$ is uniformly convex and verifies axiom (a).

DEFINITION 3.1. Let $\omega \subset X$ be an open set.

(i) $\mathscr{U}_{\omega} \subset F$ is defined

$$\mathscr{U}_{\omega} = \{u \in F \mid u \geqq 1 \text{ a.e. } \xi \text{ on } \omega\} .$$

(ii) The *capacity* of ω is a nonnegative real number or $+\infty$ given by

$$\operatorname{cap} \omega = \inf \{\|u\| \mid u \in \mathscr{U}_{\omega}\} .$$

(iii) If $\mathscr{U}_{\omega} \neq \varnothing$, the unique element of minimum norm of $\mathscr{U}_{\omega}$ is called the *capacitary element associated with* ω.

Using axiom (a) it is easy to show $\mathscr{U}_{\omega}$ is closed and convex, thus (iii) follows. In case $\mathscr{U}_{\omega} = \varnothing$, then $\operatorname{cap} \omega = +\infty$. If $\omega_1 \subset \omega_2$, then $\mathscr{U}_{\omega 1} \supset \mathscr{U}_{\omega_2}$ so $\operatorname{cap} \omega_1 \leqq \operatorname{cap} \omega_2$.

DEFINITION 3.2. For open $\omega \subset X$, the set $P_{\omega} \subset F'$ is the closure of the set of pure potentials u^{f} where $f \geqq 0$ is a bounded measurable function with compact support contained in ω, and with $\int f d\xi = 1$.

It is immediate that P_{ω} is closed and convex.

DEFINITION 3.3. For open $\omega \subset X$, the *dual capacity* of ω is

$$\operatorname{dualcap} \omega = \begin{cases} \sup \{1/\|z\| \mid z \in P_{\omega}\} & \text{for } P_{\omega} \neq \varnothing \\ 0 \quad \text{for} \quad P_{\omega} = \varnothing \end{cases}$$

(Convention $1/0 = +\infty$.)

REMARK. Definitions 3.1 and 3.3 are slightly different from their analogs used by Deny [4]. The change is required by technical problems due to the weaker assumptions. The change is not serious since it is clear that sets of zero exterior capacity are the same with either definition. Further, the exterior capacity herein is a true capacity in the sense of Brelot [2], (see § 4).

LEMMA 3.4. *Let $\{\omega_i\}_{i\in I}$ be a family of open subsets of X directed upward by containment, with $\{\text{cap}\,\omega_i\}_{i\in I}$ a bounded set of real numbers. For each $i \in I$ denote by $u_i \in F$ the capacitary element associated with ω_i. Then*

(i) *$\omega = \bigcup_{i\in I}\omega_i$ has a capacitary element u,*

(ii) *u is the limit of the net $\{u_i\}_{i\in I}$.*

Proof. By Lemma 1.3 with $E_i = \mathscr{U}_{\omega_i}$, $x_i = u_i$, and $x = u$, it follows that $\bigcap_{i\in I}\mathscr{U}_{\omega_i} \neq \varnothing$. Now $\mathscr{U}_\omega = \bigcap_{i\in I}\mathscr{U}_{\omega_i}$. In fact, $v \in \bigcap \mathscr{U}_{\omega_i}$ entails $v \geqq 1$ a.e. ξ on ω_i for each $i \in I$, i.e., if $A_i = \{x \in \omega_i \mid v(x) < 1\}$, then $\xi(A_i) = 0$ for each $i \in I$. Let $A = \{x \in \omega \mid v(x) < 1\}$, and compact $K \subset A$. Since $K \subset \omega = \bigcup \omega_i$, there is a finite subcover: $K \subset \bigcup_{j=1}^n \omega_{ij}$. Since the family $\{\omega_i\}_{i\in I}$ is directed upward, there exists $i \in I$ with $K \subset \omega_i$, so $K \subset A_i$ and $\xi(K) = 0$. Thus $\xi(A) = 0$ and $\bigcap \mathscr{U}_{\omega_i} \subset \mathscr{U}_\omega$. The reverse containment is immediate. Lemma 1.3 gives the result.

In the proof of the following theorem it will be made clear that $\xi(\omega) = 0$ entails $\text{cap}\,\omega = 0$ for open ω. Let $T{:}F' \to F$ denote the duality map. Since F is uniformly convex, F' is smooth so T is unique.

THEOREM 3.5. *For open $\omega \subset X$,*

(i) dualcap $\omega = \text{cap}\,\omega$,

(ii) *if* $0 <$ dualcap $\omega < \infty$, *the set*

$$E = \{v \in P_\omega \mid 1/\|v\| = \text{dualcap}\ \omega\}$$

is a nonvoid subset of F'. Moreover,

$$T(E) = \|v\|^2 \cdot u$$

where $u \in F$ is the capacitary element associated with ω.

Proof. Case 1. $\xi(\omega) = 0$. Here $0 \in F$ is $\geqq 1$ a.e. ξ on ω so $\text{cap}\,\omega = 0$; any bounded measurable function f supported by ω verifies $\int f d\xi = 0$, so $P_\omega = \varnothing$, thus dualcap $\omega = 0$. Hence $\text{cap}\,\omega =$ dualcap ω.

In preparation for Cases 2 and 3, suppose $\xi(\omega) > 0$. Let $K \subset \omega$ be compact with $\xi(K) > 0$. Then $f = 1/\xi(K) \cdot I_K$ is an element of P_ω, so $P_\omega \neq \emptyset$. Since P_ω is closed and convex and F' is reflexive, it follows that P_ω has at least one element of minimum norm. Denote by E the set of all such elements. Let $v \in E$ and consider $T(v) \in F$. For any $f: X \to R$ which generates a pure potential $u^f \in P_\omega$ we have $(T(v), u^f - v) \geqq 0$. In fact, if $v = 0$, then $Tv = 0$ so $(Tv, u^f - v) = 0$. If $v \neq 0$,

$$\frac{1}{\|v\|}(Tv, u^f - v) = \lim_{t \to 0} \frac{\|v + t(u^f - v)\| - \|v\|}{t}$$
$$= \lim_{t \to 0} \frac{\|(1-t)v + tu^f\| - \|v\|}{t} \geqq 0 .$$

The limit exists by smoothness of F' and the inequality holds because $(1-t)v + tu^f \in P_\omega$ by convexity and the fact that $\|v\|$ is minimal over P_ω. Thus for all such $u^f \in P_\omega$,

$$(Tv, u^f) \geqq (Tv, v) = \|v\|^2 .$$

This inequality implies

$$(2) \qquad (Tv)(x) \geqq \|v\|^2 \text{ a.e. } \xi \text{ on } \omega .$$

Case 2. $\xi(\omega) > 0$ and $\operatorname{cap} \omega = +\infty$. This entails $\mathscr{U}_\omega = \emptyset$. Recall $v \in P_\omega$ and $1/\|v\| = \operatorname{dualcap} \omega$. If $\|v\|^2 > 0$, then

$$\{u \in F \mid u \geqq \|v\|^2 \text{ a.e. } \xi \text{ on } \omega\} = \emptyset$$

because $\mathscr{U}_\omega = \emptyset$. Hence (2) implies $\|v\|^2 = 0$, so $0 = v \in P_\omega$. Thus, $\operatorname{dualcap} \omega = +\infty$.

Case 3. $\xi(\omega) > 0$ and $0 < \operatorname{cap} \omega < +\infty$. Here $\mathscr{U}_\omega \neq \emptyset$. Let u be the capacitary element associated with ω. For any $u^f \in P_\omega$ we have $\int f d\xi = 1$, then since $u \geqq 1$ a.e. ξ on ω,

$$1 \leqq \int u f d\xi = (u, u^f) .$$

But any $v \in P_\omega$ with $1/\|v\| = \operatorname{dualcap} \omega$ is the limit of a sequence of such elements u^f, so

$$1 \leqq (u, v) \leqq \|u\| \|v\| .$$

Thus, $\|v\| \neq 0$ and

$$(3) \qquad \frac{\|Tv\|}{\|v\|} \leqq \|u\| \|v\| \frac{\|Tv\|}{\|v\|^2} = \|u\| .$$

But $Tv \geqq \| v \|^2$ a.e. on ω implies $Tv/\| v \|^2 \in \mathscr{U}_\omega$. Thus by the uniqueness of u as the element of minimum norm in $\mathscr{U}_\omega$, (3) implies $Tv/\| v \|^2 = u$, which verifies (ii). Finally,

$$\operatorname{cap} \omega = \| u \| = \| Tv \| / \| v \|^2 = 1/\| v \| = \operatorname{dualcap} \omega$$

which verifies (i).

4. Exterior capacity and capacitability.

DEFINITION 4.1. For any $E \subset X$, the *exterior capacity* of E is defined by

$$\operatorname{cap}_e E = \inf \{ \operatorname{cap} \omega \mid \omega \supset E, \omega \text{ open} \} .$$

Observe that cap_e is defined on all subsets of X, and that for ω open, $\operatorname{cap} \omega = \operatorname{cap}_e \omega$.

DEFINITION 4.2. Any $E \subset X$ is *cap_e-capacitable* or merely *capacitable*, if

$$\operatorname{cap}_e E = \sup \{ \operatorname{cap}_e K \mid E \supset K, K \text{ compact} \} .$$

It will be shown that cap_e verifies

(i) cap_e is increasing, i.e., $E_1 \subset E_2$ implies $\operatorname{cap}_e E_1 \leqq \operatorname{cap}_e E_2$.

(ii) For any increasing sequence of sets $\{E_n\}$,

$$\lim_{n \to \infty} \operatorname{cap}_e E_n = \operatorname{cap}_e \bigcup_{n=1}^{\infty} E_n .$$

(iii) For any decreasing sequence of compact sets $\{K_n\}$,

$$\lim_{n \to \infty} \operatorname{cap}_e K_n = \operatorname{cap}_e \bigcap_{n=1}^{\infty} K_n .$$

These are precisely the three conditions which must be verified in order that cap_e be *true capacity*; it then follows that K-analytic subsets of σ-compact sets in X are capacitable, see [2, Chapter I part II] and [3, Chapter VI]. In this section (i) and (iii) are indicated for cap_e. That (ii) holds is shown in § 7.

PROPOSITION 4.3. *The set function* cap_e *verifies condition* (i) *for true capacity.*

The proof follows immediately from the fact that cap is increasing on open sets.

DEFINITION 4.4. A set function G is *continuous on the right on* *compact sets* if for any compact K and $\varepsilon > 0$ there is an open neighborhood $U \supset K$ such that K' compact and $K \subset K' \subset U$ imply $G(K') \leqq G(K) + \varepsilon$.

The notion "continuity on the right" is due to Choquet [3, pp. 147, 174]. The following lemma is from [2, p. 12].

LEMMA 4.5. *Let G be an increasing set function on the subsets of a Hausdorff space. If G is continuous on the right on compact sets, then G satisfies condition* (iii) *for true capacity.*

PROPOSITION 4.6. *The set function* cap_e *verifies condition* (iii) *for true capacity.*

Proof. Let $K \subset X$ be compact and $\varepsilon > 0$. By definition of cap_e there exists an open $\omega \supset K$ with $\operatorname{cap} \omega \leqq \operatorname{cap}_e K + \varepsilon$. Let a compact K' satisfy $K \subset K' \subset \omega$. Then $\operatorname{cap}_e K' \leqq \operatorname{cap} \omega \leqq \operatorname{cap}_e K + \varepsilon$. Lemma 4.5 now applies with $G = \operatorname{cap}_e$.

5. **Some properties of cap_e. Capacitability of open sets.** The lemmas of this section lead to the proposition that open sets are capacitable. Moreover, the results of these lemmas are used in § 7.

DEFINITION 5.1. For any $E \subset X$, the set $\mathscr{U}_E \subset F$ is defined by

$$\mathscr{U}_E = (\bigcup_{\omega \supset E} \mathscr{U}_\omega)^-$$

the union being over all open supersets of E. (Here the bar denotes closure.)

LEMMA 5.2. (i) $\mathscr{U}_E \neq \emptyset$ *iff* $\operatorname{cap}_e E < \infty$.

(ii) *For any $E \subset X$, $\mathscr{U}_E$ is closed and convex.*

(iii) *In case $E = V$ is open, then $\mathscr{U}_E$ is identical to $\mathscr{U}_V$ of Definition* 3.1.

Proof. (i) $\mathscr{U}_E \neq \emptyset$ iff for some open $\omega \supset E$, $\mathscr{U}_\omega \neq \emptyset$ iff for some open $\omega \supset E$, $\infty > \operatorname{cap} \omega$ iff $\infty > \operatorname{cap}_e E$.

(ii) Corollary 1.5 applies with $\{E_i\}_{i \in I} = \{\mathscr{U}_\omega\}_{\omega \supset E}$. Thus $\mathscr{U}_E$ is the closed convex hull of $\bigcup \mathscr{U}_\omega$.

(iii) If $E = V$ is open, then $\mathscr{U}_V \supset \mathscr{U}_\omega$ for all open $\omega \supset V$. Thus

$$\mathscr{U}_V = \mathscr{U}_V^- \supset (\bigcup_{\omega \supset V} \mathscr{U}_\omega)^- = (\bigcup_{\omega \supset E} \mathscr{U}_\omega)^- = \mathscr{U}_E .$$

Conversely, $\mathscr{U}_E \supset \bigcup_{\omega \supset E} \mathscr{U}_\omega \supset \mathscr{U}_V$.

As a result of (ii) of the above lemma, we can give the following definition.

DEFINITION 5.3. For any $E \subset X$ with $\mathscr{U}_E \neq \emptyset$, the *exterior capacitary element associated with* E, $u_E \in F$, is the unique element of minimum norm of $\mathscr{U}_E$.

LEMMA 5.4. *Let* $E \subset X$ *with* $\mathscr{U}_E \neq \emptyset$. *Then*

(i) $\|u_E\| = \operatorname{cap}_e E$.

(ii) *If* $\{\omega_i\}_{i \in I}$ *is any family of open sets in* X *directed downward by inclusion with each* $\omega_i \supset E$, $\operatorname{cap} \omega_i < \infty$ *and* $\operatorname{cap}_e E = \inf\{\operatorname{cap} \omega_i \mid i \in I\}$, *then* $u_E = \lim u_i$, *where* u_i *denotes the capacitary element associated with* ω_i.

Proof. (i) Apply Corollary 1.5 with $\{E_i\} = \{\mathscr{U}_\omega\}$, $H = \mathscr{U}_E$.

(ii) Apply Lemma 1.4 with $K = \mathscr{U}_E$. By (i) above, $\|x\| = \|u_E\| = \operatorname{cap}_e E$. By hypothesis,

$$\operatorname{cap}_e E = \inf\{\operatorname{cap} \omega_i \mid i \in I\} = \inf\{\|x_i\| \mid i \in I\}$$

in the notation of Lemma 1.4,

$$u_E = x = \lim x_i = \lim u_i .$$

LEMMA 5.5. *For any* $E \subset X$ *with* $\operatorname{cap}_e E < \infty$, *there exists a decreasing* sequence *of open sets* $\{\omega_n\}_{n=1}^\infty$ *with each* $\omega_n \supset E$ *and* $u_E = \lim u_n$, *where* $\{u_n\}_{n=1}^\infty$ *is the corresponding* sequence *of capacitary elements.*

Proof. From the family of all open supersets of E with finite capacity, one uses an easy induction argument to construct a decreasing sequence $\{\omega_n\}$ with the property $\operatorname{cap}_e E = \lim \operatorname{cap} \omega_n$. The result follows by Lemma 5.4.

For the purposes of the next corollary, we assume $\xi(E \cap K) = 0$ for all compact K implies $\xi E = 0$.

COROLLARY 5.6. *For* $E \subset X$ *with* $\operatorname{cap}_e E < \infty$, $u_E \geqq 1$ *a.e.* ξ *on* E. *In case* $\operatorname{cap}_e E = 0$, *then* $\xi(E) = 0$.

Proof. Let $\{\omega_n\}$, $\{u_n\}$ be as in Lemma 5.5. Then $\lim u_n = u_E$. Let $K \subset E$ be compact. Axiom (a) assures $\lim_{n\to\infty} \int_K |u_n - u_E| \, d\xi = 0$. Thus for some subsequence u_m, $\lim u_m = u_E$ pointwise a.e. ξ on K. But $u_m \geqq 1$ a.e. ξ on ω_n, and hence a.e. ξ on K. But $u_m \geqq 1$ a.e. ξ on ω_m, hence on $E \subset \omega_m$. Thus $u_E = \lim u_m \geqq 1$ a.e. ξ on $E \cap K$, i.e., $u_E \geqq 1$ a.e. ξ on E.

In case $\operatorname{cap}_e E = 0$, we have $\|u_E\| = 0$, so for any compact $K \subset X$,

$$0 \leqq \int_{E\cap K} u_E d\xi \leqq A(\overline{E\cap K}) \, \| u_E \| = 0$$

(here $A(\overline{E\cap K})$ is the constant of axiom (a)). Thus, since $u_E \geqq 1$ a.e. ξ on $E\cap K$, it follows that $\xi(E\cap K) = 0$.

PROPOSITION 5.7. *Any open $\omega \subset X$ is capacitable.*

In view of the fact that cap_e is increasing, for any $E\subset X$ if $\sup\{\mathrm{cap}_e K \mid E\supset K \text{ compact}\} = +\infty$, then $\mathrm{cap}_e E = +\infty$, so E is capacitable. Thus it suffices to consider open ω satisfying $\sup\{\mathrm{cap}_e K \mid \omega \supset K \text{ compact}\} < \infty$. Using Lemmas 1.3 and 3.4 and Corollary 5.6, a proof similar to that given by Deny [4, pp 1-05, 1-06] for the Hilbert space case will suffice.

6. Denumerable sub-additivity of cap and cap_e. In this section and the remainder of the article we assume that the uniformly convex space $F(X, \xi)$ satisfies axiom (c) as well as axiom(a). In this section the normal contraction "modulus", i.e., $u \to |u|$, is the only contraction needed, so the full strength of axiom (c) is not required.

LEMMA 6.1. *For any finite family of open subsets of X we have*

$$\mathrm{cap} \bigcup_{i=1}^{n} \omega_i \leqq \sum_{i=1}^{n} \mathrm{cap}\, \omega_i \, .$$

Proof. Without loss of generality, each $\mathscr{U}_{\omega_i} \neq \varnothing$. Let $u_i \in F$ be the capacitary element associated with ω_i, $i = 1, \cdots, n$. By axiom (c) $u \in \mathscr{U}_{\omega_i}$ implies $|u| \in \mathscr{U}_{\omega_i}$ and $\| \, |u| \, \| \leqq \|u\|$. Thus $u_i = |u_i| \geqq 0$ a.e. ξ. Hence $\sum_{i=1}^{n} u_i \geqq 1$ a.e. ξ on $\bigcup_{i=1}^{n} \omega_i$ so $\sum_i^n u_i \in \mathscr{U}_{\cup\omega_i}$, and

$$\mathrm{cap} \bigcup_{\omega_i} \leqq \| \textstyle\sum u_i \| \leqq \sum \| u_i \| = \sum \mathrm{cap}\, \omega_i \, .$$

REMARK. It is the last inequality in the above proof which makes our modified definition of capacity desirable.

LEMMA 6.2. *For any denumerable family of open subsets of X,*

$$\mathrm{cap} \bigcup_{i=1}^{\infty} \omega_i \leqq \sum_{i=1}^{\infty} \mathrm{cap}\, \omega_i \, .$$

Proof. Assume $\mathscr{U}_{\omega_i} \neq \varnothing$. Put $0_n = \bigcup_{i=1}^{n} \omega_n$, $n = 1, 2, \cdots$. Then $\{0_n\}$ is strictly increasing. If $\lim \mathrm{cap}\, 0_n = \infty$, then

$$\infty = \lim \mathrm{cap}\, 0_n = \lim_n \mathrm{cap} \bigcup_{i=1}^{n} \omega_i$$

$$\leqq \lim_n \sum_{i=1}^{n} \mathrm{cap}\, \omega_i = \sum_{i=1}^{\infty} \mathrm{cap}\, \omega_i \, ,$$

the inequality holds by Lemma 6.1. The result follows.

If $\lim \operatorname{cap} 0_n < \infty$, the hypotheses of Lemma 3.4 are satisfied by $\{0_n\}$. Thus,

$$\operatorname{cap} \bigcup_{i=1}^{\infty} \omega_i = \operatorname{cap} \bigcup_{n=1}^{\infty} 0_n = \lim \operatorname{cap} 0_n .$$

But

$$\lim \operatorname{cap} 0_n = \lim_n \operatorname{cap} \bigcup_{i=1}^{n} \omega_i$$
$$\leqq \lim_n \sum_{i=1}^{n} \operatorname{cap} \omega_i = \sum_{i=1}^{\infty} \operatorname{cap} \omega_i ,$$

the inequality holds by Lemma 6.1.

PROPOSITION 6.3. *The set function* cap_e *is denumerably subadditive, i.e., for a sequence of sets* $\{E_n\}_{n=1}^{\infty}$

$$\operatorname{cap}_e \bigcup_{n=1}^{\infty} E_n \leqq \sum_{n=1}^{\infty} \operatorname{cap}_e E_n .$$

Proof. For each n, choose an open $\omega_n \supset E_n$ with $\operatorname{cap} \omega_n \leqq \operatorname{cap}_e E_n + \varepsilon/2^n$, $\varepsilon > 0$ preassigned. Then

$$\operatorname{cap}_e \bigcup E_n \leqq \operatorname{cap} \bigcup \omega_n \leqq \sum \operatorname{cap} \omega_n ,$$

the first inequality holds since cap_e is increasing, the second by Lemma 6.2. By choice of ω_n, $\sum \operatorname{cap} \omega_n \leqq \sum \operatorname{cap}_e E_n + \varepsilon$.

7. Quasi-continuous functions; exterior capacity is a true capacity. In this section definitions and results which lead to Theorem 7.12 are listed. Several proofs are omitted, but using the earlier results in this article, proofs similar to those in [4] can readily be supplied.

DEFINITION 7.1. A function $f: X \rightarrow R$ is *quasi-continuous* if for each $\varepsilon > 0$ there exists an open $\omega \subset X$ with $\operatorname{cap} \omega < \varepsilon$ and the restriction $f|_{X-\omega}$ is continuous.

DEFINITION 7.2. A statement is true *quasi-everywhere* (quasi-everywhere on a subset $A \subset X$) if it is true for all $x \in X - E$ ($x \in A - E$) and $\operatorname{cap}_e E = 0$. The abbreviation is q.e. (q.e. on A).

By Corollary 5.6, q.e. implies a.e. ξ. It is emphasized that "q.e." depends not merely on the measure space (X, ξ), but on the function space $F(X, \xi)$. Examples are given in § 8.

PROPOSITION 7.3. *Let* $f: X \rightarrow R$ *be quasi-continuous,* $V \subset X$ *open, and* $a \in R$ *constant. Then* $f \leqq a$ a.e. ξ *on* V *implies* $f \leqq a$ q.e. *on* V.

The proof requires axioms (a) and (c), uniform convexity of F, and relies heavily on Theorem 3.5.

COROLLARY 7.4. *Let f, g be quasi-continuous functions. Then $f = g$ a.e. ξ implies $f = g$ q.e. Consequently, since q.e. always implies a.e. ξ, f and g are quasi-continuous representatives of the same element $u \in F$ iff $f = g$ q.e.*

Proof. Since $f = g$ a.e. ξ implies $f - g \leqq 0$ a.e. ξ and $g - f \leqq 0$ a.e. ξ, Proposition 7.3 gives $f - g \leqq 0$ q.e. and $g - f \leqq 0$ q.e., so $f = g$ q.e.

DEFINITION 7.5. An element $u \in F$ is *continuous* if u has a continuous representative.

LEMMA 7.6. *Let u denote any continuous representative of a continuous element of F and $\| u \|$ the norm of that element of F. Then for any $a > 0$,*

$$\operatorname{cap}\{x \mid u(x) > a > 0\} \leqq \| u \| / a .$$

NOTE. In [4], the right hand side of the above inequality is $\| u \|^2/a^2$, due to the different definition of capacity.

At this point axiom (b) is assumed. Consequently, from here on we will be concerned with a *uniformly convex BD space, $D(X, \xi)$.* Observe that because axiom (b) requires ξ to be dense in X, i.e., $\xi(\omega) > 0$ for nonvoid open ω, each continuous $u \in D$ has exactly one continuous representative, which is also denoted u.

PROPOSITION 7.7. *Every $u \in D$ has a quasi-continuous representative ($q = c$ rep).*

Proof. By axiom (b) there exists a sequence $\{u_k\}$ in $\mathscr{C} \cap D$ converging to u. By passing to a subsequence we may assume

$$(4) \qquad \sum_{k=1}^{\infty} 2^k \| u_{k+1} - u_k \| < +\infty .$$

For each $k = 1, 2, \cdots$ put

$$e_k = \{x \mid |u_{k+1}(x) - u_k(x)| > 2^{-k}\} .$$

By Lemma 7.6 and axiom (c)

$$\operatorname{cap} e_k \leqq 2^k \| \, |u_{k+1} - u_k| \, \| \leqq 2^k \| u_{k+1} - u_k \| .$$

Put $\omega_j = \bigcup_{k=j}^{\infty} e_k$. Then $\{\omega_j\}_{j=1}^{\infty}$ is a decreasing sequence of open sets and by (4)

$$\operatorname{cap} \omega_j \leqq \sum_{k=j}^{\infty} 2^k \| u_{k+1} - u_k \| \longrightarrow 0$$

as $j \to \infty$; $\operatorname{cap}_e (\bigcap \omega_j) = 0$ as a result, so $\xi(\bigcap \omega_j) = 0$.

Clearly $\{u_k(x)\}$ is a convergent sequence of reals for $x \in X - \bigcap_{j=1}^{\infty} \omega_j$. Put

$$u^*(x) = \begin{cases} \lim u_k(x) & \text{for } x \in X - \bigcap \omega_j \\ 0 \quad \text{for } x \in \bigcap \omega_j . \end{cases}$$

The convergence $u_k \to u^*$ is uniform on the complement of any ω_j, so u^* is continuous there; thus u^* is quasi-continuous since $\operatorname{cap} \omega_j \to 0$.

Finally, if u denotes any representative of $u \in D$, we show $u^* = u$ a.e. ξ. It suffices to show that for all j, $u^* = u$ a.e. ξ on $X - \omega_j$. To this end, let f be a bounded measurable function with compact $\mathscr{S}(f) \subset X - \omega_j$. Then

$$\int u f d\xi = \lim_k \int u_k f d\xi = \int u^* f d\xi ,$$

the first equality holds since $u_k \to u$ in D and $\int (\cdot) f d\xi \in D'$ by axiom (a); the second equality follows from the uniform convergence $u_k \to u^*$ on $X - \omega_j$. Thus $u = u^*$ a.e. ξ on $X = \omega_j$ and u^* is a $q = c$ rep of u.

LEMMA 7.8. *If u^* is any quasi-continuous representative of an element $u \in D$, then for any $a > 0$,*

$$\operatorname{cap}_e \{x \mid u^*(x) \geqq a > 0\} \leqq \| u \| / a .$$

PROPOSITION 7.9. *Let $\{v_n\}$ be a sequence in D converging to $v \in D$; let v_n^*, v^* be any quasi-continuous representatives of v_n, v respectively. Then there exists a subsequence $\{v_{n_k}^*\}$ of $\{v_n^*\}$ converging to v^* quasi-everywhere.*

LEMMA 7.10. *Let $E \subset X$ with $\operatorname{cap}_e E < +\infty$. If $u \in D$ has a quasi-continuous representative $u^* \geqq 1$ q.e. on E and $u \geqq 0$ a.e. ξ on X, then $u \in \mathscr{U}_E$ (Definition 5.1).*

Proof. Let $u \in D$ satisfy the hypothesis, and u^* be a $q = c$ rep of u. By adjusting u^* on a set of exterior capacity zero and observing that $\lim_n (1 + (1/n))u = u$ and $\mathscr{U}_E$ is closed, we see that the lemma will be proved if we assume $u^* > 1$ everywhere on E, $\geqq 0$ a.e. ξ on X, and show $u \in \mathscr{U}_E$.

Let $\varepsilon > 0$, $\omega_\varepsilon \subset X$ open with $\operatorname{cap} \omega_\varepsilon < \varepsilon$ and $u^* |_{X - \omega_\varepsilon}$ continuous. Consider the open set

$$\Omega_\varepsilon = \{x \mid u^*(x) > 1\} \bigcup \omega_\varepsilon ,$$

and the capacitary element v_ε associated with ω_ε. Since $v_\varepsilon \geqq 1$ a.e. ξ on ω_ε and by axiom (c) $v_\varepsilon \geqq 0$ a.e. ξ on X, it follows that $u + v_\varepsilon \geqq 1$ a.e. ξ on Ω_ε, i.e., $u + v_\varepsilon \in \mathscr{U}_{\Omega_\varepsilon}$. But $\Omega_\varepsilon \supset E$ so $u + v_\varepsilon \in \mathscr{U}_E$. Letting $\varepsilon \to 0$ we have $\| v_\varepsilon \| = \operatorname{cap} \omega_\varepsilon \to 0$ so $u = \lim_\varepsilon u + v_\varepsilon \in \mathscr{U}_E$.

LEMMA 7.11. *Let $E \subset X$ with $\operatorname{cap}_e E < \infty$. Then u_E, the exterior capacitary element associated with E, is $\geqq 0$ a.e. ξ on X. Moreover, any quasi-continuous representative verifies $u_E^* \geqq 1$ q.e. on E.*

Proof. By axiom (c), the capacitary element associated with an open set is $\geqq 0$ a.e. ξ. Thus in the notation of Lemma 5.5, $\lim u_n = u_E$ and $u_n \geqq 0$ a.e.ξ. Axiom (a) assures that the cone of nonnegative elements in D is closed. Therefore, $u_E \geqq 0$ a.e. ξ.

Let u_E^*, u_n^* be $q = c$ reps of u_E, u_n respectively, $n = 1, 2, \cdots$. By Proposition 7.3, $u_n^* \geqq 1$ a.e. ξ on ω_n implies $u_n^* \geqq 1$ q.e. on ω_n. Proposition 7.9 implies $u_E^*(x) = \lim_k u_{n_k}^*(x)$ q.e., so $u_E^* \geqq 1$ q.e. on $\bigcap_{k=1}^\infty \omega_{n_k} \supset E$.

THEOREM 7.12. *The set function* cap_e *is a true capacity.*

Proof. In view of Propositions 4.3 and 4.6, it remains to show that cap_e verifies condition (ii) for a true capacity (see § 4). Let a sequence $\{E_n\}$ of sets verify $E_n \subset E_{n+1}$, $n = 1, 2, \cdots$ and put $E = \bigcup_{n=1}^\infty E_n$. Clearly $\operatorname{cap}_e E \geqq \lim_n \operatorname{cap}_e E_n$. We prove the reverse inequality.

If $\lim_n \operatorname{cap}_e E_n = +\infty$, equality holds. Assume $\operatorname{cap}_e E_n \leqq M < +\infty$ for $n = 1, 2, \cdots$. Lemma 7.11 assures $u_n \geqq 0$ a.e. ξ where u_n denotes the exterior capacitary element associated with E_n. Put $\mathscr{U} = \bigcap_{n=1}^\infty \mathscr{U}_{E_n}$. Since $\{E_n\}$ is increasing, $\{\mathscr{U}_{E_n}\}$ is decreasing; further $\| u_n \| = \operatorname{cap}_e E_n \leqq M$. Thus Lemma 1.3 applies: $u = \lim u_n$ is the unique element of minimum norm of $\mathscr{U}$. Observe that $u \geqq 0$ a.e. ξ since $u_n \geqq 0$ a.e. ξ. If u_n^* is a $q = c$ rep of u_n, by applying Lemma 7.11 and adjusting u_n^* on a set of exterior capacity zero, we assume $u_n^* \geqq 1$ everywhere on E_n. Proposition 7.9 assures $u^* = \lim_k u_{n_k}^*$ q.e., so $u^* \geqq 1$ q.e. on $E = \bigcup_k E_{n_k}$. Thus by Lemma 7.10, $u \in \mathscr{U}_E$, so

$$\operatorname{cap}_e E = \inf \{\| v \| \mid v \in \mathscr{U}_E\} \leqq \| u \| = \lim \operatorname{cap}_e E_n .$$

8. Conditions under which $\xi(E) = 0$ implies $\operatorname{cap}_e (E) = 0$. In this section and the next we investigate the nature of sets of exterior capacity zero. In this section a connection is made with quasi-continuous functions; § 9 deals with measures $\mu \in D'$. The results of this section were motivated in part by the work of Thomas [7]; it is not hard to show that the union of the equivalence classes of a space $F(X, \xi)$ which is reflexive and satisfies axioms (a) and (c) forms a semi-norm space $\mathscr{E}(X, p)$ of [7].

To emphasize that "$\text{cap}_e E = 0$" depends on the functions in $D(X, \xi)$ and not merely on ξ, we give two brief examples of BD spaces over the same measure space but with diverse notions of capacity. In both examples $X = (0, 1)$, ξ is Lebesgue measure, and $1 < p < \infty$.

EXAMPLE 8.1. $D = L^p(X, \xi)$. We show that $\xi E = 0$ implies $\text{cap}_e E = 0$ for $E \subset (0, 1)$. In fact, given $\xi E = 0$, cover E with an open ω verifying $\xi\omega < \varepsilon$, $\varepsilon > 0$ preassigned. The indicator I_ω is an L^p function, and clearly

$$\text{cap}\, \omega = \left(\int I_\omega{}^p d\xi\right)^{1/p} < \varepsilon^{1/p} .$$

Thus, $\text{cap}_e E = \inf \{\text{cap}\, \omega \mid \omega \supset E, \omega \text{ open}\} = 0$.

EXAMPLE 8.2. $D(X, \xi)$ is the space of equivalence classes of R-valued functions on $(0, 1)$, each class containing an absolutely continuous representative satisfying $\lim_{x\to 0} u(x) = \lim_{x \to 1} u(x) = 0$. The norm is defined by

$$\| u \|^p = \int | u' |^p d\xi < \infty .$$

Here the prime denotes derivative which may be taken in the ordinary sense in the case of the absolutely continuous representatives, or generally taken in the sense of distributions.

That D is a BD space is an easy exercise. We show any open interval $\omega = (a, b)$ with $0 < a < b < 1$ verifies $\text{cap}\, \omega > 2^{1/p}$, from which it follows that $\text{cap}_e E \geqq 2^{1/p}$ for all nonvoid $E \subset (0, 1)$. In fact, let $u \in \mathscr{U}_\omega$, i.e., $u \in D$, $u \geqq 1$ on (a, b). (Here we are actually considering the absolutely continuous representative of u.) Then

$$\| u \|^p = \int_0^1 | u' |^p \, d\xi \geqq \int_0^a | u' |^p \, d\xi + \int_b^1 | u' |^p \, d\xi .$$

By Hölder's inequality, $(q = p/p - 1)$

$$\left(\int_0^a | u' |^p \, d\xi\right)^{1/p} \geqq a^{-1/q} \int_0^a | u' | \, d\xi = a^{-1/q} V_0^a(u) \geqq a^{-1/q} > 1 .$$

Here the variation $V_0^a(u) \geqq 1$ since $u(a) \geqq 1$ and $\lim_{x\to 0} u(x) = 0$. Similarly $\int_b^1 | u' | \, d\xi > 1$. Thus $\| u \| > 2^{1/p}$, so $\text{cap}\, \omega > 2^{1/p}$.

It is clear from this example that $\xi E = 0$ does *not* generally imply $\text{cap}_e E = 0$. Conversely, Corollary 5.6 assures $\text{cap}_e E = 0$ does entail $\xi E = 0$. The next proposition gives one condition under which $\xi E = 0$ and $\text{cap}_e E = 0$ are equivalent. We consider a uniformly convex BD space $D(X, \xi)$, and the indicator I_E for $E \subset X$. (In Propositions 8.3 and 8.4 it is assumed that $\xi(E \cap K) = 0$ for all compact K implies $\xi E = 0$; thus Corollary 5.6 applies.)

PROPOSITION 8.3. *If I_E is quasi-continuous, then $\xi E = 0$ iff $\text{cap}_e E = 0$.*

Proof. Assume I_E quasi-continuous and $\xi E = 0$. Then I_E is a $q = c$ rep of $0 \in D$. Since the null function is a $q = c$ rep of $0 \in D$, Corollary 7.4 assures $I_E = 0$ q.e., so $\text{cap}_e E = 0$. Corollary 5.6 gives the converse.

As the following result indicates, in 8.1 all representatives of all elements in the space are quasi-continuous, but in 8.2 the only $q = c$ reps are the absolutely continuous representatives of each element.

PROPOSITION 8.4. *In order that for all $E \subset X$ $\xi E = 0$ implies $\text{cap}_e E$, it is necessary and sufficient that all representatives of all elements of D be quasi-continuous.*

Proof. Necessity. Assume $\xi E = 0$ implies $\text{cap}_e E = 0$. Let u and u^* be two representatives of the same element of D, u^* quasi-continuous (u^* exists by Proposition 7.7). Put $E = \{x \mid u(x) \neq u^*(x)\}$. We have $\xi E = 0$ so the hypothesis entails $\text{cap}_e E = 0$. Thus, $u = u^*$ q.e., so u is quasi-continuous because u^* is.

Sufficiency. If $\xi E = 0$, then I_E is a representative of $0 \in D$, so by hypothesis I_E is quasi-continuous. Proposition 8.3 applies.

9. Sets of exterior capacity zero and pure potentials. It has been shown in previous sections that, roughly speaking, sets of zero capacity are smaller than sets of ξ-measure zero: $\text{cap}_e E = 0$ implies $\xi E = 0$. In this section we consider the question "how small is a set of zero exterior capacity?" More precisely, we give the following analog to a classical result: $\text{cap}_e E = 0$ iff E is cap_e-capacitable and $\mu E = 0$ for all pure potentials u^μ. It is assumed $D(X, \xi)$ is a uniformly convex BD space.

For any open $\omega \subset X$, the characteristic function I_ω is lower semicontinuous. Consequently, for any Radon measure $\mu \geqq 0$, we have by definition

$$\mu(\omega) = \mu^*(I_\omega) = \sup \{\mu(\varphi) \mid \varphi \in \mathscr{C}, \varphi \leqq I_\omega\} .$$

The next lemma shows that the supremum can be taken over a smaller set. We use the normal contraction $T_\varepsilon: R \to R$ defined for $\varepsilon > 0$ by $T_\varepsilon(x) = x - \varepsilon$ if $x \geqq \varepsilon$, $T_\varepsilon(x) = x + \varepsilon$ if $x \leqq -\varepsilon$ and $T(x) = 0$ if $|x| < \varepsilon$.

LEMMA 9.1. *For any open $\omega \subset X$ and any Radon measure $\mu \geqq 0$*

$$\mu(\omega) = \sup \{\mu(\varphi) \mid \varphi \in \mathscr{C} \cap D, 0 \leqq \varphi \leqq I_\omega, \mathscr{S}(\varphi) \subset \omega\} .$$

Proof. Let $\Gamma = \{\varphi \mid \varphi \in \mathscr{C} \cap D, 0 \leqq \varphi \leqq I_\omega, \mathscr{S}(\varphi) \subset \omega\}$. It suffices to show that Γ is upward directed and $I_\omega = \sup \Gamma$ (see, for example, [5], Proposition 4. 5. 1). That Γ is upward directed is immediate: $\varphi, \psi \in \Gamma$ implies $\varphi \vee \psi = 1/2(\varphi + \psi + |\varphi - \psi|) \in \Gamma$ because $\mathscr{C} \cap D$ is a vector space closed under normal contractions (axiom (c)).

To see $I_\omega = \sup \Gamma$, let $p \in \omega$, $\psi \in \mathscr{C}$ with $\psi(p) = 1, 0 \leqq \psi \leqq 1$ on X and $\mathscr{S}(\psi) \subset \omega$. Such a ψ exists since $\{p\}$ is compact and X is locally compact. Since $\mathscr{C} \cap D$ is dense in $\mathscr{C}$, given $\varepsilon, 0 < \varepsilon < 1/2$, there exists $\varphi \in \mathscr{C} \cap D$ with $|\varphi - \psi| < \varepsilon$ on X. An easy calculation shows $T_\varepsilon \varphi \in \Gamma$. Finally,

$$\begin{aligned} 1 - T_\varepsilon\varphi(p) &= \psi(p) - T_\varepsilon\varphi(p) \\ &= \psi(p) - \varphi(p) + \varepsilon < 2\varepsilon . \end{aligned}$$

Letting ε tend to 0,

$$I_\omega(p) = \sup \{\varphi(p) \mid \varphi \in \Gamma\} ,$$

so $I_\omega = \sup \Gamma$.

For open $\omega \subset X$, recall $P_\omega \subset D'$ given in Definition 3.2.

LEMMA 9.2. *For any open $\omega \subset X$ and $\mu \geqq 0$ for which $u^\mu \in P_\omega$, $\mathscr{S}(\mu) \subset \bar{\omega}$ holds.*

Proof. We show that any $p \in X - \bar{\omega}$ has a μ-negligible neighborhood. Let U be an open neighborhood of p not meeting $\bar{\omega}$. Let $\varphi \in \mathscr{C} \cap D$ with $\varphi \leqq I_U$, $\mathscr{S}(\varphi) \subset U$. By definition of $u^\mu \in P_\omega$, there exists a sequence f_n of bounded measurable functions supported by ω such that

$$\int \varphi d\mu = \lim_n \int \varphi f_n d\xi = 0 .$$

Lemma 9.1 applied to U gives $\mu(U) = 0$.

THEOREM 9.3. *Let $E \subset X$. The following two conditions are equivalent:*

(i) *E is* cap_e*-capacitable and* $\mu E = 0$ *for every Radon measure* $\mu \geqq 0$ *generating a pure potential* $u^\mu \in D'$.

(ii) $\mathrm{cap}_e E = 0$.

Proof. (i) implies (ii). First, we prove the contrapositive for compact E. Suppose $\mathrm{cap}_e E > 0$, i.e., for some $\alpha > 0$, $\mathrm{cap}\,\omega \geqq \alpha$ for every open $\omega \supset E$. For every such ω

$$\alpha \leqq \mathrm{dualcap}\,\omega = 1/\inf\{\|z\| \mid z \in P_\omega\}$$

and therefore $1/\alpha \geqq \inf\{\|z\| \mid z \in P_\omega\}$. Now P_ω is convex and closed in D' and $\omega \subset \Omega$ implies $P_\omega \subset P_\Omega$ for open Ω. Thus $\{P_\omega \mid \omega \supset E,\ \omega \text{ open}\}$ satisfies the hypothesis of Lemma 1.3 (i) (here D' is reflexive because D is uniformly convex), so there exists $z_0 \in \bigcap_{\omega \supset E} P_\omega$.

Now $z_0 \neq 0$. In fact, E is compact so some open $\omega \supset E$ is relatively compact. Therefore, since $z_0 \in P_\omega$,

$$+\infty > \mathrm{cap}\,\omega = \mathrm{dualcap}\,\omega \geqq 1/\|z_0\|\ ,$$

so $\|z_0\| \geqq 1/\mathrm{cap}\,\omega > 0$. Thus since $z_0 \in P_\omega$, it follows that $z_0 = u^\mu$ for some Radon measure $\mu > 0$. We show $\mu E \geqq 1$. That $\mathscr{C} \cap D$ is dense in $\mathscr{C}$ assures the existence of $\varphi \in \mathscr{C} \cap D$ verifying $0 \leqq 1 - \varphi < \varepsilon$ on the compact $\bar{\omega}$. Let $U \subset X$ be open with

$$E \subset U \subset \bar{U} \subset \omega\ .$$

Such U exists because E is compact and X is locally compact Hausdorff. For all $u^f \in P_\omega$ with $0 \leqq f$ measurable, bounded and $\mathscr{S}(f) \subset U$, we have

$$\int_U \varphi f d\xi \geqq (1 - \varepsilon) \int_U f d\xi = 1 - \varepsilon\ ,$$

since $\varphi \geqq 1 - \varepsilon$ on $\bar{\omega}$. By definition of $z_0 = u^\mu \in P_U$, there exists a sequence $\{f_n\}$ of such functions so that

$$1 - \varepsilon \leqq \lim_n \int_U \varphi f_n d\xi = \int_{\bar{U}} \varphi d\mu$$

by Lemma 9.2, thus

$$1 - \varepsilon \leqq \int_\omega \varphi d\mu \leqq \int I_\omega d\mu = \mu(\omega)\ .$$

Therefore,

$$1 - \varepsilon \leqq \inf\{\mu(\omega) \mid E \subset \omega \text{ open}\} = \mu(E)$$

so $\mu E \geqq 1$, and our result holds for compact sets E.

For the general case, E capacitable means

$$\mathrm{cap}_e\, E = \sup\{\mathrm{cap}_e\, K \mid E \supset K \text{ compact}\} = 0$$

since $\mu(E) = 0$ assures $\mu(K) = 0$ for $K \subset E$.

(ii) implies (i). For any open $\omega \subset X$ and any pure potential u^μ, we establish the inequality

$$\mu(\omega) \leqq \| u^\mu \| \,\mathrm{cap}\, \omega \; . \tag{7}$$

In fact, by Lemma 9.1,

$$\mu(\omega) = \sup\{\mu(\varphi) \mid \varphi \in \mathscr{C} \cap D,\ 0 \leqq \varphi \leqq I_\omega,\ \mathscr{S}(\varphi) \subset \omega\} \; .$$

If $\mathrm{cap}\, \omega = +\infty$, the inequality holds. Assume $\mathrm{cap}\, \omega < +\infty$, let $u \in D$ denote the capacitary element associated with ω; then $I_\omega \leqq u$ a.e. ξ. Since u^μ is a positive form on D, we have for any $\varphi \in \Gamma$ (see Lemma 9.1),

$$\begin{aligned} \mu(\varphi) &= (\varphi, u^\mu) \leqq (u, u^\mu) \\ &\leqq \| u \| \, \| u^\mu \| = \| u^\mu \| \,\mathrm{cap}\, \omega \; . \end{aligned}$$

Taking the supremum over all $\mu(\varphi)$, $\varphi \in \Gamma$, inequality (7) is proved.

Now, assume $\mathrm{cap}_e\, E = 0$. Since $\mathrm{cap}_e\, K = 0$ for all compact $K \subset E$, it is immediate that E is capacitable. By definition,

$$0 = \mathrm{cap}_e\, E = \inf\{\mathrm{cap}\, \omega \mid E \subset \omega \text{ open}\} \; .$$

By (7)

$$\begin{aligned} \mu(E) &= \inf\{\mu(\omega) \mid E \subset \omega \text{ open}\} \\ &\leqq \| u^\mu \| \inf\{\mathrm{cap}\, \omega \mid E \subset \omega \text{ open}\} = 0 \end{aligned}$$

for all pure potentials u^μ.

10. Quasi-continuous representatives and pure potentials. In this section we indicate that by considering only the quasi-continuous representatives from each equivalence class $[u] \in D$, we get a "refined" space of equivalence classes of functions, the new equivalence relation being equality q.e. rather than equality a.e. ξ. An application of Theorems 9.3 and 10.1 give the important Corollary 10.2. Every representative in the "refined" space is measurable and summable with respect to any measure generating a pure potential and the "correct" integral formula holds. Our measure theoretic notation follows that of [5, §§ 4.5, 4.6].

THEOREM 10.1. *Every $u \in D$ has a quasi-continuous representative u^* such that*

(i) *there exists some σ-compact subset of X outside of which u^* vanishes, and*

(ii) *for every pure potential* $u^{\mu} \in D'$, *we have* $u^* \in \mathscr{L}^1(X, \mu)$ *and*

$$(u, u^{\mu}) = \int u^* d\mu .$$

Proof. Refer to Proposition 7.7; we show that u^* constructed in that proof verifies (i) and (ii). We have

$$e_k = \{x \in X \mid u_{k+1}(x) - u_k(x) \mid > 2^{-k}\}$$

and

$$\begin{aligned}\mu(e_k) &\leqq \mu(2^k \mid u_{k+1} - u_k \mid) = 2^k(\mid u_{k+1} - u_k \mid, u^{\mu}) \\ &\leqq 2^k \| \mid u_{k+1} - u_k \mid \| \cdot \| u^{\mu} \| \leqq 2^k \| u_{k+1} - u_k \| \cdot \| u^{\mu} \| ,\end{aligned} \tag{8}$$

where u^{μ} is an arbitrary pure potential. The last quantity tends to zero as k increases because

$$\sum_{k=1}^{\infty} 2^k \| u_{k+1} - u_k \| < + \infty . \tag{9}$$

Further, since $\omega_j = \bigcup_{k=j}^{\infty} e_k$, we have $\mu(\omega_j) \leqq \sum_{k=j}^{\infty} \mu(e_k)$ which tends to zero as j increases by (8) and (9). Thus $\mu(\bigcap_{j=1}^{\infty} \omega_j) = 0$. Also

$$u^*(x) = \begin{cases}\lim_k u_k(x) & \text{for } x \in X - \bigcap \omega_j \\ 0 & \text{for } x \in \bigcap \omega_j .\end{cases}$$

Hence,

$$\{x \mid u^*(x) \neq 0\} \subset \bigcup_{k=1}^{\infty} \{x \mid u_k(x) \neq 0\} \subset \bigcup_{k=1}^{\infty} \mathscr{S}(u_k)$$

which establishes (i).

For (ii), let $E = \bigcap_{j=1}^{\infty} \omega_j$; $\mu E = 0$. Then

$$\mu^*(\mid u^* - u_k \mid) \leqq \mu^*(\mid u^* - u_k \mid \cdot I_{X-E}) + \mu^*(\mid u^* - u_k \mid \cdot I_E)$$

because the upper integral $\mu^*(\cdot)$ is sub-additive. But $\mu E = 0$, so $\mu^*(\mid u^* - u_k \mid \cdot I_E) = 0$. Thus

$$\begin{aligned}\mu^*(\mid u^* - u_k \mid) &\leqq \mu^*(\lim_j \mid u_j - u_k \mid \cdot I_{X-E}) \\ &\leqq \mu^*(\lim_j \mid u_j - u_k \mid) \leqq \underline{\lim}_j \, \mu^*(\mid u_j - u_k \mid)\end{aligned}$$

by Fatau's lemma. But $\mid u_j - u_k \mid \in \mathscr{C} \cap D$, so $\mu^*(\mid u_j - u_k \mid) = \mu(\mid u_j - u_k \mid)$. Therefore,

$$\begin{aligned}\mu^*(\mid u^* - u_k \mid) &\leqq \underline{\lim}_j \, \mu(u_j - u_k) \\ &= \underline{\lim}_j(\mid u_j - u_k \mid, u^{\mu}) \leqq \underline{\lim}_j \| u_j - u_k \| \cdot \| u^{\mu} \|\end{aligned}$$

which tends to zero as k increases because $\{u_k\}$ is Cauchy in D. Thus $u^* \in \mathscr{L}^1(X, \mu)$ and

$$\int u^* d\mu = \lim_k \int u_k d\mu = \lim_k (u_k, u^\mu) = (u, u^\mu) .$$

COROLLARY 10.2. *Every quasi-continuous representative v of every $u \in D$ verifies $v \in \mathscr{L}^1(X, \mu)$ and $(u, u^\mu) = \int v d\mu$ for every pure potential u^μ.*

Proof. Let u^* be as in Theorem 10.1. Then $u^* = v$ a.e. ξ and both are quasi-continuous. Thus Corollary 7.4 implies that $u^* = v$ q.e. Let

$$E = \{x \in X \mid u^*(x) \neq v(x)\} ;$$

then $\text{cap}_e\, E = 0$. According to Theorem 9.3, $\mu E = 0$, so

$$\int v d\mu = \int u^* d\mu = (u, u^\mu) .$$

REMARK. The theorem and corollary give a very strong result. For an arbitrary representative $\tilde{u}$ of u, the formula

$$(u, u^\mu) = \int \tilde{u} d\mu$$

does *not* hold in general unless μ is absolutely continuous with respect to ξ. However, we can select any quasi-continuous representative u^* of u and the formula does hold, not just for one μ, but for all μ simultaneously.

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