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COTORSION THEORIES

STEPHEN D. BRONN

In this paper A is a ring with unit, and Mod-A denotes the category of unitary right A-modules. The aim of the paper is to dualize the concept of torsion and develop the corresponding idea of cotorsion.

One generalization of torsion was given by Goldman, using what he called a kernel functor. These kernel functors are here dualized to give cokernel functors. Cokernel functors are categorized over Mod-A.

The final section investigates what information the cotorsion functors can reveal about the homological properties of the rings under discussion.

1. Definition. An I-functor is a pair (F, λ) where F is an additive covariant functor from Mod-A to Mod-A and λ is a natural transformation from the identity functor on Mod-A to F.

Thus if M and N are A-modules and $f \in \mathrm{Hom}_A(M, N)$ we have the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle\lambda_M}\downarrow & & \downarrow{\scriptstyle\lambda_N} \\ F(M) & \xrightarrow{F(f)} & F(N)\,. \end{array}$$

That is $\lambda_N f = F(f)\lambda_M$.
An A-module M is said to be:

(i) F-reduced if λ_M is a monomorphism.
(ii) F-divisible if $\lambda_M = 0$.
(iii) F-cotorsion if λ_M is an isomorphism.
(iv) F-d-strong if D_M = cokernel of λ_M is F-divisible.

In addition the I-functor (F, λ) is said to be:

(a) epi if λ_M is an epimorphism for every $M \in$ Mod-A.
(b) idempotent if $F(M)$ is F-cotorsion for every $M \in$ Mod-A.
(c) restricted idempotent if $F(M)$ is F-cotorsion whenever M is F-reduced.
(d) d-strong if every $M \in$ Mod-A is F-d-strong.

The cotorsion completion functor of Matlis [4] is an example of a d-strong I-functor. This I-functor is idempotent if and only if the homological dimension of Q (A is an integral domain and Q is the quotient field of A in this case) is one as an A-module.

If A is a commutative ring and S is a multiplicatively closed set of elements from A then the localization of every module at S is an

I-functor. If every element of S is a nonzero divisor this I-functor is idempotent and d-strong.

The following proposition follows directly from the definitions.

PROPOSITION 1.1. *Let (F, λ) be an I-functor.*

(a) *Every F-cotorsion module is F-reduced.*

(b) *Every submodule of an F-reduced module is also F-reduced.*

(c) *Every quotient module of an F-divisible module is also F-divisible.*

(d) $\operatorname{Hom}_A(M, N) = 0$ *whenever M is F-divisible and N is F-reduced.*

(e) *The additive condition is unnecessary if (F, λ) is an epi I-functor or if (F, λ) is idempotent and d-strong.*

PROPOSITION 1.2. *Let (F, λ) be an I-functor and M be an F-d-strong A-module. For every A-module N we denote by β_N the group homorphism from* $\operatorname{Hom}_A(F(M), N)$ *to* $\operatorname{Hom}_A(M, N)$ *defined by composition with λ_M.*

(a) *If N is F-reduced β_N is a monomorphism.*

(b) *If N is F-cotorsion β_N is an isomorphism.*

Proof. (a) Suppose that N is F-reduced and that g is in the kernel of β_N. Thus $g\lambda_M = 0$. Let $u_M: F(M) \to D_M$ be the cokernel of λ_M. There exists $h \in \operatorname{Hom}_A(D_M, N)$ such that $hu_M = g$. By 1.1

$$\operatorname{Hom}_A(D_M, N) = 0$$

and therefore $h = 0$. Hence $g = 0$ and so β_N is a monomorphism.

(b) By the preceding part we need only show that β_N is onto if N is F-cotorsion. Let $g \in \operatorname{Hom}_A(M, N)$, since N is F-cotorsion λ_N has an inverse λ_N^{-1}. Let $h = \lambda_N^{-1}F(g)$. Now $h\lambda_M = \lambda_N^{-1}F(g)\lambda_M = \lambda_N^{-1}\lambda_N g = g$ hence β_N is onto.

PROPOSITION 1.3. *Let J be a directed set and B_i, $i \in J$, be a family of A-modules indexed by J. Whenever (F, λ) is an I-functor on* Mod-A *then:*

(a) $\varprojlim_{i \in J} B_i$ *is F-reduced if each B_i, $i \in J$, is F-reduced.*

(b) $\varinjlim_{i \in J} B_i$ *is F-divisible if each B_i, $i \in J$, is F-divisible.*

(c) $\varprojlim_{i \in J} B_i$ *is F-cotorsion if each B_i, $i \in J$, is F-cotorsion and if (F, λ) is d-strong and restricted idempotent.*

Proof. Let $M = \varprojlim_{j \in J} B_j$ and $N = \varinjlim_{j \in J} B_j$ with respect to the defining homomorphisms $P_i: M \to B_i$ and $q_i: B_i \to N$ for $i \in J$.

(a) Suppose that each B_i, $i \in J$, is F-reduced. If $X \in \text{Mod-}A$ and $h \in \mathrm{Hom}_A(X, M)$ such that $\lambda_M h = 0$ then $0 = F(P_i)\lambda_M h = \lambda_{B_i} P_i h$ for each $i \in J$. But λ_{B_i} is a monomorphism thus $P_i h = 0$ for each $i \in J$ and hence $h = 0$ which means that λ_M is a monomorphism.

(b) Suppose that each B_i, $i \in J$, is F-divisible. Now $\lambda_N q_i = F(q_i)\lambda_{B_i} = 0$ for each $i \in J$ and therefore $\lambda_N = 0$.

(c) Suppose that each B_i, $i \in J$, is F-cotorsion and that (F, λ) is d-strong and restricted idempotent. Thus λ_{B_i} has an inverse $\lambda_{B_i}^{-1}$ for each $i \in J$ and so there exists $h \in \mathrm{Hom}_A(F(M), M)$ such that $P_i h = \lambda_{B_i}^{-1} F(P_i)$ for each $i \in J$. Now $P_i h \lambda_M = \lambda_{B_i}^{-1} F(P_i)\lambda_M = \lambda_{B_i}^{-1}\lambda_{B_i} P_i = P_i$ for each $i \in J$ and thus $h\lambda_M = 1_M$. By (a) M is F-reduced and since (F, λ) is restricted idempotent it follows that $F(M)$ is F-cotorsion and thus by 1.2 $\beta_{F(M)}$ is an isomorphism. Since $\lambda_M h \lambda_M = \lambda_M 1_M = \lambda_M = 1_{F(M)}\lambda_M$ it follows that $\lambda_M h = 1_{F(M)}$ and thus λ_M is an isomorphism and M is F-cotorsion.

We now make a definition which allows us to compare I-functors.

Definition. If (F, λ) and (G, α) are I-functors on Mod-A and μ is a natural transformation from F to G such that $\mu\lambda = \alpha$ we say that μ is an I-morphism. If in addition μ_M is an isomorphism for each $M \in \text{Mod-}A$ we say that μ is an I-isomorphism and that (F, λ) and (G, α) are equivalent I-functors.

Theorem 1.4. *Let (F, λ) and (G, α) be I-functors on* Mod-A *where (F, λ) is d-strong and $G(M)$ is F-cotorsion for every* $M \in \text{Mod-}A$. *There exists an I-morphism μ from (F, λ) to (G, α).*

Proof. Let $M \in \text{Mod-}A$, now $G(M)$ is F-cotorsion so by 1.2 there exists a unique $\mu_M \in \mathrm{Hom}_A(F(M), G(M))$ such that $\mu_M\lambda_M = \alpha_M$. Suppose now that $f \in \mathrm{Hom}_A(M, N)$. Thus $\mu_N F(f)\lambda_M = \mu_N\lambda_N f = \alpha_N f = G(f)\alpha_M = G(f)\mu_M\lambda_M$. But $G(N)$ is F-cotorsion hence by 1.2 $\mu_N F(f) = G(f)\mu_M$ which means that μ is an I-morphism.

2. The purpose of this section is to show that $F(A)$ is a ring for most I-functors (F, λ).

Theorem 2.1. *Let (F, λ) be an I-functor on* Mod-A *such that A is F-d-strong and $F(A)$ is F-cotorsion.*

(a) *$F(A)$ is a ring with unit and λ_A is a ring homomorphism.*

(b) *Every F-cotorsion module* $M \in \text{Mod-}A$ *is also a right $F(A)$-module.*

(c) *Whenever M and N are right $F(A)$-modules and N is F-reduced as a right A-module then* $\mathrm{Hom}_A(M, N) = \mathrm{Hom}_{F(A)}(M, N)$.

(d) *$F(A)$ is commutative if A is commutative.*

Proof. Let M be any F-cotorsion right A-module and let $x \in M$. Define $u_x \in \operatorname{Hom}_A(A, M)$ by $u_x(r) = xr$ for every $r \in A$. By 1.2 there exists $w_x \in \operatorname{Hom}_A(F(A), M)$ such that $w_x\lambda_A = u_x$.

(i) Clearly $u_x + u_y = u_{x+y}$ and so by 1.2 $w_x + w_y = w_{x+y}$ for every $x, y \in M$.

(ii) Let $x \in M, s \in F(A)$ and set $y = w_x(s)$. $w_x w_s \lambda_A(r) = w_x(sr) = w_x(s)r = yr = u_y(r) = w_y\lambda_A(r)$ for every $r \in A$. Thus by 1.2 $w_x w_s = w_y$.

Now $F(A)$ is F-cotorsion so by (i) and (ii) $F(A)$ becomes a ring under the multiplication $xy = w_x(y)$ where $x, y \in F(A)$. By the same taken M is a right $F(A)$-module.

If $r, s \in A$ let $x = \lambda_A(r)$, then $\lambda_A(rs) = \lambda_A(r)s = xs = u_x(s) = w_x\lambda_A(s) = \lambda_A(r)\lambda_A(s)$ and therefore λ_A is a ring homomorphism. Clearly $\lambda_A(1)$ is the unit of $F(A)$.

Suppose now that M and N are right $F(A)$-modules and that N is F-reduced when considered as a right A-module. Let $f \in \operatorname{Hom}_A(M, N)$ and $x \in M$. Define $h, g \in \operatorname{Hom}_A(F(A), N)$ by $g(s) = f(x)s$ and $h(s) = f(xs)$ for $s \in F(A)$. It is easily seen that $g\lambda_A = h\lambda_A$ so by 1.2 $g = h$. That is f is a right $F(A)$-module homomorphism and so $\operatorname{Hom}_A(M, N) = \operatorname{Hom}_{F(A)}(M, N)$.

Now assume that A is commutative. Let $r \in A, x \in F(A)$ and set $y = \lambda_A(r)$. Define $g \in \operatorname{Hom}_A(F(A))$ by $g(s) = s\lambda_A(r) = sy = w_s(y) = u_s(r) = sr$ for $s \in F(A)$. Now $g\lambda_A = w_y\lambda_A$ and so by 1.2 $g = w_y$ and therefore $\lambda_A(r)s = s\lambda_A(r)$ for every $r \in A, s \in F(A)$.

Define $h_x \in \operatorname{Hom}_A(F(A), F(A))$ by $h_x(s) = sx - xs$ where $x \in F(A)$. Now $h_x\lambda_A = 0$ by the previous paragraph and so by 1.2 $h_x = 0$ which means that $F(A)$ is commutative. This completes the proof of the theorem.

DEFINITION. Let (F, λ) be an I-functor such that A is F-d-strong and $F(A)$ is F-cotorsion. By 2.1 $F(A)$ is a ring with unit $\lambda_A(1)$ where 1 is the unit of A. We define a new I-functor $(\bar{F}, \bar{\lambda})$ on Mod-A by $F(M) = M \otimes_A F(A)$ for every $M \in$ Mod-A and $\bar{\lambda}(y) = y \otimes \lambda_A(1)$ for every $y \in M$.

THEOREM 2.2. *Let (F, λ) be an idempotent, d-strong I-functor on* Mod-A. *(F, λ) and $(\bar{F}, \bar{\lambda})$ are equivalent I-functors on* Mod-A *if and only if $\bar{F}(M)$ is F-cotorsion for every module* $M \in$ Mod-A.

Proof. If (F, λ) and $(\bar{F}, \bar{\lambda})$ are equivalent I-functors then $F(M)$ and $\bar{F}(M)$ are isomorphic for every $M \in$ Mod-A. But $F(M)$ is F-cotorsion and thus $\bar{F}(M)$ is F-cotorsion.

Conversely suppose that $\bar{F}(M)$ is F-cotorsion for every $M \in$ Mod-A. By 1.4 there exists an I-morphism μ from (F, λ) to $(\bar{F}, \bar{\lambda})$. By 2.1 $F(M)$ is a right $F(A)$-module for every module $M \in$ Mod-A. Thus there exists $\alpha_M \in \mathrm{Hom}_A(\bar{F}(M), F(M))$ such that $\alpha_M(y \otimes s) = \lambda_M(y)s$ for every $y \in M, s \in F(A)$. Now $\alpha_M \mu_M \lambda_M = \lambda_M$ and $F(M)$ is F-cotorsion thus by 1.2 $\alpha_M \mu_M = 1_{F(M)}$ for every $M \in$ Mod-A.

Let $y \in M$ and $s \in F(A)$ $\mu_M \alpha_M(y \otimes s) = \mu_M(\lambda_M(y)s = \mu_M(\lambda_M(y))s$ by 2.1 thus $\mu_M \alpha_M = 1_{\bar{F}(M)}$ and hence μ is an I-isomorphism.

3. In this section the kernel functor of Goldman [3] is dualized. Stenstrom (6) studied a particular type of this kernel functor in one attempt to extend the work of Matlis [4].

DEFINITION. A cokernel functor on Mod-A is an epi I-functor (F, λ) on Mod-A such that whenever $g \in \mathrm{Hom}_A(M, N)$ is an epimorphism then the following diagram is a pushout

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\lambda_M"'] & N \arrow[d, "\lambda_N"] \\ F(M) \arrow[r, "F(g)"] & F(N) \end{tikzcd} .$$

PROPOSITION 3.1. *Every cokernel functor is idempotent and d-strong.*

Proof. Let (F, λ) be a cokernel functor on Mod-A. (F, λ) is clearly d-strong since it is an epi I-functor. Suppose that $M \in$ Mod-A and $N = F(M)$. Now $F(\lambda_M)\lambda_M = \lambda_N \lambda_M$ thus $F(\lambda_M) = \lambda_N$ since λ_M is an epimorphism. This means that

$$\begin{tikzcd} M \arrow[r, "\lambda_M"] \arrow[d, "\lambda_M"'] & F(M) \arrow[d, "\lambda_N"] \\ F(M) \arrow[r, "\lambda_N"] & F(F(M)) \end{tikzcd}$$

is a pushout and therefore λ_N is an isomorphism. Hence (F, λ) is idempotent.

PROPOSITION 3.2. *Let (F, λ) be an epi I-functor on* Mod-A. *The following statements are equivalent*:

(i) *(F, λ) is a cokernel functor.*

(ii) *F is a right exact functor.*

(iii) *(F, λ) is idempotent and any homomorphic image of an F-cotorsion module is also F-cotorsion.*

Proof. The equivalence of (i) and (ii) follows from Mitchell [5] Chapter 1, Proposition 13.2*.

Suppose that (F, λ) is a cokernel functor. By 3.1 (F, λ) is idempotent. If $g \in \mathrm{Hom}_A(M, N)$ is an epimorphism and M is F-cotorsion then

$$\begin{array}{ccc} M & \xrightarrow{\quad g \quad} & N \\ \downarrow{\scriptstyle \lambda_M} & & \downarrow{\scriptstyle \lambda_N} \\ F(M) & \xrightarrow{F(g)} & F(N) \end{array}$$

is a pushout where λ_M is an isomorphism. Thus by Mitchell [5] Chapter 1, Propositions 7.2* and 20.2* λ_N is an isomorphism. This shows that (i) implies (iii).

Conversely assume (iii). Let $g \in \mathrm{Hom}_A(M, N)$ be an epimorphism and let $u: G \to M$ be the kernel of g. Let $v: F(M) \to X$ be the cokernel of $\lambda_M u$. Since $F(M)$ is F-cotorsion it follows that X is also F-cotorsion. Since g is the cokernel of u there exists $h \in \mathrm{Hom}_A(N, X)$ such that $hg = v\lambda_M$. Thus by 1.2 there exists $f \in \mathrm{Hom}_A(F(N), X)$ such that $f\lambda_N = h$. Therefore, $fF(g) = v$ and so $F(g): F(M) \to F(N)$ is the cokernel of $\lambda_M u$. Hence by Mitchell [5] Chapter 1, Proposition 13.2*

$$\begin{array}{ccc} M & \xrightarrow{\quad g \quad} & N \\ {\scriptstyle \lambda_M}\downarrow & & \downarrow{\scriptstyle \lambda_N} \\ F(M) & \xrightarrow{F(g)} & F(N) \end{array}$$

is a pushout and so (F, λ) is a cokernel functor.

THEOREM 3.3. *If* (F, λ) *is a cokernel functor on* Mod-A *then* (F, λ) *and* $(\bar{F}, \bar{\lambda})$ *are equivalent I-functors.*

Proof. Let $J =$ kernel of λ_A. By 3.1 and 2.1 $F(A)$ is a ring and J is a 2-sided ideal of A. Also $F(A)$ is ring isomorphic to A/J.

Let M be any free right $F(A)$-module. M can be embedded in a direct product of copies of $F(A)$. By 1.3 a direct product of copies of $F(A)$ is F-cotorsion and so by 1.1 M is F-reduced. But λ_M is an epimorphism thus M is F-cotorsion.

If N is any right $F(A)$-module then N is the homomorphic image of a free $F(A)$-module M and so by 3.2 N is F-cotorsion. If $U \in$ Mod-A then $\bar{F}(U)$ is a right $F(A)$-module and so $\bar{F}(U)$ is F-cotorsion. Thus by 2.2 (F, λ) and $(\bar{F}, \bar{\lambda})$ are equivalent I-functors.

If J is any 2-sided ideal of A then $M \to M \otimes_A A/J$ is easily seen to define a cokernel functor on Mod-A. Combining this with 3.3 we have a complete classification of all cokernel functors.

4. We now investigate the relationship between homological

properties of $F(A)$ and those of A where (F, λ) is an I-functor on Mod-A, much in the same manner as Turnidge [7].

LEMMA 4.1. *Let (F, λ) be a restricted idempotent, d-strong I-functor such that A is F-reduced. If every F-reduced right A-module is flat then A is left semi-hereditary.*

Proof. Every direct product of F-reduced modules is F-reduced by 1.3 and submodules of F-reduced modules are also F-reduced by 1.1. Thus every torsionless right A-module is F-reduced since A is F-reduced and therefore every torsionless right A-module is flat. Hence by [2, Thm. 4.1] A is left semi-hereditary.

We will need to refer to a restricted idempotent, d-strong I-functor where A is F-reduced frequently throughout this section. We therefore call such an I-functor *special* for easy reference.

LEMMA 4.2. *Suppose that (F, λ) is a special I-functor on* Mod-A. *Every F-reduced right A-module is also $\bar{F}$-reduced.*

Proof. Let $M \in$ Mod-A be F-reduced. Since (F, λ) is restricted idempotent, $F(M)$ is F-cotorsion and hence by 2.1 is a right $F(A)$-module. Thus there exists $u_M \in \mathrm{Hom}_A(\bar{F}(M), F(M))$ such that $u_M(y \otimes r) = \lambda_M(y)r$ for every $y \in M$ $r \in F(A)$. That is $u_M \bar{\lambda}_M = \lambda_M$ and since λ_M is a monomorphism so is $\bar{\lambda}_M$. Therefore M is $\bar{F}$-reduced.

The following theorem investigates the weak dimension (WD) of $\bar{F}$-reduced modules if the global weak dimensions (GWD) of $F(A)$ and A are known.

THEOREM 4.3. *Let (F, λ) be a special I-functor on* Mod-A *such that $F(A)$ is flat as a right A-module. If* GWD $F(A) \leqq m$ *and* GWD $A \leqq n + 1$ *where m and n are nonnegative integers such that $m \leqq n$ then* WD $M \leqq n$ *for every $\bar{F}$-reduced right A-module M.*

Proof. Let $M \in$ Mod-A be $\bar{F}$-reduced. Since GWD $F(A) \leqq m$ thus $\bar{F}(M) = M \otimes_A F(A)$ has weak dimension $\leqq m$ as an $F(A)$-module. Hence by [1, Prop. VI 4.12] $M \otimes_A F(A)$ has weak dimension $\leqq m$ as an A-module.

Let $B =$ cokernel $\bar{\lambda}_M : M \rightarrow M \otimes_A F(A)$. This gives rise to exact sequences

$$\mathrm{Tor}^A_{k+1}(B, X) \longrightarrow \mathrm{Tor}^A_k(M, X) \longrightarrow \mathrm{Tor}^A_k(M \otimes_A F(A), X) \longrightarrow \mathrm{Tor}^A_k(B, X)$$

for every nonnegative integer k and left A-module X. If $k > n$ then $k + 1 > n + 1$ and $k > m$. Thus $\mathrm{Tor}^A_{k+1}(B, X) = 0 = \mathrm{Tor}^A_k(M \otimes_A F(A),$

X) so $\mathrm{Tor}_k^A(M, X) = 0$ and therefore WD $M \leqq n$.

COROLLARY 4.4. *Let (F, λ) be a special I-functor on* Mod-A *such that $F(A)$ is flat as a right A-module and* GWD $F(A) = 0$. *The following statements are equivalent*:

(i) *A is left semi-hereditary.*

(ii) GWD $A \leqq 1$.

(iii) *Every F-reduced right A-module is flat.*

(iv) *Every $\bar{F}$-reduced right A-module is flat.*

Proof. (i) $\Rightarrow$ (ii) follows from [1, Prop. VI 2.9]

(ii) $\Rightarrow$ (iv) is a consequence of 4.3.

(iv) $\Rightarrow$ (iii) is immediate from 4.2.

(iii) $\Rightarrow$ (i) is immediate from 4.1.

THEOREM 4.5. *Let (F, λ) be a special I-functor on* Mod-A. *If $F(A)$ is projective as a right A-module and is a semi-simple Artinian ring, the following statements are equivalent*:

(i) *A is right hereditary.*

(ii) *M is projective for every F-reduced $M \in$* Mod-A.

Proof. Since (F, λ) is special every right ideal of A is F-reduced by 1.1. Thus (ii) $\Rightarrow$ (i) is immediate.

(i) $\Rightarrow$ (ii). Let $M \in$ Mod-A be F-reduced. By 4.2 M is $\bar{F}$-reduced so we have an exact sequence

$$0 \longrightarrow M \longrightarrow M \otimes_A F(A) \longrightarrow B \longrightarrow 0 .$$

Now $F(A)$ is semi-simple Artinian so $M \otimes_A F(A)$ is a projective $F(A)$-module. $F(A)$ is a projective A-module and thus $M \otimes_A F(A)$ is a projective A-module. Therefore, by [1, I Thm. 5.4] M is a projective A-module.

We now investigate a relationship between the global dimension (GD) of $F(A)$ and the injective dimension (ID) of F-cotorsion modules over a commutative ring.

THEOREM 4.6. *Let A be a commutative ring and (F, λ) a special I-functor on* Mod-A *such that $F(A)$ is flat as an A-module. If* GD $F(A) \leqq n$ *where n is a nonnegative integer then* ID $M \leqq n$ *for every F-cotorsion $M \in$* Mod-A. *In addition if* GWD $F(A) \leqq m$ *where m is a nonnegative integer then* WD $M \leqq m$ *for every F-cotorsion $M \in$* Mod-A.

Proof. Let $M \in$ Mod-A be F-cotorsion. By 2.1 M is an $F(A)$-module and by [1, Prop. VI 4.1.3 and 4.1.2] we have isomorphisms $\mathrm{Ext}_{F(A)}^k(X \otimes_A F(A), M) \cong \mathrm{Ext}_A^k(X, M)$ and $\mathrm{Tor}_k^A(M, X) \cong \mathrm{Tor}_k^{F(A)}(M,$

$X \otimes_A F(A))$ for every $X \in$ Mod-A. Since GD $F(A) \leqq n$ and GWD $F(A) \leqq m$ it follows that ID $M \leqq n$ and WD $M \leqq m$.

COROLLARY 4.7. *Let A be a commutative ring and (F, λ) a special I-functor on* Mod-A *such that $F(A)$ is flat as an A-module and such that A/I is F-cotorsion for every nonzero ideal I of A. Then* GWD $A \leqq$ GWD $F(A)$.

Proof. Assume GWD $F(A) = m$. Then by 4.6 WD $A/I \leqq m$ for every ideal I of A. Hence GWD $A \leqq m$.

An example of a special I-functor of the type in the preceding corollary is the cotorsion completion functor of Matlis [4] which is given by $M \rightarrow \mathrm{Ext}^1_A(K, M)$ for every $M \in$ Mod-A where A is an integral domain and $K = Q/A$ where Q is the quotient field of A.

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SOUTHERN COLORADO STATE COLLEGE