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CROSS-SECTIONS OF DECOMPOSITIONS

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The following question was raised by R. H. Bing: "Is it true that if G is a monotone decomposition of E^3 into straight line intervals and one-point sets, then E^3/G is homeomorphic to E^3?" In his paper "Point-like decompositions of E^3" he described a possible counter example. This example has the interesting property that it has many tame cross-sections, but if its decomposition space is homeomorphic to E^3, its set of nondegenerate elements would have to form a wild Cantor set. This suggests that it would be interesting to study the connection between the embedding of a cross-section and the embedding of the set of nondegenerate elements in the decomposition space.

1. Introduction. Most of the terminology and notation used in this paper is standard. The reader is referred to [1], [3], [4], and [6].

If S is a 2-sphere in E^3, then by Int S we will mean the bounded component of E^3-S and by Ext S, the unbounded component.

Let G be an upper semi-continuous decomposition of E^3 and let H be the set of all nondegenerate elements of G. We will say that a set $R \subset E^3$ is a cross-section of G if (i) $R \cap h$ is a singleton for each $h \in H$, and (ii) the natural map P restricted to R is homeomorphism onto $\overline{P(H)}$. We note that cross-sections exist only for certain decompositions. A simple example may be constructed as follows: Let $a_n = 1/n$, for $n = 1, 2, \cdots$ and let $b_n = -1/n$ for $n = 1, 2, \cdots$. Let the set of nondegenerate elements of our decomposition consist of the closed interval from (0, 1, 0) to (0, -1, 0), the closed interval from (a_n, 1/2, 0) to (a_n, 1, 0) for each positive integer n, and the closed interval from (b_n, $-1/2$, 0) to (b_n, -1, 0) for each positive integer n.

II. Cross-sections of decompositions. The following question naturally arises: How are the embeddings of a cross-section R and $\overline{P(H)}$ related when E^3/G is homeomorphic to E^3? We will give some partial results to this question.

THEOREM 1. *Let G be an upper semi-continuous decomposition of E^3 into points and straight line intervals pointing in only a countable number of directions whose lengths are bounded away from zero such that $P(H)$ is a compact 0-dimensional set. If there exists a cross-section C of G then C is tame.*

Proof. In the special case where the elements of H point in only one direction, we can easily show the tameness by a modification of the proof of Theorem 2 of [7].

Suppose that $H = \bigcup_{n=1}^{\infty} H_n$ where the elements of H_n are all parallel and if $h_1 \in H_i$ and $h_2 \in H_j$ where $j \neq i$ then h_1 is not parallel to h_2. Let C_n be the set of all points $c \in C$ such that $c \in h$ for some $h \in H_n$. Let G_n be the upper semi-continuous decomposition of E^3 whose only nondegenerate elements are the elements of H_n and let P_n be the natural map. Then E^3/G_n is homeomorphic to E^3 and $P_n(H_n)$ is tame in E^3/G_n. So by the special case C_n is tame and by Corollary 2 to Theorem 3 of [7], C is tame.

The following two lemmas will be stated without proof. Their proofs are similar to that of Lemma A of [7] and use standard techniques. Lemma B is similar to Theorem 2.3 of [3].

LEMMA A. *Let G be an upper semi-continuous decomposition of E^3 such that $P(H)$ is a compact 0-dimensional set. Let $h \in H$ and suppose that there exist 2-spheres S_1 and S_2 such that $h \subset \operatorname{Int} S_1 \cap \operatorname{Int} S_2$ and $(S_1 \cup S_2) \cap (\cup H) = \emptyset$. Then there exists a 2-spheres S such that $h \subset \operatorname{Int} S$, $S \cup \operatorname{Int} S \subset S_1 \cup \operatorname{Int} S_1$, and if $k \in H$ then $k \subset \operatorname{Int} S$ iff $k \subset \operatorname{Int} S_1 \cap \operatorname{Int} S_2$.*

LEMMA B. Let *$S_1, S_2, \cdots, S_n$ be a finite collection of 2-sphere whose interiors cover $\cup H$ and which miss $\cup H$. Then there exists a finite collection of 2-spheres $R_1, R_2, \cdots, R_n$ such that $R_1 = S_1$, $(R_i \cup \operatorname{Int} R_i) \cap (R_j \cup \operatorname{Int} R_j) = \emptyset$ if $i \neq j$, and $h \subset \operatorname{Int} R_i$ iff $h \subset \operatorname{Int} S_i$ and $h \cap \operatorname{Int} S_j = \emptyset$ for $j < i$.*

THEOREM 2. *Let C be a wild Cantor set in E^3 with the property that if x and y are distinct points of C, then there exist disjoint 2-spheres S_1 and S_2 such that $(S_1 \cup S_2) \cap C = \emptyset$, $x \in \operatorname{Int} S_1 \cap \operatorname{Ext} S_2$ and $y \in \operatorname{Int} S_2 \cap \operatorname{Ext} S_1$. Then there exists a monotone decomposition G of E^3 such that C is a cross-section for G, E^3/G is homeomorphic to E^3 and $P(\bar{H})$ is tame.*

Proof. Let C be a wild Cantor set in E^3 with the required property. For each $x \in C$ we choose a 2-sphere $S_1(x)$ as follows:

Let $N_1(x)$ be a 2-sphere of radius 1/2, centered at x. Let $C_1(x) = \{t \in C \mid t \notin \operatorname{Int} N_1(x)\}$. Then for each $y \in C_1(x)$ choose disjoint 2-spheres $S(y)$ and $R(y)$ such that $(S(y) \cup R(y)) \cap C = \emptyset$, $x \in \operatorname{Int} S(y) \cap \operatorname{Ext} R(y)$, and $y \in \operatorname{Int} R(y) \cap \operatorname{Ext} S(y)$. Now choose a set $y_1, y_2, \cdots, y_n$ of elements of $C_1(x)$ such that $\{\operatorname{Int} R(y_1), \operatorname{Int} R(y_2), \cdots, \operatorname{Int} R(y_n)\}$ covers $C_1(x)$. We now apply Lemma A to get a 2-sphere $S_1(x)$ such that $x \in \operatorname{Int} S_1(x)$, $S_1(x) \cap C = \emptyset$, $C_1(x) \subset \operatorname{Ext} S_1(x)$ and $S_1(x) \subset S(y_i) \cup \operatorname{Int} S(y_i)$ for $i = 1, 2, \cdots, n$. Therefore, there exists a finite collection of points $x_1, x_2, \cdots,$

$x_{m(1)}$ of C such that $C \subset \text{Int}\, S_1(x_1) \cup \text{Int}\, S_1(x_2) \cup \cdots \cup \text{Int}\, S_1(x_{m(1)})$. We replace $\mathscr{S}_1 = \{S_1(x_1), S_1(x_2), \cdots, S_1(x_{m(1)})\}$ by another collection of 2-spheres $\mathscr{T}_1 = \{T_{11}, T_{12}, \cdots, T_{1n(1)}\}$ satisfying the conclusions of Lemma B with respect to $\mathscr{S}_1$.

We will now proceed to construct a sequence $\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3, \cdots$ of finite covers of C. Suppose that $\mathscr{T}_{k-1}$ has been chosen. For each point $x \in C$ we choose a 2-sphere $N_k(x)$ centered at x with radius $1/2^k$. We then proceed to choose $\mathscr{T}_k$ by the same process as in the construction of $\mathscr{T}_1$. We note that if $y_1, y_2 \in T_{kj} \cap C$ then $d(y_1, y_2) < 1/2^{k-1}$ since $T_{jk} \cap C \subset N_k(x)$ for some $x \in C$. Now for $x \in C$ we define h_x to be $\bigcap_{k=1}^{\infty} (T_{ki} \cup \text{Int}\, T_{ki})$ where T_{ki} is the 2-sphere in T_k whose interior contains x. Let G be the decomposition of E^3 whose only nondegenerate elements are the nondegenerate elements of $\{h_x \mid x \in C\}$. It follows easily that G is upper semi-continuous and it is clear that C is a cross-section for G. A theorem of Harrold [5] shows that E^3/G is homeomorphic to E^3 and from the criteria of [3], we see that $\overline{P(H)}$ is tame.

The Cantor set constructed in [2] is an example of a wild Cantor set satisfying the hypothesis of Theorem 2.

We can note that if C is a wild Cantor set in E^3 which does not satisfy the condition of Theorem 2, also, if C is a cross-section of a decomposition G whose decomposition space is homeomorphic to E^3 then $P(H_G)$ is a wild Cantor set which does not satisfy the condition of Theorem 2.

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