PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 1, 1973

$\widetilde{HD}$-MINIMAL BUT NO HD-MINIMAL

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Let U_{HD}^k (resp. $U_{\widetilde{HD}}^k$) be the class of Riemannian n-manifolds ($n \geqq 2$) on which there exist k non-proportional HD-minimal (resp. $\widetilde{HD}$-minimal) functions. The purpose of the present paper is to construct a Riemannian n-manifold $n \geqq 3$ which carries a unique (up to constant factors) $\widetilde{HD}$-minimal function but no HD-minimal functions. Thus the inclusion relation

$$U_{HD}^1 \subset U_{\widetilde{HD}}^1$$

is strict for $n \geqq 3$. By welding k copies of this Riemannian n-manifold, it is then established that the inclusion relation

$$U_{HD}^k \subset U_{\widetilde{HD}}^k$$

is strict for all $k \geqq 1$ and $n \geqq 3$. The problem still remains open for $n = 2$.

1. An HD-function (harmonic and Dirichlet-finite) ω on a Riemannian n-manifold M is called HD-minimal on M if ω is positive on M and every HD-function ω' with $0 < \omega' \leqq \omega$ reduces to a constant multiple of ω on M. Let $\{\omega_n\}$ be a sequence of positive HD-functions on M. If the sequence $\{\omega_n\}$ decreases on M, the limit function is harmonic on M by Harnack's inequality. Such a harmonic function is called an $\widetilde{HD}$-function on M, and $\widetilde{HD}$-minimality can be defined as in the case of HD-minimal functions.

These functions were introduced by Constantinescu and Cornea [1] and systematically studied by Nakai [6]. In particular the following characterization by Nakai is important (loc. cit., cf. also Kwon-Sario [5]):

(i) A Riemannian n-manifold M carries an HD-minimal function ω if and only if the Royden harmonic boundary Δ_M of M contains a point p, isolated in Δ_M. In this case $\omega(p) > 0$ and $\omega \equiv 0$ on $\Delta_M - \{p\}$.

(ii) A Riemannian n-manifold M carries an $\widetilde{HD}$-minimal function ω if and only if the Royden harmonic boundary Δ_M of M has a point p of positive harmonic measure. These are corresponded such that $\limsup_{x \in M, x \to p} \omega(x) > 0$ and $\limsup_{x \in M, x \to q} \omega(x) = 0$ for almost all $q \in \Delta_M - \{p\}$ with respect to a harmonic measure on Δ_M.

Since an isolated point of Δ_M has a positive harmonic measure, the above characterization yields the inclusion

$$U_{HD}^k \subset U_{\widetilde{HD}}^k$$

for all $k \geqq 1$.

For the notation and terminology we refer the reader to the monograph by Sario-Nakai [7].

2. Let $n \geqq 3$. Denote by M_0 the punctured Euclidean n-space $R^n - 0$ with the Riemannian metric tensor

$$g_{ij}(x) = |x|^{-4}(1 + |x|^{n-2})^{4/(n-2)}\delta_{ij}\ , \qquad 1 \leqq i,\ j \leqq n$$

where $|x| = [\sum_{i=1}^{n} (x^i)^2]^{1/2}$ for $x = (x^1, x^2, \cdots, x^n) \in M_0$.

For each pair (m, l) of positive integers m, l, set

$$H_{ml} = \{8^k x \in M_0 \mid |x| = 1 \text{ and } x^1 \geqq 0\}\ ,$$

where $k = 2^{m-1}(2l-1) - 1$, and $ax = (ax^1, ax^2, \cdots, ax^n)$ for $x = (x^1, x^2, \cdots, x^n) \in M_0$ and real a. Let M_0' be the slit manifold obtained from M_0 by deleting all the closed hemispheres H_{ml}. Take a sequence $\{M_0'(l)\}_1^\infty$ of copies of M_0'. For each fixed $m \geqq 1$ and subsequently for fixed $j \geqq 0$ and $1 \leqq i \leqq 2^{m-1}$, connect $M_0'(i + 2^m j)$, crosswise along all the hemispheres $H_{ml}(l \geqq 1)$, with $M_0'(i + 2^{m-1} + 2^m j)$.

The resulting Riemannian n-manifold N is an infinitely sheeted covering manifold of M_0. Let π: $N \to M_0$ be the natural projection.

The following result is essential to our problem (Kwon [4]):

THEOREM 1. *A function $u(x)$ is harmonic on N if and only if $[1 + |\pi(x)|^{2-n}]u(x)$ is Δ_e-harmonic (harmonic with respect to the Euclidean structure) on N. In particular every bounded harmonic function $u(x)$ on the submanifold*

$$G = \left\{x \in N \mid |\pi(x)| > \frac{1}{3}\right\}$$

is constant on $\pi^{-1}(x)$ for each $x \in M_0$ whenever it continuously vanishes on

$$\partial G = \left\{x \in N \mid |\pi(x)| = \frac{1}{3}\right\}.$$

3. For each integer $l \geqq 1$, consider the subset of N:

$$N_l = [M_0'(l)] \cup \left[\bigcup_{i \neq l} G_i\right]$$

where

$$G_i = \left\{x \in M_0'(i) \mid |\pi(x)| > \frac{1}{3}\right\}.$$

It is obvious that

$$G = \bigcup_{i=1}^{\infty} G_i$$

and the Riemannian n-manifold G is an infinitely sheeted covering manifold of the annulus $\{x \in M_0 \mid 1/3 < |x| < \infty\}$.

We are now ready to state our main result:

THEOREM 2. *The Riemannian n-manifold* G $(n \geqq 3)$ *carries a unique (up to constant factors)* $\widetilde{HD}$*-minimal function but no HD-minimal functions. Thus the inclusion*

$$U^1_{HD} \subset U^1_{\widetilde{HD}}$$

is strict for Riemannian manifolds of dim $\geqq 3$.

The proof will be given in 4 – 5.

4. For $m \geqq 1$ construct $u_m \in HBD(N_m)$, the class of bounded HD-functions on N_m, such that $0 \leqq u_m \leqq 1$ on N, $u_m \equiv 0$ on $\bigcup_{i=1}^{m-1} [M_0'(i) - G_i]$, and $u_m \equiv 1$ on $\bigcup_{i=m+1}^{\infty} [M_0'(i) - G_i]$. Clearly $u_m \geqq u_{m+1}$ on N and therefore the sequence $\{u_m\}$ converges to an $\widetilde{HD}$-function u on G, uniformly on compact subsets of G. It is easy to see that $0 \leqq u < 1$ on G and $u \mid N - G \equiv 0$. Since

$$u_m(x) \geqq \frac{|\pi(x)|^{n-2} - 3^{2-n}}{|\pi(x)|^{n-2} + 1}$$

on G by maximum principle and Theorem 1, it follows that $0 < u < 1$ on G. Note that $\lim_{|\pi(x)| \to \infty} u_m(x) = 1$.

We claim that the function u is $\widetilde{HD}$-minimal on G. In fact, let $v \in \widetilde{HD}(G)$ be such that $0 < v \leqq u$ on G. In view of

$$0 \leqq \limsup_{x \in G, x \to y} v(x) \leqq \limsup_{x \in G, x \to y} u(x) = 0$$

for all $y \in \partial G$, v can be continuously extended to N by setting $v \equiv 0$ on $N - G$. By Theorem 1 v attains the same value at all the points in N which lie over the same point in M_0. Thus we may assume that u, v are bounded harmonic functions on $\pi(G) = \{\pi(x) \mid x \in G\}$ such that u, $v \equiv 0$ on $\pi(\partial G)$.

Again by Theorem 1, $(1 + |x|^{2-n})v(x)$ is Δ_e-harmonic on $\pi(G)$. In view of the fact that Δ_e-harmonicity is invariant by the Kelvin transformation, the function

$$\frac{1}{3^{n-2}|x|^{n-2}}(1 + 3^{2(n-2)}|x|^{n-2})v\left(\frac{x}{9|x|^2}\right)$$

is Δ_e-harmonic on M_0 for $0 < |x| < 1/3$ and continuously vanishes for

$|x| = 1/3$. Therefore, there exists a constant $a \geqq 0$ such that

$$v\left(\frac{x}{9|x|^2}\right) = \frac{3^{n-2}a}{1 + 3^{2(n-2)}|x|^{n-2}}$$

on M_0 for $0 < |x| < 1/3$ (cf., e.g. Helms [3, p. 81]). Thus

$$\lim_{x \to 0} v\left(\frac{x}{9|x|^2}\right) = 3^{n-2}a$$

exists and $v = 3^{n-2}au$ on G, as desired.

5. Suppose that there exists another $\widetilde{HD}$-minimal function ω on G. Choose a point $q \in \Delta_{M,G}$, the Royden harmonic boundary of G, such that q has a positive harmonic measure and

$$\limsup_{x \in G, x \to q'} \omega(x) = 0$$

for almost all $q' \in \Delta_{M,G} - \{q\}$ relative to a harmonic measure for G. Let j: $G^* \to \bar{G} \subset N^*$ be the subjective continuous mapping such that $j \mid G$ is the identity mapping and $f(x) = f(j(x))$ for all $x \in G^*$, the Royden compactification of G, and $f \in M(N)$, the Royden algebra of N. Here $\bar{G}$ is the closure of G in N^*. Note that a Borel set $E \subset \partial G$ has a positive harmonic measure if and only if $j^{-1}(E)$ has a positive harmonic measure (cf. Sario-Nakai [7, p. 192]). Therefore, $j(q) \notin \partial G$ and $\partial G \subset j(\Delta_{M,G})$.

For each $m \geqq 1$, $u_m(q) = u_m(j(q)) = 1$ since $j(q) \in \overline{\partial G} - \partial G$. Thus it is not difficult to see that $0 < \omega \leqq \beta u_m$ on G, where

$$\beta = \limsup_{x \in G, x \to q} \omega(x) > 0 .$$

Therefore, $0 < \omega \leqq \beta u$ on G and ω is a constant multiple of u on G as in 4.

It remains to show that u is not HD-minimal on G. If it were, u would have a finite Dirichlet integral. But u has a continuous extension to $G \cup \partial G$ with $u \mid \partial G \equiv 0$. Then by Theorem 1 u must attain the same value at all the points in G which lie over the same point in $\pi(G)$, a contradiction.

This completes the proof of Theorem 2.

6. Let G' be the Riemannian n-manifold obtained from G by deleting two disjoint closed subsets B, C, where

$$B = \left\{x \in M_0'(1) \mid |x| = \frac{9}{24} \text{ and } x^1 \geqq 0\right\} ,$$

$$C = \left\{x \in M_0'(1) \mid |x| = \frac{11}{24} \text{ and } x^1 \geqq 0\right\} .$$

For each $k \geqq 2$ take k copies $G_1, G_2, \cdots, G_k$ of G', and identify, crosswise, B_i with C_{i+1} for $1 \leqq i \leqq m$. Here we set $C_{n+1} = C_1$. Then it is easy to see that the resulting Riemannian n-manifold $G^{(k)}$ has exactly k non-proportional $\widetilde{HD}$-minimal functions but no HD-minimal functions.

COROLLARY. *For all* $k \geqq 1$ *the strict inclusion*

$$U^k_{HD} < U^k_{\widetilde{HD}}$$

holds for Riemannian manifolds of $\dim \geqq 3$.

7. For the sake of completeness we shall sketch a proof of Theorem 1. In view of the simple relation

$$\Delta u = |x|^{n+2}(1 + |x|^{n-2})^{-(n+2)/(n-2)} \cdot \Delta_e[(1 + |\pi x|^{2-n})u] ,$$

it suffices to show the latter half.

For each integer $k \geqq 0$ let U_k be a component of the open set

$$\{x \in N \mid 2^{3k-1} < |\pi(x)| < 2^{3k+1}\} ,$$

and S_k a compact subset of U_k which lie over the set

$$\{x \in M_0 \mid |x| = 2^{3k}\} .$$

Since U_k is a magnification of U_0 and the Δ_e-harmonicity is invariant under a magnification, it is not difficult to see that there exists a constant q, $0 < q < 1$, such that

$$|u(x)| \leqq q \cdot \sup\{|u(x)| \mid x \in U_k\}$$

on S_k for any harmonic function u on U_k which changes sign on S_k. Note that q is independent of k.

Let u be a harmonic function on G such that $|u| \leqq 1$ and it continuously vanishes on ∂G. For each $m \geqq 1$, denote by π_m the cover transformation of G which interchanges the sheets of G: the points in $G \cap M_0'(i + 2^m j)$ are interchanged with points, with the same projection, in $M_0'(i + 2^{m-1} + 2^m j)$ for $j \geqq 0$ and $1 \leqq i \leqq 2^{m-1}$. Define v_m on G by

$$v_m(x) = \frac{1}{2}[u(x) - u(\pi_m(x))] .$$

Clearly v_m is harmonic on G, $|v_m| \leqq 1$, and v_m changes sign on S_k, $k = 2^{m-1}(2l - 1) - 1$. Therefore,

$$\max\{|v_m(x)| \mid x \in S_k\} \leqq q$$

for all $l \geqq 1$. By induction on l, we derive that $|v_m| \leqq q^l$ on $S_{k'}$, where $k' = 2^{m-1} - 1$. Letting $l \to \infty$, we conclude that $v_m \equiv 0$ on G, as desired.

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Received August 21, 1972 and in revised form January 17, 1973.

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