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THE SEIFERT AND VAN KAMPEN THEOREM VIA REGULAR COVERING SPACES

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The Seifert and Van Kampen theorem has lately been phrased as the solution to a universal mapping problem. There is given here an analogous theorem for regular covering spaces, regarded as principal bundles with discrete structure groups. The universal covering space of a union of two spaces is built up from the universal covering spaces of the two subspaces by an application of the associated bundle and clutching constructions. When all spaces are semi-locally simply connected, the Seifert and Van Kampen theorem is a consequence.

The technique of building a covering space piece by piece was effectively exploited by Neuwirth [10] to construct nonsimply connected covering spaces. We give an alternative approach to constructing a regular covering space of a base B which is either the union of open sets B_1 and B_2 with connected intersection B_0, or which is an adjunction space $B_1 \cup_f B_2$, where f glues a closed subspace B_0 of B_1 to B_2. We assume that all spaces are connected, and that there is given a regular covering space ξ_i of B_i for $i = 0, 1$, and 2, together with morphisms of covering spaces $\xi_0 \rightarrow \xi_i$, $i = 1, 2$. By regarding a regular covering space as a principal bundle with discrete structure group and applying the associated bundle and clutching constructions, we obtain a regular covering space ξ of B as the pushout of $\xi_1 \leftarrow \xi_0 \rightarrow \xi_2$. The structure group of ξ is the pushout of the structure groups of ξ_0, ξ_1, and ξ_2. One may obtain in this fashion the universal covering space of B from universal covering spaces of B_0, B_1, and B_2, or one may obtain the universal abelian covering space (i.e., that one having the maximal possible abelian structure group) from the universal abelian covering spaces of B_0, B_1, and B_2. The proof is by universal mapping arguments. In contrast to Neuwirth [10], the Seifert and Van Kampen theorem, under the hypotheses that all base spaces are locally connected and semi-locally simply connected, is a corollary. It is interesting that local homotopy conditions in a neighborhood of B_0, such as those assumed by Van Kampen and others ([15], [11], and [2]), turn out to be unnecessary, provided the space B_1 is paracompact. On the other hand, the Van Kampen formulae may apply in cases where B_0 is not connected or one of the B_i does not have a universal covering space, neither case being included in our results.

The only difficult arguments involve the validity of the clutching construction [14], [1], and [7] for $B_1 \cup_f B_2$. That is to say, one must establish the existence of a local product structure for any bundle formed by clutching. In the alternative case of open B_1 and B_2, with $B = B_1 \cup B_2$, the clutching construction easily yields a locally trivial object. Thus the person interested in the simplest route to a form of Seifert and Van Kampen will consider only the case of open B_1 and B_2, with $B = B_1 \cup B_2$.

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1. Universality of the associated bundle. If ξ is a principal bundle with structure group G, and $u: G \to K$ is a continuous group homomorphism, then G acts on K on the left via u, and the associated bundle construction yields a new bundle with structure group K. This associated bundle is the solution to a certain universal mapping problem in the category of all principal bundles.

All spaces are assumed to be regular. Recall [14] that a principal bundle ξ consists of *bundle space* E_ξ, a *base space* B_ξ, a *projection* $p_\xi: E_\xi \to B_\xi$. There is also given a topological *structure group* G_ξ which acts on E_ξ freely from the right and such that p_ξ is equivalent map of E_ξ onto the space of orbits of G_ξ. It is assumed that p_ξ is *locally trivial.* This means that there is an open covering $\{V_i\}_{i \in I}$ of B_ξ by sets called *coordinate neighborhoods,* and for each $i \in I$ there exists a map (= continuous function) s_i: $V_i \to E_\xi$. This map is assumed to be a *section* of ξ *over* V_i, that is, for any point b of V_i, $p_\xi(s_i(b)) = b$. Local triviality is the condition that the function S_i defined below is a homeomorphism.

$$S_i\colon V_i \times G_\xi \longrightarrow p_\xi^{-1}(V_i) \ , \qquad S_i(b, g) = s_i(b)\cdot g \ .$$

Here $s_i(b)\cdot g$ denotes the right translate of $s_i(b)$ by an element g of G. The functions S_i are called *coordinate functions.* The coordinate function determined by any section of ξ is a homeomorphism.

LEMMA 1.1. *If ξ is a principal bundle and f and h are maps of a space X into E_ξ such that $p_\xi^\circ f = p_\xi^\circ h$, then there is a unique map $t: X \to G_\xi$ such that for any point x of X, equation* (1) *holds*

$$f(x) = h(x)\cdot t(x) \ . \tag{1}$$

Proof. For any point x of X, equation (1) determines $t(x)$ uniquely, by the freeness of the action of G_ξ on E_ξ. The map t is continuous by local triviality of p_ξ.

DEFINITION 1.2. Let ξ and ξ' be principal bundles. A *morphism*

$h: \xi \to \xi'$ is a map of E_ξ into $E_{\xi'}$ such that there exists a function $h_G: G_\xi \to G_{\xi'}$, which for any point x in E_ξ, and element g of G_ξ, satisfies equation (2)

$$h(x \cdot g) = h(x) \cdot h_G(g) . \tag{2}$$

By the lemma, h_G is unique and continuous. It is an easy consequence of (2) that h_G is a homomorphism. Note that (2) also implies that h maps a fibre $p_\xi^{-1}(b)$ into a fibre $p_\xi^{-1}(h_B(b))$. Let $h_B: B \to B'$ be the unique and continuous function such that for any point x of E_ξ, equation (3) holds.

$$p_{\xi'}(h(x)) = h_B(p_\xi(x)) . \tag{3}$$

Let ξ be a principal bundle, let K be a topological group, and let $u: G_\xi \to K$ be a continuous homomorphism. Then G_ξ acts on K on the left via u, and the "weakly" associated bundle with fibre K will be denoted by α_u. See [14, §§ 8.7, 9.1]. The bundle space of α_u is usually denoted by $E \times_G K$, where $E = E_\xi$. It is formed as the quotient space of $E \times K$ by the relation which identifies a point (x, k) with $(x \cdot g, u(g^{-1})k)$, for every element g of G_ξ. The equivalence class of (x, k) is denoted $\langle x, k \rangle$. The action of K on $E \times_G K$ is defined by the rule $\langle x, k_1 \rangle \cdot k_2 = \langle x, k_1 k_2 \rangle$. The base space of α_u is that of ξ, $B_{\alpha_u} = B_\xi$. The projection is defined by the rule $p_{\alpha_u}\langle x, k \rangle = p_\xi(x)$. There is a natural map $u^\#: E \to E \times_G K$ defined by the rule $u^\#(x) = \langle x, e \rangle$, where e is the identity element of K. If u is understood, then it will be convenient to write $\xi \times_G K$ for α_u.

THEOREM 1.3. *Let ξ be a principal bundle and let $u: G_\xi \to K$ be a continuous homomorphism of topological groups. Then $u^\#$ is a morphism of principal bundles such that $u^\#_G = u$. If $h: \xi \to \xi'$ is a morphism of principal bundles and $v: K \to G_{\xi'}$ is a continuous homomorphism such that $h_G = v \circ u$, then there exists a unique morphism*

$$(h, v)^\#: \xi \times {}_G K \to \xi'$$

such that (4) *and* (5) *hold.*

$$h = (h, v)^\# \circ u^\# \tag{4}$$

$$v = (h, v)^\#_G . \tag{5}$$

Proof. For x in E_ξ and g in G_ξ, $u^\#(x \cdot g) = \langle x \cdot g, e \rangle = \langle x \cdot g, e \rangle = \langle x, u(g) \rangle = \langle x, e \rangle \cdot u(g) = u^\#(x) \cdot u(g)$. This proves that $u^\#$ is a morphism and $u^\#_G = u$. The conditions (4) and (5) on $(h, v)^\#$ are equivalent by (2) to defining

$$(h, v)^\# \langle x, k \rangle = h(x) \cdot v(k) , \qquad \langle x, k \rangle \in E \times {}_G K .$$

The uniqueness of $(h, v)^{\#}$ follows also from this rule, so the proof is complete.

Roughly that theorem says that $h\colon \xi \to \xi'$ factors through α_u if and only if h_G factors through K, and the former factorization is determined by the latter. In this sense the associated bundle is the solution to a universal mapping problem.

2. The clutching construction. The clutching construction ([14], p. 97) is extended. There are given spaces B_0, B_1, and B_2 such that B_0 is a subspace of B_1 with suitable properties relative to bundles. There is given a map $f\colon B_0 \to B_2$ by means of which B_1 is attached to B_2 to form $B = B_1 \cup_f B_2$ ([3], p. 127 f). There is given a common structure group K for principal bundles ξ_0, ξ_1, and ξ_2 with respective base spaces B_0, B_1, and B_2 and there are given morphisms $j\colon \xi_0 \to \xi_1$ and $h\colon \xi_0 \to \xi_2$ such that $j_B\colon B_0 \to B_1$ is the inclusion map and $h_B = f$, and such that j_G and h_G are the identity homomorphism of K. If E_i is the bundle space of ξ_i for $i = 0, 1$, and 2, and let $E = E_1 \cup_h E_2$. The projection p of E onto B is defined by functoriality of the attaching construction, and likewise by functoriality K acts on E freely from the right. In order to conclude that $\xi = (E, p, B, K)$ is a principal bundle it suffices to prove that the projection is locally trivial. We write $\xi_1 \cup_h \xi_2$ for ξ.

DEFINITION 2.1. For any ξ_1 and ξ_2 as above, and for a closed subset A of B_1, the set, $\mathscr{G}(A, \xi_1)$, of *germs* of sections of ξ_1 *over* A is defined as follows. An element of $\mathscr{G}(A, \xi_1)$ is an equivalence class of sections of ξ_1 over neighborhoods of A, where two such sections, s and t, are defined to be equivalent if for some open neighborhood V of A both s and t are defined throughout V and their restrictions to V are equal. For a closed subset A of B_0, likewise there is defined $\mathscr{G}(A, \xi_0)$, and restriction of sections induces a function

$$j^*\colon \mathscr{G}(A, \xi_1) \longrightarrow \mathscr{G}(A, \xi_0) .$$

Let ξ_2 be a principal bundle, and let $f\colon B_0 \to B_{\xi_2}$ be a map. Let $f^*(\xi)$ be the induced principal bundle ($f^{-1}(\xi)$ in [14]) with base B_0, and let $\bar{f}\colon f^*(\xi_2) \to \xi_2$ be the canonical morphism. Recall that $\bar{f}_B = f$, $\bar{f}_G$ is the identity homomorphism, $\bar{f}_G\colon G_{f^*(\xi_2)} = G_{\xi_2}$, and it is easy to see that

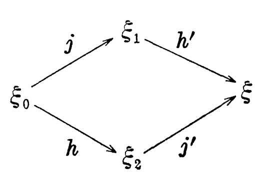

DIAGRAM I

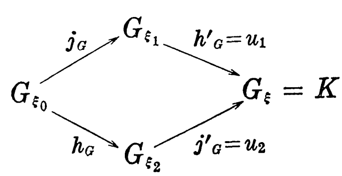

DIAGRAM II

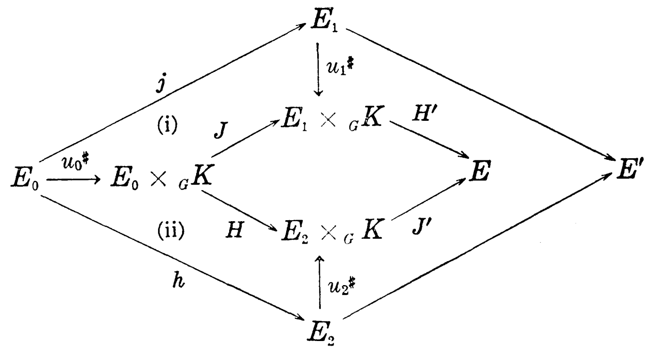

DIAGRAM III

these two conditions characterize $f^*(\xi_2)$. Recall further that for any section $s\colon V_2 \to E_{\xi_2}$, where V_2 is a subset of B_{ξ_2}, there exists a unique section denoted $f^*(s)\colon f^{-1}(V_2) \to E_{f^*(\xi_2)}$ such that $s \circ f = \bar{f}{}^{\circ} f^*(s)$.

THEOREM 2.2. *Let ξ_0, ξ_1, ξ_2, f, j, h, and ξ be as above. Let*

$$h'\colon E_1 \longrightarrow E_1 \cup_h E_2$$

$$j'\colon E_2 \longrightarrow E_1 \cup_h E_2$$

be the canonical maps. Under any of the conditions (2A), (2B), *or* (2C) *stated below, ξ is a principal bundle and h' and j' are morphisms of principal bundles which fill in the pushout diagram for j and h in the category of principal bundles* (diagram I).

(2A) *B_1 and B_2 are open subspaces of a common space $B = B_1 \cup B_2$, and $B_0 = B_1 \cap B_2$. The inclusion of B_0 into B_2 is $h_B = f$.*

(2B) *B_1 and B_2 are closed subspaces of a common space $B = B_1 \cup B_2$, and $B_0 = B_1 \cap B_2$. The inclusion of B_0 into B_2 is $h_B = f$. Further for each point b of B_0 the restriction function of $\mathscr{G}(b, \xi_1)$ to $\mathscr{G}(b, \xi_0)$ is onto.*

(2C) *B_0 is a closed subspace of B_1, and $h_B = f$ is arbitrary. Further for every closed subset A of B_0, the restriction function of $\mathscr{G}(A, \xi_1)$ to $\mathscr{G}(A, \xi_0)$ is onto.* (Cf. 4.2 *and* 4.3).

Proof. For properties of the attaching construction see [3, pp. 127–129]. The canonical maps h' and j' are the restrictions to E_1 and E_2, resp., of the quotient projection of the free union, $E_1 + E_2$, onto $E_1 \cup_h E_2$. Assuming that ξ is a principal bundle, then h' and j' fill in the pushout diagram ([9], p. 10) for j and h, since they do so as maps in the category of topological spaces. It remains to show that p is locally trivial. In case (2A), the coordinate neighborhoods for p_{ξ_1} and p_{ξ_2} are trivially coordinate neighborhoods for p. In case (2B), the coordinate neighborhoods for p_{ξ_1} and p_{ξ_2} in the complements of B_0 in B_1 and B_2, resp., are trivially coordinate neighborhoods for p. For an arbitrary point b of B_1, let V_2 be an open neighborhood of b relative to B_2, and let s_2 be a section of ξ_2 over V_2. Then $f^*(s_2)$ is the restriction of s_2 to a section, s_0, of ξ_0 over $V_2 \cap B_0$. By hypothesis, the germ of s_0 over b extends to a germ of a section s_1 of ξ_1 over a neighborhood V_1 of b relative to B_1. By cutting down V_1 and V_2, if necessary, we may assume that s_1, s_0, and s_2 all are defined on $V_1 \cap V_2$ and all agree there. Then $s_1 \cup s_2$ is a section of ξ over $V_1 \cup V_2$, and the coordinate function defined by $s_1 \cup s_2$ is a homeomorphism, by the functoriality of the attaching construction. In case (2C), the coordinate neighborhoods for p_{ξ_1} in the complement of B_0 in B_1 are trivially coordinate neighborhoods for p. It remains to find coordinate neighborhoods of the points b of B_2. So, let s_2 be a section of ξ_2 over an open neighborhood, V_2, of b. Let A_2 be a closed neighborhood of b contained in V_2, and let $A = f^{-1}(A_2)$. Again $f^*(s_2)$ is a section, s_0, of ξ_0 in a neighborhood of A, and by hypothesis, the germ of s_0 over A extends to a germ of a section, s_1, of ξ_1 over a neighborhood, V_1, of A relative to B_1. By cutting down V_1 and V_2, if necessary, we may assume that s_1 and s_0 are both defined on $V_1 \cap B_0$ and both agree there. Then $s_1 \cup_f s_2$ is a section of ξ (defined by functoriality of the attaching construction), the coordinate function defined by it being a homeomorphism by functoriality of the attaching construction. This completes the proof of theorem.

3. Universal covering spaces. The Seifert and Van Kampen theorem for regular covering spaces is the statement that pushouts exist in the category of regular covering spaces provided suitable conditions are satisfied by the base spaces. The construction does not generalize to locally compact principal fibre bundles since it would rely on the existence of pushouts in the category of locally compact structure groups.

A *regular covering space* of a connected base space B is a principal bundle with base B and a discrete structure group. The bundle space of a regular covering space is not assumed to be connected. The *universal covering space* of B (if it exists) is the regular covering space such that for a fixed element x_0 of E_ξ, ξ has the *universality* property: for any regular covering space ξ', for any map $f\colon B \to B_{\xi'}$, and for any point x' of $E_{\xi'}$ such that $p_{\xi'}(x') = f(p_\xi(x_0))$, there exists a unique morphism $f^-\colon \xi \to \xi'$ such that $(f^-)_B = f$, and $f^-(x_0) = x'$. It is not hard to conclude that the universal covering space of a given base space B is unique up to isomorphism, that its bundle space is connected, that the universality property does not depend upon the choice of fixed element x_0, and that it suffices to verify the universality property relative to covering spaces ξ' with the same base B and for f equal to the identity of B. It is also known that if a regular covering space of a base B has a path connected simply connected bundle space, then it is the universal covering space of B, and its structure group is naturally isomorphic to the fundamental group of B. There exists an example ([13], p. 84 and [6]) of a connected, locally path connected metric space B such that B has a universal covering space which is not simply connected.

We now consider circumstances similar to those of paragraph 2. There are given spaces B_0, B_1, and B_2 such that B_0 is a subspace of B_1. There is given a map, f, of B_0 into B_2, by means of which B_1 is attached to B_2 to form $B = B_1 \cup_f B_2$. There are given regular covering spaces ξ_0, ξ_1, and ξ_2 with respective base spaces B_0, B_1, and B_2. *No assumption is made that the structure groups are isomorphic.* There are given morphisms, j and h, of ξ_0 to ξ_1, and of ξ_0 to ξ_2, respectively, such that j_B is the inclusion map of B_0 into B_1, and such that $h_B = f$. Let E_i be the bundle space of ξ_i, for $i = 0, 1$, and 2. There exists a discrete group, K, and there exist morphisms, u_1 and u_2, of G_{ξ_1} and G_{ξ_2}, respectively, into K filling in the pushout diagram for j_G and h_G (diagram II). Let $u_0 = u_1 \circ j_G = u_2 \circ h_G$. Using these homomorphisms define the associated bundles $\xi_i \times_G K = \alpha_{u_i}$ for $i = 0, 1$, and 2. Then j and h induce morphisms

$$J\colon E_0 \times_G K \longrightarrow E_1 \times_G K\,, \qquad J(\langle x, k\rangle) = \langle j(x), k\rangle$$

$$H\colon E_0 \times_G K \longrightarrow E_2 \times_G K\,, \qquad H(\langle x, k\rangle) = \langle h(x), k\rangle\ .$$

Evidently $J_G = H_G =$ the identity homomorphism of K. Then define

$$E = (E_1 \times_G K) \bigcup\nolimits_H (E_2 \times_G K)$$

$$\xi = (\xi_1 \times_G K) \bigcup\nolimits_H (\xi_2 \times_G K)\ .$$

There are natural maps H' and J' induced by the clutching construction filling in a pushout diagram of spaces (the inside cell of diagram III). Let $h' = H' \circ u_1^\#$ and $j = J' \circ u_2^\#$. Then diagram I is a pushout diagram, provided that ξ is a principal bundle. The theorem will be that this will be so in cases (3A), (3B), and (3C).

(3A) B_1 and B_2 are open subspaces of a common space $B = B_1 \cup B_2$, and $B_0 = B_1 \cap B_2$. The inclusion of B_0 into B_2 is $h_B = f$.

(3B) B_1 and B_2 are closed subspaces of a common space $B = B_1 \cup B_2$, and $B_0 = B_1 \cap B_2$. The inclusion of B_0 into B_2 is $h_B = f$.

(3C) B_0 is a closed subspace of B_1, B_1 is paracompact, and $h_B = f$ is arbitrary.

LEMMA 3.1. *Case* (3B) *implies case* (2B), *and case* (3C) *implies* *case* (2C), *for the principal bundles* $\xi_i \times_G K$ *over* B_i, $i = 0, 1, 2$.

Proof. In case (3B) note that since K is discrete, then for a point b of B_0 any two sections of $\xi_0 \times_G K$ which are defined over neighborhoods of b must have the same germ over b. It follows that the restriction function of $\mathscr{S}(b, \xi_1 \times_G K)$ to $\mathscr{S}(b, \xi_0 \times_G K)$ is an isomorphism, and case (2B) holds. In case (3C), we regard all regular covering spaces to be sheaves of sets. Let A be any closed subset of B_0. By a standard theorem of sheaf theory ([4], p. 150), any section over A extends to a section over a neighborhood of A, and since two sections must agree over an open set, then it follows that the restriction function of $\mathscr{S}(A, \xi_1 \times_G K)$ to $\mathscr{S}(A, \xi_0 \times_G K)$ is an isomorphism and case (2C) holds.

THEOREM 3.2. *Let* ξ_0, ξ_1, ξ_2, f, j, h, *and* ξ *be as above. Under any of the conditions* (3A), (3B), *or* (3C), *diagram I is a pushout diagram in the category of regular covering spaces, and the induced diagram II of homomorphisms of structure groups is a pushout diagram in the category of groups.*

Proof. Cells (i) and (ii) of diagram III are commutative as an application of (1.3). Let $h' = H' \circ u_1^\#$, and $j = J' \circ u_2^\#$. Since H'_G and J'_G both are the identity homomorphism of K, then

$$h'_G = (u_1^\#)_G = u_1 \ ,$$

and

$$j'_G = u_2 \ .$$

We show that h' and j' fill in the pushout diagram for j and h in the category of regular covering spaces. Suppose that for $i = 0, 1$, and 2, there are given morphisms $l_i: \xi_i \longrightarrow \xi'$ such that $l_0 = j \circ l_1 = h \circ l_2$. Since then

$$(l_0)_G = (l_1)_G \circ j_G = (l_2)_G \circ h_G$$

then there exists a unique homomorphism $v: K \rightarrow G_{\xi'}$ such that

$$v \circ h'_G = l_1 \ ,$$

and

$$v \circ j'_G = l_2 \ .$$

For each $i = 0, 1$ or 2, l_i induces a unique morphism

$$(l_i, v)^\#: \xi_i \times_G K \longrightarrow \xi'$$

such that

$$(l_i, v)^{\#}_G = v \ , \quad \text{and} \quad (l_i, v)\circ u^{\#}_i = l_i \ .$$

It follows that

$$(l_1, v)^{\#}\circ J = (l_0, v)^{\#} = (l_2, v)^{\#}\circ H \ .$$

Since H' and J' fill in the pushout diagram for J and H then $(l_1, v)^{\#}$ and $(l_2, v)^{\#}$ induce a unique morphism $l\colon \xi \to \xi'$ such that

$$(l_1, v)^{\#} = l\circ H' \ , \quad \text{and} \quad (l_2, v)^{\#} = l\circ J' \ .$$

By its construction, l satisfies

$$l\circ h' = l_1 \ , \quad \text{and} \quad l\circ j' = l_2$$

and l is the unique morphism which does so. Since l_1 and l_2 were arbitrary such that $l_1\circ j = l_2\circ h$, this completes the proof that h' and j' fill in the pushout diagram for j and h.

The Corollary 3.3 is the analogue for universal covering spaces of the Seifert and Van Kampen theorem.

COROLLARY 3.3. *Under the conditions of* 3.2, *if* ξ_i *is the universal covering space of* B_i *for* $i = 0, 1,$ *and* 2, *then* ξ *is the universal covering space of* B, *and the diagram II of induced homomorphisms of structure groups is a pushout diagram in the category of groups.*

Proof. There exist morphisms $j\colon \xi_0 \to \xi_1$, and $h\colon \xi_0 \to \xi_2$ such that $j_B =$ the inclusion of B_0 into B_1, and $h_B = f$. Then there exist pushout diagrams in the category of covering spaces and in the category of groups. (See illustration I.) Let ξ' be any regular covering space of B, and suppose there are given points x of E_ξ, such that $p_\xi(x) = p_{\xi'}(x')$. We must find a morphism $l\colon \xi \to \xi'$ such that $l(x) = x'$, and show that such a morphism is unique, in order to complete the proof. We may assume that x is so chosen that there is a point x_0 in E_{ξ_0} such that $h'(j(x_0)) = x$. For $i = 0, 1,$ and 2, let $l_i\colon \xi_i \to \xi'$ be the unique morphism such that $(l_i)_B$ is the natural map of B_i into B defined by the attaching construction, and such that $l_0(x_0) = x'$, $l_1(j(x_0)) = x'$, and $l_2(h(x_0)) = x'$. Since they agree at one point, x_0, the maps l_0, $l_1\circ j$, and $l_2\circ h$ are all equal. Let l be the unique morphism such that $l\circ h' = l_1$, and $l\circ j' = l_2$. This completes the proof.

COMMENT 3.4. Theorem 3.2 here stated and proven for all principal bundles with structure groups in the category of (discrete) groups, could be stated and be valid with no change in proof for any full subcategory of the category of principal bundles provided the corresponding category of structure groups had pushouts. The notion of universal principal bundle in that category would make sense provided the structure groups were discrete. For example, by taking the corresponding category of structure groups to be the category of abelian groups, one arrives at the notion of universal *abelian* regular covering space, and Theorems 3.2 and 3.3 in terms of *abelian* regular covering spaces would remain valid.

4. Extension properties of germs. The extension properties of germs are developed for principal bundles in general, there being interest in the validity of the clutching construction. There is a trade-off in hypotheses to be made, it being necessary to strengthen hypotheses on the base spaces in order to admit less stringent conditions on the structure group.

Throughout this section ξ_1 is a principal bundle with base B_1 and structure group K, and B_0 is a closed subspace of B_1. Let $\xi_0 = \xi_1 | B_0 = j^*(\xi_1)$ be the restriction of ξ_1 to B_0, where j is the inclusion of B_0 into B_1. Recall that for a closed subset A of B_1, one says of A that it has the *neighborhood extension property* in B_1 relative to K provided that for any map f of A into K, there exists an extension of f to a map of a neighborhood of A into K.

THEOREM 4.1. *If every closed subset of B_1 has the neighborhood extension property in B_1 relative to K, and if B_1 is paracompact, then case* (2C) *holds.*

Proof. The proof is a straightforward modification of standard arguments which conclude a global property from the corresponding local property; see [14], p. 55, or [5], Theorem 2.7. To begin, let V_0 be a closed relative neighborhood of A in B_0, and let s_0 be any section of ξ_0 over V_0. It will be shown that there exists an extension of s_0 to a section s_1 of ξ_1 over a closed neighborhood, V_1, of V_0 in B_1. That would suffice to prove the theorem. Let $\{U_i\}_{i \in I}$ be a locally finite open covering of B_1 by coordinate neighborhoods, and for each index i, let s_i be a section of ξ_1 over U_i. Let $\{V_i\}_{i \in I}$ be a closed covering of B_1 such that for each index i, V_i is contained in U_i. For any subset J of I, let

$$V_J = V_0 \cup (\bigcup \{V_i : i \in J, \text{ and } V_0 \cap V_i \neq \varnothing\}) .$$

Let $\mathscr{S}$ be the set of all pairs (J, s), where J is a subset of I, and s is an extension of s_0 to a section of ξ_1 over a closed set, N_s, which is a relative neighborhood of V_0 in V_J. $\mathscr{S}$ is partially ordered by the relation defined as $(J, s) \leqq (J', s')$ if $J \subset J'$ and s' extends s. By the Hausdorff maximal principal, there exists a maximal chain, $\mathscr{C}$, contained in $\mathscr{S}$. If we let

$$J' = \bigcup \{J: (J, s) \in \mathscr{C}\} ,$$

and

$$s' = \bigcup \{s: (J, s) \in \mathscr{C}\} ,$$

then (J', s') is the maximum of $\mathscr{C}$. If $J' = I$, then $N_{s'}$ would be a neighborhood of V_0 in B_1, and the theorem would be proven. Suppose that there were an index i not contained in J'. Then $V_i \cap V_0 \neq \emptyset$, for otherwise $(J' \cup \{i\}, s')$ would be an element of $\mathscr{S}$ greater than (J', s'). Let

$$t: N_{s'} \cap V_i \longrightarrow K$$

be the map defined by the equation

$$s'(x) = s_i(x) \cdot t(x) , \qquad x \in N_{s'} \cap V_i .$$

By the neighborhood extension property hypothesis, there is an extension, t', of t to a map of a closed neighborhood, M, of $N_{s'} \cap V_i$ into K. Then *extend* s' to a section s'' over $N_{s'} \cup (M \cap V_i)$ by letting, for any point x of $M \cap V_i$,

$$s''(x) = s_i(x) \cdot t'(x) .$$

Then $(J' \cup \{i\}, s'')$ is an element of $\mathscr{S}$ greater than (J', s'), contrary to the maximality of $\mathscr{C}$ and of (J', s''). It follows that $J' = I$. As previously observed, this proves the theorem.

COROLLARY 4.2. *If B_1 is a paracompact space, and K is a Lie group, then case* (2C) *holds.*

Proof. Since K is topologically complete and an ANR, then [8] every closed subset of B_1 has the neighborhood extension property in B_1 relative to K, and so 4.1 implies 4.2.

If the structure group, K, is not a Lie group, all is not lost.

THEOREM 4.3. *If ξ_1 has the homotopy lifting property and B_0 is a neighborhood deformation retract of B_1, and in particular if the inclusion of B_0 into B_1 is a cofibration, then case* (2C) *holds.*

Proof. This straightforward application of the homotopy lifting property is left to the reader.

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