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TENSOR AND DIRECT PRODUCTS

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Let R be an associative ring with 1, E a unitary right module, and $(F_i)_{i \in I}$ a family of unitary left modules. Let $f: E \otimes_R \prod F_i \rightarrow \prod (E \otimes_R F_i)$ be the canonical map. THEOREM. f is bijective (surjective) for all families (F_i) iff E is finitely presented (finitely generated). Theorem. If R is a Dedekind domain or is commutative artinian and every F_i is flat, then f is injective. COROLLARY. If R is a Dedekind domain or is commutative artinian, every F_i is flat and $E \otimes_R F_i$ is reduced, then $E \otimes_R \prod F_i$ is reduced. THEOREM. If R is a Dedekind domain or is commutative artinian, E_j is flat, f is injective for every E_j (e.g. E_j projective) and E is pure in $\prod E_j$, then f is injective. THEOREM. If R is a Dedekind domain and E is flat then f is injective for E iff f is injective for Hom(F, E) for all modules F. THEOREM. If R is a Dedekind domain and f is injective for E for all families (F_i) then E is reduced. THEOREM. If R is commutative and f is always injective then R must be artinian. The converse holds for serial rings.

Introduction. If E is a right module and $(F_i)_{i \in I}$ is a family of left modules over an associative ring R with 1 then it is always true that the groups $E \otimes_R (\bigoplus_{i \in I} F_i)$ and $\bigoplus_{i \in I} (E \otimes_R F_i)$ are isomorphic. However, it is not hard to see that the groups $E \otimes_R \prod_{i \in I} F_i$ and $\prod_{i \in I} (E \otimes_R F_i)$ are not necessarily isomorphic (e.g. if R is the integers let E be the rationals, I the natural numbers and, for a fixed prime p, F_n the cyclic group of order p^n. This example is found in ([8], p. 257, Ex. 10).

It is our purpose to study the relationship between the groups $E \otimes_R \prod_{i \in I} F_i$ and $\prod_{i \in I} (E \otimes_R F_i)$. We will do so in terms of the natural homomorphism

$$f: E \otimes_R \prod_{i \in I} F_i \longrightarrow \prod_{i \in I} (E \otimes_R F_i)$$

which sends a generator $x \otimes (y_i)_{i \in I}$ of $E \otimes_R \prod_{i \in I} F_i$ onto $(x \otimes y_i)_{i \in I}$ in $\prod_{i \in I} (E \otimes_R F_i)$.

In §§1 and 2 we investigate the properties of f over an arbitrary ring. It is relatively easy to show that f is surjective (bijective) for all families $(F_i)_{i \in I}$ if and only if E is finitely generated (finitely presented). In §§3 and 4 we study the more difficult problem of when f is injective. In §3 the ring is a Dedekind domain. In §4 the ring is commutative artinian.

We will always denote by R an associative ring with 1, by E a right R-module, and by $(F_i)_{i \in I}$ a family of left R-modules. Modules are all unitary.

We will often use the notions of flatness and purity. Briefly, a left R-module F is said to be flat if the functor $(\ \ )\otimes_R F$ is exact. A submodule E' of a right R-module E is said to be pure if $(\ \ )\otimes_R F$ is exact on $0 \to E' \xrightarrow{\text{incl}} E \xrightarrow{\text{proj}} E/E' \to 0$ for every right module F.

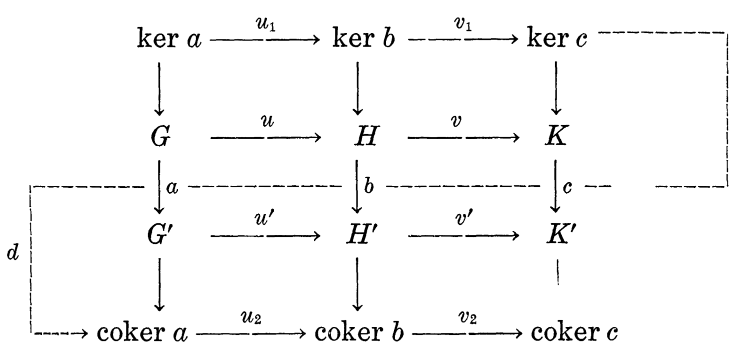
Over a domain it is well known that torsion free modules are flat iff the domain is Prüfer. It is also known that over a domain, E' is pure in E means $rE' = E' \cap rE$ for all $r \in R$ iff the domain is Prüfer. For this last fact see [13].

Many of our arguments involve a diagram chase. At these points we usually will be able simply to cite the following.

Snake Lemma [2, Ch. 1, §1, No. 4, Prop. 2]. Suppose

$$\begin{array}{ccccc} G & \xrightarrow{u} & H & \xrightarrow{v} & K \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ G' & \xrightarrow{u'} & H' & \xrightarrow{v'} & K' \end{array}$$

is a commutative diagram of abelian groups with exact rows. Then this diagram can be embedded in the following commutative diagram with top and bottom row not necessarily exact.



The following are true of this diagram

(i) $v_1 \circ u_1 = 0$. If u' is injective then the top row is exact.

(ii) $v_2 \circ u_2 = 0$. If v is surjective the bottom row is exact.

(iii) If u' is injective and v is surjective there exists a unique homomorphism d such that the sequence (u_1, v_1, d, u_2, v_2) is exact.

For convenience we will omit writing the indexing set whenever there is no possibility of confusion. For example, we will write $\prod F_i$, $\bigoplus F_i$ and (F_i), omitting the index set I. fg and fp will denote, respectively, finitely generated and finitely presented. Instead of writing $E \otimes_R F$ we will simply write $E \otimes F$. Often an obvious homomorphism will not be defined explicitly or even named. Z will be the ring of integers, Q the field of rational numbers, and, for a positive integer

n, $Z(n)$ will denote the cyclic group of order n.

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1. Surjectivity and bijectivity.

PROPOSITION 1.1. *If E is fg and projective then f is bijective.*

Proof. If E' is fg and free then for some positive integer n, $E' \approx R^n$. We have the canonical isomorphisms

$$E' \otimes \prod F_i \approx R^n \otimes \prod F_i \approx (\prod F_i)^n$$

and

$$\prod (E' \otimes F_i) \approx \prod (R^n \otimes F_i) \approx \prod F_i^n .$$

So f is just the canonical isomorphism $(\prod F_i)^n \approx \prod F_i^n$.

If E is fg and projective then E is a direct summand of such an E'. We show later (Prop. 2.1) that if f is injective for E' then f is injective for any summand of E'.

PROPOSITION 1.2. *f is surjective (for all families (F_i)) iff E is fg.*

Proof. Suppose E is fg, n is a positive integer, and $R^n \to E \to 0$ is exact. Consider the commutative diagram with exact rows

$$\begin{array}{ccccc} R^n \otimes \prod F_i & \longrightarrow & E \otimes \prod F_i & \longrightarrow & 0 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\ \prod (R^n \otimes F_i) & \longrightarrow & \prod (E \otimes F_i) & \longrightarrow & 0 . \end{array}$$

f' is bijective (Prop. 1.1). The snake lemma tells us f is surjective.

On the other hand, suppose E is not fg. Let $I = E$ (E considered as a set), $F_i = {}_RR$. Write $E = \{x_i\}$. Then $(x_i \otimes 1)_{i \in I} \in \prod (E \otimes F_i)$ is not in imf. So f is not surjective.

REMARK 1. In order to show E fg all we need to show is that f is surjective for all families (F_i) with $F_i = {}_RR$.

REMARK 2. The part of Proposition 1.2 that supposes E fg is found in [3, Ch. II, Ex. 2].

If g and h are injective homomorphisms of right, respectively, left R-modules, having domains E and F, respectively, we denote im $(g \otimes h)$ as $[E \otimes F]$.

PROPOSITION 1.3. *f is bijective (for all families (F_i)) iff E is fp.*

Proof. Suppose E is fp and $0 \to K \to L \to E \to 0$ is a finite presentation of E. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [K \otimes \prod F_i] & \longrightarrow & L \otimes \prod F_i & \longrightarrow & E \otimes \prod F_i & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f & & \\ 0 & \longrightarrow & \prod [K \otimes F_i] & \longrightarrow & \prod (L \otimes F_i) & \longrightarrow & \prod (E \otimes F_i) & \longrightarrow & 0 . \end{array}$$

f_1 is surjective because K is fg (Prop. 1.2). f_2 is bijective because L is fg and free (Prop. 1.1). By the snake lemma, therefore, f is bijective.

On the other hand, suppose f is bijective for all families (F_i). Since f is surjective, E is fg (Prop. 1.2). So we have an exact sequence $0 \to K \to L \to E \to 0$ where L is fg and free. By Remark 1 following Prop. 1.2 to show K fg it suffices to show f is surjective for K for all families (F_i), $F_i = {}_R R$.

For such a family consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & K \otimes \prod F_i & \longrightarrow & L \otimes \prod F_i & \longrightarrow & E \otimes \prod F_i & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f & & \\ 0 & \longrightarrow & \prod (K \otimes F_i) & \longrightarrow & \prod (L \otimes F_i) & \longrightarrow & \prod (E \otimes F_i) & \longrightarrow & 0 . \end{array}$$

f_2 is bijective because L is fg and free (Prop. 1.1). f is bijective by assumption. Therefore, by the snake lemma, f_1 is surjective.

REMARK 1. It is found in [3, Ch. II, Ex. 2] that if R is right noetherian and E fg then f is bijective. More generally, [2, Ch. 1, Ex. 8] states that if E is fp then f is bijective.

REMARK 2. The first half of the proof of Prop. 1.3 shows that if f is injective for E then f is injective for any quotient of E by a fg submodule.

2. Injectivity.

PROPOSITION 2.1. *Suppose J is a set and E_j is a right R-module for every $j \in J$. Let $E \approx \bigoplus E_j$. Then f is injective for E iff f is injective for every E_j. If E' is pure in E and f is injective for E then f is injective for E'.*

Proof. It is easy to see that if J is finite and f is injective for every E_j then f is injective for E. Suppose J is infinite,

$$z = \sum_{\lambda=1}^{n} x_\lambda \otimes y_\lambda \in E \otimes \prod F_i ,$$

and $f(z) = 0$. Let J' be the support of $\{x_\lambda\}_{\lambda=1}^n$. Consider the commutative diagram

$$\begin{array}{ccc} \left(\bigoplus_{J'} E_j\right) \otimes \prod F_i & \xrightarrow{f'} & \prod\left(\left(\bigoplus_{J'} E_j\right) \otimes F_i\right) \\ \downarrow g & & \downarrow h \\ \left(\bigoplus_{J} E_j\right) \otimes \prod F_i & \xrightarrow{f} & \prod\left(\left(\bigoplus_{J} E_j\right) \otimes F_i\right) \end{array}$$

where g and h are the obvious injective homomorphisms. If $z' = \sum_{\lambda=1}^n x_\lambda \otimes y_\lambda \in (\bigoplus_{J'} E_j) \otimes \prod F_i$ then $g(z') = z$. But since J' is finite, f' is injective. Therefore $z' = 0$, hence $z = 0$. So f is injective.

For the remainder of the proposition suppose f is injective for E and E' is pure in E. Consider the commutative diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & E' \otimes \prod F_i & \longrightarrow & E \otimes \prod F_i \\ & & \downarrow f' & & \downarrow f \\ 0 & \longrightarrow & \prod (E' \otimes F_i) & \longrightarrow & \prod (E \otimes F_i)\ . \end{array}$$

Since f is injective the snake lemma says f' is injective.

REMARK. It is not hard to see that if E' is a pure submodule of E and f is injective for E' and E/E' then f is injective for E. Merely apply the snake lemma to the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' \otimes \prod F_i & \longrightarrow & E \otimes \prod F_i & \longrightarrow & E/E' \otimes \prod F_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod (E' \otimes F_i) & \longrightarrow & \prod (E \otimes F_i) & \longrightarrow & \prod (E/E' \otimes F_i) & \longrightarrow & 0\ . \end{array}$$

PROPOSITION 2.2. *If E is the quotient of a projective by a fg submodule then f is injective.*

Proof. If E is free and $E \approx \bigoplus R$ then we have the canonical isomorphisms

$$E \otimes \prod F_i \approx (\bigoplus R) \otimes \prod F_i \approx \bigoplus \prod F_i$$

and

$$\prod (E \otimes F_i) \approx \prod (\bigoplus R \otimes F_i) \approx \prod \bigoplus F_i\ .$$

So f is the injection $\bigoplus \prod F_i \to \prod \bigoplus F_i$.

If E is projective then E is a summand of a free module. Hence

f is injective for E (Prop. 2.1). The proposition now follows from Remark 2 following Prop. 1.3.

We want to use the following result of Matlis.

THEOREM 2.3. (Matlis, [12].) *If R is right noetherian and E is injective then E is the direct sum of indecomposable, injective submodules.*

It will be useful to consider f when every F_i of the family (F_i) is flat. Such a family will be called a flat family.

PROPOSITION 2.4. *f is injective for all flat families (F_i) iff R is right noetherian and f is injective for all indecomposable injective E for all flat families (F_i).*

Proof. Suppose f is injective for all flat families (F_i) and E is a right ideal of R. Recall that to show E is fg it suffices to show f is surjective for E for all families (F_i) with $F_i = {}_RR$ (Remark 1, following Prop. 1.2). So suppose (F_i) is such a family. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & E \otimes \prod F_i & \longrightarrow & R \otimes \prod F_i & \longrightarrow & R/E \otimes \prod F_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ 0 & \longrightarrow & \prod (E \otimes F_i) & \longrightarrow & \prod (R \otimes F_i) & \longrightarrow & \prod (R/E \otimes F_i) & \longrightarrow & 0 \,. \end{array}$$

f_1 is bijective. f_2 is injective by supposition. By the snake lemma f is surjective.

For the converse, suppose (F_i) is a flat family and $E' = \bigoplus E_j$ is the injective envelope of E where E_j is indecomposable injective. By Prop. 2.1, f is injective for E'. Consider the commutative diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & E \otimes \prod F_i & \longrightarrow & E' \otimes \prod F_i \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ 0 & \longrightarrow & \prod (E \otimes F_i) & \longrightarrow & \prod (E' \otimes F_i) \,. \end{array}$$

(The top row is exact because over a right noetherian ring the direct product of flat left modules is flat, [3, Ch. VI, Ex. 4].) Since f' is injective the snake lemma says f is also injective.

3. Dedekind domains. Let K be the field of fractions of a domain R. If R is a Dedekind domain the indecomposable injective R-modules are K and, for primes P, the P-primary components of K/R [9, Thm. 7]. Adopting the notation that is standard when R

is the ring of integers we will denote the P-primary component of K/R as $R(P^\infty)$.

In order to establish that Dedekind domains possess the property of Prop. 2.4 we prove the following lemma.

LEMMA 3.1. *If S is a multiplicative subset of an arbitrary commutative ring R, $E = S^{-1}R$ and (F_i) is a family of torsion free R-modules then f is injective.*

Proof. If $0 \in S$ then $S^{-1}R = 0$. So assume $0 \notin S$. We have the canonical isomorphisms [2, Ch. II, §2, no. 7, Prop. 18], $S^{-1}R \otimes \prod F_i \approx S^{-1}(\prod F_i)$ and $\prod (S^{-1}R \otimes F_i) \approx \prod (S^{-1}F_i)$. So we can think of f as the canonical map $S^{-1}(\prod F_i) \longrightarrow \prod (S^{-1}F_i)$. Suppose $f((y_i)/s) = (y_i/s) = 0$. Then for every $i \in I$ there exists $t_i \in S$ such that $t_i y_i = 0$. Since $t_i \neq 0$, $y_i = 0$. Hence f is injective.

PROPOSITION 3.2. *If R is a Dedekind domain then f is injective for all flat families (F_i).*

Proof. By Prop. 2.4 and Prop. 2.1 it suffices to show f is injective for K and K/R. But since we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R \otimes \prod F_i & \longrightarrow & K \otimes \prod F_i & \longrightarrow & K/R \otimes \prod F_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & \prod (R \otimes F_i) & \longrightarrow & \prod (K \otimes F_i) & \longrightarrow & \prod (K/R \otimes F_i) & \longrightarrow & 0 \end{array}$$

where f_1 is bijective we know from the snake lemma that f_2 is injective iff f_3 is injective. That f_2 is injective is Lemma 3.1. The proof is now complete.

Since a Prüfer domain is Dedekind iff it is noetherian we have an immediate corollary.

COROLLARY 3.3. *If R is a Prüfer domain then f is injective for all flat families (F_i) iff R is Dedekind.*

There is an interesting generalization of Prop. 3.2. We need the following lemma whose proof is immediate from the definitions.

LEMMA 3.4. *If R is a Prüfer domain and (F_i, g_i^j) is an inverse system of flat R-modules then $\varprojlim F_i$ is a pure submodule of $\prod F_i$.*

THEOREM 3.5. *If R is a Dedekind domain and (F_i, g_i^j) is an* *inverse system of flat R-modules then the canonical homomorphism*

$$F\colon E \otimes \varprojlim F_i \longrightarrow \varprojlim (E \otimes F_i)$$

is injective.

Proof. Clearly $(E \otimes F_i, \mathrm{id} \otimes g_i^j)$ is an inverse system and F is a homomorphism. To show F injective consider the commutative diagram

$$\begin{array}{ccc} E \otimes \varprojlim F_i & \xrightarrow{F} & \varprojlim (E \otimes F_i) \\ \downarrow u & & \downarrow v \\ E \otimes \prod F_i & \xrightarrow{f} & \prod (E \otimes F_i) \end{array}$$

where u and v are the obvious maps. u is injective (Lemma 3.4). f is injective (Prop. 3.2). By the Snake lemma F must also be injective.

REMARK. Suppose R is any ring for which f is injective for all flat families (F_i). Then it is easy to see that F is injective for the inverse system of flat left modules (F_i) iff $\varprojlim F_i$ is pure in $\prod F_i$. Just chase the following commutative diagram

$$\begin{array}{ccc} E \otimes \varprojlim F_i & \longrightarrow & E \otimes \prod F_i \\ \downarrow F & & \downarrow f \\ \varprojlim (E \otimes F_i) & \longrightarrow & \prod (E \otimes F_i) \end{array} .$$

In Prop. 2.2 we established that f is injective whenever E is projective. We will see later (Thm. 3.13) that this cannot be extended to the case when E is flat. But when R is a Dedekind domain we do have the following generalization.

THEOREM 3.6. *If R is a Dedekind domain, $(E_j)_{j \in J}$ is a family of flat R-modules such that f is injective for every E_j* (e.g. E_j *is projective), and E is isomorphic to a pure submodule of $\prod E_j$, then f is injective for all families (F_i).*

Proof. Consider the following diagram where all maps are the obvious ones

$$\begin{array}{ccccc} \prod_J E_j \otimes \prod_I F_i & \xrightarrow{f} & \prod_I (\prod_J E_j \otimes F_i) & \xrightarrow{f_1} & \prod_I \prod_J (E_j \otimes F_i) \\ \downarrow{\scriptstyle f_2} & & & & \\ \prod_J (E_j \otimes \prod_I F_i) & & & & \\ \downarrow{\scriptstyle f_3} & & & & \\ \prod_J \prod_I (E_j \otimes F_i) & & & & \\ \downarrow{\scriptstyle f_4} & & & & \\ \prod_I \prod_J (E_j \otimes F_i) \,. & & & & \end{array}$$

f_1 and f_2 are injective (Prop. 3.2) and f_3 is injective by hypothesis. Therefore, $f_4 \circ f_3 \circ f_2 = f_1 \circ f$ is injective. Hence f is injective. Now apply Prop. 2.1.

REMARK 1. We have not used any specific property of Dedekind domains in Thm. 3.6. The theorem holds for any ring for which f is injective for flat families (F_i).

REMARK 2. If in Thm. 3.6, J is infinite then $\prod_J Z$ is not free (e.g. [8, Thm. 19.2]). Indeed it is not hard to see that in any presentation of $\prod_J Z$ there are infinitely many defining relations [8, p. 95, Ex. 6]. Thus Thm. 3.6 genuinely enlarges the class of modules E as described in Prop. 2.2 for which f is injective.

COROLLARY 3.7. *If R is a Dedekind domain, $(E_j)_{j \in J}$ is an inverse system of flat R-modules such that f is injective for every E_j* (e.g. *E_j is projective*), *and $E = \varprojlim E_j$ then f is injective.*

Proof. By Lemma 3.4, $\varprojlim E_j$ is pure in $\prod E_j$. Now apply Thm. 3.6.

There is a characterization of modules E for which f is injective in terms of groups of homomorphisms. First a crucial result of D. Lazard.

THEOREM 3.8. (Lazard, [11].) *A right R-module E is flat iff E is the direct limit of fg free modules.*

DEFINITION 3.9. If R is commutative and E and F are R-modules we will call the module $\mathrm{Hom}_R(E, F)$ the F-dual of E.

The R-dual of E is commonly called, simply, "the dual of E". Henceforth we will omit the R when writing $\mathrm{Hom}_R(E, F)$.

THEOREM 3.10. *If R is a Dedekind domain and E is a flat R-module then f is injective for E iff f is injective for all E-duals.*

Proof. If f is injective for all E-duals then f is injective for the E-dual of R, $\text{Hom}(R, E) \approx E$.

Suppose, on the other hand, that f is injective for E. To show f is injective for all E-duals it suffices to consider the E-duals of flat modules since if T is the torsion submodule of an arbitrary module F then $\text{Hom}(F, E) \approx \text{Hom}(F/T, E)$. But if F is flat then $F \approx \varinjlim F_j$ where F_j is fg and free, say of rank n_j. Therefore, $\text{Hom}(F, E) \approx \text{Hom}(\varinjlim F_j, E) \approx \varprojlim \text{Hom}(F_j, E)$ where $\text{Hom}(F_j, E) \approx E^{n_j}$. Since f is injective for $\text{Hom}(F_j, E)$ (Prop. 2.1) and $\text{Hom}(F_j, E)$ is flat, f is injective for $\varprojlim \text{Hom}(F_j, E)$ (Cor. 3.7). Hence f is injective for $\text{Hom}(F, E)$.

REMARK 1. Suppose R is a Dedekind domain, (E_j) is family of flat modules, E' is a pure submodule of the flat module E, and f is injective for E and every E_j. Then (a) f is injective for $\bigoplus E_j$ (Prop. 2.1), (b) f is injective for $\prod E_j$ (Thm. 3.6), and (c) f is injective for E' (Prop. 2.1). By Thm. 3.10 we also know (a′) f is injective for all $\bigoplus E_j$-duals, (b′) f is injective for all $\prod E_j$-duals, and (c′) f is injective for all E'-duals. ((a) and (a′) are actually special cases of (c) and (c′) respectively.) It is perhaps worthwhile to note that (a′), (b′), and (c′) can be proved directly by noting the following where $F \approx \varprojlim F_\lambda$ is an arbitrary flat module, F_λ is free and of finite rank n_λ.

(i) $\text{Hom}(F, \bigoplus E_j)$ is pure in $\text{Hom}(F, \prod E_j)$.

(ii) f is injective for $\text{Hom}(F, \prod E_j) \approx \prod \text{Hom}(F, E_j) \approx \prod \varprojlim E_j^{n_\lambda}$ by Cor. 3.7 and Thm. 3.6.

(iii) $\text{Hom}(F, E') \approx \varprojlim (E')^{n_\lambda}$ is pure in $\text{Hom}(F, E) \approx \varprojlim E^{n_\lambda}$ and f is injective for $\varprojlim E^{n_\lambda}$ by Cor. 3.7.

REMARK 2. It is known that the Z-dual of a countable direct product of copies of Z is free [6]. Thus $E = \text{Hom}(F, Z)$ when F is such a product does not enlarge beyond our previous knowledge our class of flat groups E for which f is injective.

In general when E is flat and f is injective for E it seems largely unknown what the E-duals are. ($\text{Hom}(\prod^\infty Z, Z) \approx \bigoplus^\infty Z$ is the only example we know.) Thus it is unclear how and if Thm. 3.10 does in fact enlarge the class of flat groups E for which f is injective. The identity of such E-duals seems a matter of general and independent interest.

It is an easy consequence of [3, Ch. I, Ex. 8] that any right module over a right noetherian ring possesses a maximal injective submodule. The complement of this maximal injective submodule has no injective submodules. Therefore, over a Dedekind domain, an arbitrary module E can be written as a direct sum $E_1 \oplus E_2 \oplus E_3$ where E_1 is a direct sum of $R(P^\infty)$'s, E_2 is a direct sum of quotient fields K, and E_3 has no injective submodules. Since injectivity $\equiv$ divisibility over a Dedekind domain [9, Thm. 6], it is consistent with terminology in abelian groups to say E_3 is reduced. We claim that if f is injective for E then E is reduced, i.e., $E_1 \oplus E_2 = 0$. We need the following lemmas.

LEMMA 3.11. *If R is a Dedekind domain, P is a prime ideal and $a \in R - P$ then R/P^n is a-divisible for every positive integer n.*

Proof. Define $(P^n: a) = \{b \in R: ba \in P^n\}$. To show

$$g: R/P^n \longrightarrow R/P^n$$

given by $b + P^n \longrightarrow ab + P^n$ is bijective note that $(P^n: a) = P^n$. Thus g is injective. But R/P^n has a (finite) composition series. Hence g must be surjective.

LEMMA 3.12. *If R is a Dedekind domain with prime ideal P, $F = \prod_{n=1}^{\infty}(R/P^n)$, and h is the canonical homomorphism $R \otimes F \rightarrow K \otimes F$ then h is not surjective.*

Proof. Note that $K \otimes F$ is divisible. If im h were divisible then for an arbitrary $0 \neq p \in P$ there would exist $1 \otimes (b_n)_{n=1}^{\infty} \in R \otimes F$ such that $1 \otimes (1 - pb_n)_{n=1}^{\infty} \in \ker h$. But ker h is torsion [3, Ch. VII, Prop. 4.6]. (One can see this directly.) Thus there is $0 \neq t \in R$ such that $t(1 - pb_n) = 0$ for all n. Setting $x_n = 1 - pb'_n$ where $b'_n \in R$ and $b'_n + P^n = b_n$, we have $tx_n \in P^n$. Let m be the largest integer such that $t \in P^m$. If $t \notin P$, let $m = 0$. We claim $x_{m+1} \in P$.

If not there is $y \in R$ such that $x_{m+1}y - 1 = s \in P$. So $yx_{m+1}t = (1 + s)t = t + st \in P^{m+1}$. Thus $t \in P^{m+1}$. Contradiction.

Since $x_{m+1} = 1 - pb'_{m+1}$, $1 = x_{m+1} + pb'_{m+1} \in P$. Contradiction. So im h could not have been divisible.

THEOREM 3.13. *If R is a Dedekind domain and f is injective for E then E must be reduced.*

Proof. By Prop. 2.1 it suffices to show for a given prime ideal P there exists a family (F_i) such that f is not injective for K and f is not injective for $R(P^\infty)$. We do this by choosing $F_n = R/P^n$, $n =$

1, 2, $\cdots$, and showing $K \otimes F_n = R(P^\infty) \otimes F_n = 0$ but $K \otimes \prod_{n=1}^\infty F_n \neq 0 \neq R(P^\infty) \otimes \prod_{n=1}^\infty F_n$.

Since $R(P^\infty)$ and K are divisible, $R(P^\infty) \otimes F_n = K \otimes F_n = 0$.

Set $F = \prod_{n=1}^\infty F_n$ and denote by T the torsion submodule of F. We have the exact sequence

$$K \otimes F \longrightarrow K \otimes F/T \longrightarrow 0$$

where K and F/T are nonzero flat modules. Since $K \otimes F/T \neq 0$, $K \otimes F \neq 0$ by exactness.

Suppose Q is a prime different from P, m is a positive integer and $a \in Q^m - P$. Then F_n is a-divisible for every n (Lemma 3.11). So F is a-divisible. Hence $R(Q^\infty) \otimes F = 0$. Therefore, $R(P^\infty) \otimes F \approx K/R \otimes F$.

To show $K/R \otimes F \neq 0$ consider the exact sequence

$$R \otimes F \xrightarrow{h} K \otimes F \longrightarrow K/R \otimes F \longrightarrow 0$$

where h is the obvious map. Lemma 3.12 says h is not surjective. By exactness $K/R \otimes F \neq 0$.

REMARK. Over a Dedekind domain a reduced module that is not flat is either (1) a direct sum of a flat module and a finite direct sum of modules of type R/P^n where P is prime, or (2) has an infinite proper chain of direct summands each summand itself a finite direct sum of modules of type R/P^n [9, Thm. 9].

Since f is injective for any module of type R/P^n (Prop. 1.3), injectivity for modules of type (1) reduces to injectivity of reduced flat modules. Although we have some important examples of reduced flat modules for which f is injective, f need not be injective for such a module. This is shown by Prop. 3.15.

It is also true that f need not be injective for modules of type (2). This is shown by Prop. 3.14.

PROPOSITION 3.14. *Suppose p is a prime integer and*

$$E = \prod_{n=1}^\infty Z(p^n) .$$

Then f is not injective for E. (Obviously E is of type (2).)

Proof. Choose a prime q different from p. Let $F = \prod_{n=1}^\infty Z(q^n)$ and denote by S and T the torsion subgroups of E and F respectively. Consider the exact sequence

$$E \otimes F \longrightarrow E/S \otimes F/T \longrightarrow 0 .$$

Since E/S and E/T are nonzero and flat, $E/S \otimes F/T \neq 0$. By exactness, $E \otimes F \neq 0$.

To complete the proof note that $E \otimes Z(q^n) = 0$ for every integer n.

REMARK. It is easy to check that $\prod Z(p)$ where p ranges over the primes is another group of type (2) for which f is not injective.

PROPOSITION 3.15. *Suppose E is a reduced torsion free abelian group that is divisible for a prime integer p. Then f is not injective for E.*

Proof. Since E is p-divisible, $E \otimes Z(p^n) = 0$ for all positive n. Therefore $\prod_{n=1}^{\infty} (E \otimes Z(p^n)) = 0$.

If $F = \prod_{n=1}^{\infty} Z(p^n)$ and T is the torsion subgroup of F consider the exact sequence

$$E \otimes F \longrightarrow E \otimes F/T \longrightarrow 0 .$$

Since E and F/T are nonzero and torsion free $E \otimes F/T \neq 0$. By exactness, $E \otimes F \neq 0$.

Since $E \otimes \prod_{n=1}^{\infty} Z(p^n) \neq 0$ and $\prod_{n=1}^{\infty} (E \otimes Z(p^n)) = 0$,

$$f \colon E \otimes \prod_{n=1}^{\infty} Z(p^n) \longrightarrow \prod_{n=1}^{\infty} (E \otimes Z(p^n))$$

is not injective.

REMARK 1. There are many examples of reduced torsion free groups that are divisible for a prime p. For a prime q different from p, the q-adic integers are such a group. The integers localized at q, Z_q, is another common example.

Concerning Z_q, if E is any torsion free group of rank 1 whose type is $(k_1, k_2, \cdots)$, then E is reduced and divisible for p iff the k_n corresponding to p is ∞ but at least one other k_n is finite. (For the notion of type and a discussion of torsion free groups of rank 1 see [7].)

REMARK 2. We do not know if for every prime integer p, E is not p-divisible then f is injective. However, we have since shown this is true for rank 1 torsion free groups.

We have following interesting corollary to Thm. 3.13.

COROLLARY 3.16. *If R is a Dedekind domain and (F_i) is a flat family such that $E \otimes F_i$ is reduced then $E \otimes \prod F_i$ is reduced.*

Proof. $\prod (E \otimes F_i)$ is reduced and f is injective (Prop. 3.2). Therefore $E \otimes \prod F_i$ is reduced.

REMARK. A special case is when E is reduced. Then $E \otimes \prod R$ is reduced where the product is arbitrary

4. Commutative artinian rings. Matlis [12, Thm. 3.11] has shown that over a commutative artinian ring all indecomposable injective modules are fg. And we have seen that f is injective for all flat families (F_i) iff R is right noetherian and f is injective for all flat families (F_i) whenever E is indecomposable injective (Prop. 2.4). Therefore, by Prop. 1.3 we know the following

THEOREM 4.1. *Over a commutative artinian ring f is injective for all flat families (F_i).*

REMARK 1. It is possible to prove this theorem by purely elementary methods, without resorting to Matlis' result.

REMARK 2. Thm. 4.1 endows commutative artinian rings with the property that was necessary to prove Thm. 3.6. (See Remark 1, following that theorem.) We have, therefore, the following

THEOREM 4.2. *If R is commutative artinian, (E_j) is a family of flat modules, f is injective for E_j, and E is a pure submodule of $\prod E_j$, then f is injective for all families (F_i).*

REMARK. Thm. 4.2 can be derived in a different way by noting that over a commutative artinian ring every flat module is projective [1] and that the direct product of projective modules is projective [4].

We do not know that if R is commutative artinian then f is always injective. But if f does have this property R must be commutative artinian. To prove this we need two lemmas. A multiplicative subset of R will always be assumed to possess 1 and not 0. If $r \in R$ then (r) will denote the ideal generated by r.

LEMMA 4.3. *Suppose S is a multiplicative subset of the commutative ring R. Then*

$$S \cap \left(\bigcap_{s \in S} (s)\right) \neq \varnothing \text{ iff } S^{-1}R \otimes \prod_{s \in S} R/(s) = 0 .$$

Therefore, if f is injective we have $S \cap (\bigcap_{s \in S} (s)) \neq \varnothing$ for all multiplicative subsets S of R.

Proof. For convenience we will write $R/(s)$ as $\bar{s}$ and omit the index set of products and intersections when this set is obviously S. Note that $S^{-1}R \otimes \prod \bar{s} \approx S^{-1}(\prod \bar{s})$, naturally.

If $t \in S \cap (\cap (s))$ and $x/u \in S^{-1}(\prod \bar{s})$ then $x/u = tx/tu = 0/tu = 0$. So $S \cap (\cap (s)) \neq \emptyset$ implies $S^{-1}R \otimes \prod \bar{s} = 0$.

If, on the other hand, $S^{-1}R \otimes \prod \bar{s} = 0$ then $(1 + (s))s \in S/1 = 0$ in $S^{-1}(\prod \bar{s})$. So there exists $t \in S$ such that $t \in \cap (s)$. Hence

$$S \cap (\cap (s)) \neq \emptyset .$$

For the rest, note that for $s \in S$, $S^{-1}R \otimes \bar{s} = 0$. So $\prod (S^{-1}R \otimes \bar{s}) = 0$. Hence if f is injective, it must be true that $S^{-1}R \otimes \prod \bar{s} = 0$. Therefore, by the first part of the lemma, $S \cap (\cap (s)) \neq \emptyset$.

LEMMA 4.4. *If R is a commutative ring such that for every prime ideal P, $(R - P) \cap (\bigcap_{s \notin P} (s)) \neq \emptyset$, then every prime ideal is maximal. In particular, if R is a domain then R is a field.*

Proof. If $t \in (R - P) \cap (\bigcap_{s \notin P} (s))$ then $t + P \in \bigcap_{s \notin P} ((s) + P)/P$. So $t + P \in ((t^2) + P)/P$. Hence there exists $r \in R$ such that $t - rt^2 = t(1 - rt) \in P$. So $1 - rt \in P$. This says $t + P$ is a unit in R/P.

If $s \in R - P$ then $t + P \in ((s) + P)/P$. Since $t + P$ is a unit in R/P, $(s) + P = R$. Choose $b \in R$ such that $sb - 1 \in P$. Then $(s + P)(b + P) = 1 + P$, i.e., $s + P$ is a unit in R/P. Hence P is maximal.

THEOREM 4.5. *If R is a commutative ring and f is always injective then R is artinian.*

Proof. By Lemma 4.3 $(R - P) \cap (\bigcap_{s \notin P} (s)) \neq \emptyset$ for all prime ideals P. Therefore, every prime ideal is maximal (Lemma 4.4). But R must be noetherian (Prop. 2.4). Since R is noetherian with every prime ideal maximal, it is a standard result that R must be artinian.

Since an artinian domain is necessarily a field we have the following immediate corollary.

COROLLARY 4.6. *If R is a domain and f is always injective then R is a field.*

As already remarked we do not know if the converse to Thm. 4.5 is true. There are, however, important classes of artinian rings over which f is injective. For example, suppose R is a proper quotient of a Dedekind domain. Then any R-module can be considered as a module, say E, over a Dedekind domain, this module having nontrivial annihilator. A classical result of Prüfer [10, Thm. 6] tells us that E is a direct sum of cyclic submodules. Since f is injective on each summand (Prop. 1.3), f is injective on all of E (Prop. 2.1).

A proper quotient of a Dedekind domain is a direct sum of rings each of whose lattices of ideals is finite and totally ordered by inclusion.

Any ring, not necessarily commutative, whose left and right free modules of rank 1 have unique composition series has been called a serial ring [5].

THEOREM 4.7. (Eisenbud and Griffith, [5].) *If R is a serial ring then any right R-module is a direct sum of submodules with unique composition series.*

In particular every module over a serial ring is a direct sum of finitely generated submodules. We have as an immediate consequence of this and Propositions 1.3 and 2.1, the following.

PROPOSITION 4.8. *If R is a serial ring then f is always injective.*

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