PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 2, 1973

SELF-ADJOINT EXTENSIONS OF SYMMETRIC DIFFERENTIAL OPERATORS

ARNOLD VILLONE

Let $\mathscr{H}$ denote the Hilbert space of square summable analytic function on the unit disk, and consider those formal differential operators

$$L = \sum_{i=0}^{n} p_i D^i$$

which give rise to symmetric operators in $\mathscr{H}$. This paper is devoted to a study of when these operators are actually self-adjoint or admit of self-adjoint extensions in $\mathscr{H}$. It is shown that in the first order case the operator is always self-adjoint. For $n > 1$ sufficient conditions on the p_i are obtained for the existence of self-adjoint extensions. In particular a condition on the coefficients is obtained which insures that the operator has defect indices equal to the order of L.

Let $\mathscr{A}$ denote the space of functions analytic on the unit disk and $\mathscr{H}$ the subspace of square summable functions in $\mathscr{A}$ with inner product

$$(f, g) = \int_{|z|<1}\int f(z)\overline{g(z)}dxdy .$$

A complete orthonormal set for $\mathscr{H}$ is provided by the normalized powers of z,

$$e_n(z) = [(n+1)/\pi]^{1/2} z^n , \qquad n = 0, 1, \cdots .$$

From this it follows that $\mathscr{H}$ is identical with the space of power series $\sum_{n=0}^{\infty} a_n z^n$ which satisfy

$$\sum_{n=0}^{\infty} |a_n|^2/(n+1) < \infty . \tag{1.1}$$

Consider the formal differential operator

$$L = p_n D^n + \cdots + p_1 D + p_0 ,$$

where $D = d/dz$ and the p_i are in $\mathscr{H}$. We now associate two operators as follows. Let $\mathscr{D}_0$ denote the span of the e_n and $\mathscr{D}$ the set of all f in $\mathscr{H}$ for which Lf is in $\mathscr{H}$, and define T_0 and T as

$$T_0 f = Lf \qquad f \in \mathscr{D}_0$$
$$Tf = Lf \qquad f \in \mathscr{D} .$$

It is shown in [2] that T_0 and T are both densely defined operators in $\mathscr{H}$, $T_0 \subseteqq T$ and T is closed. Moreover, T_0 is symmetric if and only if

$$(Le_n, e_m) = (e_n, Le_m)\ , \qquad n, m = 0, 1, \cdots . \tag{1.2}$$

Such a formal operator is said to be formally symmetric. Regarding symmetric T_0 we have the following result.

THEOREM 1.1. *If T_0 is symmetric, $T_0^* = T$ and $T^* \subseteqq T$. The closure of T_0, $S = T_0^{**} = T^*$, is self-adjoint if and only if $S = T$.*

Proof. See [2].

For f and g in $\mathscr{D}$ consider the bilinear form

$$\langle f, g\rangle = (Lf, g) - (f, Lg)\ , \tag{1.3}$$

and let $\tilde{\mathscr{D}}$ be the set of those f in $\mathscr{D}$ for which $\langle f, g\rangle = 0$ for all g in $\mathscr{D}$. Since $S = T^*$ and $\mathscr{D}(T^*) = \mathscr{D}$, S has domain $\tilde{\mathscr{D}}$.

Let $\mathscr{D}^+$ and $\mathscr{D}^-$ denote the set of all solutions of the equation $Lu = iu$ and $Lu = -iu$ respectively, which are in $\mathscr{H}$. It is known from the general theory of Hilbert space [1, p. 1227–1230] that $\mathscr{D} = \tilde{\mathscr{D}} + \mathscr{D}^+ + \mathscr{D}^-$, and every $f \in \mathscr{D}$ has a unique such representation. Let the dimensions of $\mathscr{D}^+$ and $\mathscr{D}^-$ be m^+ and m^- respectively. Clearly, m^+ and m^- cannot exceed the order of L. These integers are referred to as the deficiency indices of S, and S has self-adjoint extensions if and only if $m^+ = m^-$. Moreover, S is self-adjoint if and only if $m^+ = m^- = 0$.

2. In [2] it is shown that the general formally symmetric first order operator is given by

$$L = (cz^2 + az + \bar{c})D + (2cz + b) \tag{2.1}$$

where a and b are real. In this case it is possible to compute the solutions of $Lu = \pm iu$ explicitly and show that the solutions so obtained are not in $\mathscr{H}$. Proceeding in this manner we obtain the following result.

THEOREM 2.1. *If L is a first order formally symmetric operator, the associated operator T is self-adjoint.*

Proof. We shall show that m^+ and m^- are both zero. When $c = 0$ L is just the first order Euler operator, and hence T is self-adjoint by the corollary to Theorem 1.3 of [2]. When $c \neq 0$ we have

$$(z^2 + (a/c)z + \bar{c}/c)u' + (2z + b/c - i/c)u = 0 \tag{2.2}$$

$$(2.3) \qquad (z^2 + (a/c)z + \bar{c}/c)u' + (2z + b/c + i/c) = 0 .$$

The coefficient of u' has zeros at

$$\alpha = -a/2c + (a^2 - 4|c|^2)^{1/2}/2c .$$
$$\beta = -a/2c - (a^2 - 4|c|^2)^{1/2}/2c .$$

There are three cases to consider:

1. $a^2 < 4|c|^2$
2. $a^2 = 4|c|^2$
3. $a^2 > 4|c|^2$.

In case 1 we have $\alpha = -a/2c + iR/2c$, $\beta = -a/2c - iR/2c$ where $R = (4|c|^2 - a^2)^{1/2}$, moreover $|\alpha| = |\beta| = 1$. Every solution of (2.2) is a multiple of the fundamental solution $\phi(z) = (z - \alpha)^{-r}(z - \beta)^{-s}$ where $r = (R - 1)/R - i(b - a)/R$ and $s = (R + 1)/R + i(b - a)/R$. Hence every (nontrivial) solution of (2.2) is analytic in the open unit disc D with at least one singularity on the boundary at $z = \beta$. We now show that ϕ is not in $\mathscr{H}$, i.e., the integral $\int_D\int|\phi(z)|^2 dx\, dy$ diverges. Introduce polar coordinates at β so $z - \beta = \rho e^{i\theta}$. Let δ be less than $|\beta - \alpha|$, then there exist suitable θ_1 and θ_2 such that for $0 < \varepsilon < \delta$, the regions $W_\varepsilon = \{z | \varepsilon \leqq \rho \leqq \delta, \theta_1 \leqq \theta \leqq \theta_2\}$ lie within D and $\alpha \notin W_\varepsilon$. Now

$$(2.4) \qquad \int_D\int|\phi(z)|^2 dx\, dy \geqq \lim_{\varepsilon\to 0}\int_{W_\varepsilon}\int|(z - \alpha)^{-r}|^2|(z - \beta)^{-s}|^2 dx\, dy .$$

Since $\alpha \notin W_\varepsilon$ it follows from continuity that $|(z - \alpha)^{-r}|^2 \geqq m > 0$ for z in W_ε, all $0 < \varepsilon < \delta$. Using this and the fact that $|(z - \beta)^{-s}| = \rho^{-u}e^{v\theta}$, where $s = u + iv$, the inequality of (2.4) becomes

$$\int_D\int|\phi(z)|^2 dx\, dy \geqq \lim_{\varepsilon\to 0} m \int_{\theta_1}^{\theta_2}\int_\varepsilon^\delta \rho^{-2u+1}e^{2v\theta}d\rho\, d\theta$$
$$\geqq \lim_{\varepsilon\to 0} mk(\theta_2 - \theta_1)\int_\varepsilon^\delta \rho^{-2u+1}d\rho ,$$

where $k =$ infimum of $e^{2v\theta}$ on $\theta_1 \leqq \theta \leqq \theta_2$ which is greater than zero. But $-2u + 1 = -2(R + 1)/R + 1 = -1 - 2/R < -1$, hence the integral on the left diverges and ϕ is not square summable.

The fundamental solution for (2.3) is given by $\phi(z) = (z - \alpha)^{-r}(z - \beta)^{-s}$, where $r = (R + 1)/R - i(b - a)/R$ and $s = (R - 1)/R + i(b - a)/R$. Hence $\phi(z)$ is analytic in the open unit disc D with a singularity on the boundary at α. Let $z - \alpha = \rho e^{i\theta}$, then there exist suitable θ_1 and θ_2 such that for $0 < \varepsilon < \delta < |\alpha - \beta|$, the regions $W_\varepsilon = \{z | \varepsilon \leqq \rho \leqq \delta, \theta_1 \leqq \theta \leqq \theta_2\}$ lie within D and $\beta \notin W_\varepsilon$. As before, we obtain

$$\int_D \int |\phi(z)|^2 dx\, dy \geqq \lim_{\varepsilon \to 0} mk(\theta_2 - \theta_1) \int_\varepsilon^\delta \rho^{-2\mu+1} d\rho$$

where $|(z-\beta)^{-s}|^2 \geqq m > 0$ for all z in W_ε and $0 < \varepsilon < \delta$, k is the infimum of $e^{2v\theta}$ on $\theta_1 \leqq \theta \leqq \theta_2$ and $r = u + iv$. But $-2u + 1 = -(R+2)/R < -1$, hence the integral on the left diverges and ϕ is not square summable.

In case 2 the coefficient of u' has a double zero at $\alpha = -a/2c$ where $|\alpha|^2 = a^2/4|c|^2 = 1$. The functions $\phi_+(z) = (z-\alpha)^{-2} e^{r(z-\alpha)^{-1}}$, $r = (b - a - i)/c$ and $\phi_-(z) = (z-\alpha)^{-2} e^{r(z-\alpha)^{-1}}$, $r = (b - a + i)/c$ are fundamental solutions for (2.2) and (2.3) respectively. Let us introduce polar coordinates at $z = \alpha$ so that $z - \alpha = \rho e^{i\theta}$ and let us agree to set $\theta = 0$ so that for $|z| < 1$, the argument of $z - \alpha$ is restricted to the intervals $0 \leqq \theta < \pi/2$ and $3\pi/2 < \theta < 2\pi$. Let $r = u + iv$, then

$$\begin{aligned} |\phi_\pm(z)| &= |\rho^{-2} e^{-i2\theta} e^{(u+iv)(\cos\theta - i\sin\theta)/\rho}| \\ &= \rho^{-2} e^{(u\cos\theta + v\sin\theta)/\rho} . \end{aligned}$$

We note that u and v are not both zero, for then $b - a \pm i = 0$ where a and b are real. Now consider the function $F(\theta) = u\cos\theta + v\sin\theta$. If $u > 0$, $F(0) = u > 0$ and by continuity there exist θ_1 and θ_2 such that $F(\theta) \geqq u/2 > 0$ for $\theta_1 \leqq \theta \leqq \theta_2 < \pi/2$, similarly if $v > 0$, $F(\pi/2) = v$ and there exist θ_1 and θ_2 such that $F(\theta) \geqq v/2 > 0$ for $\theta_1 \leqq \theta \leqq \theta_2 \leqq \pi/2$. If $v < 0$, $F(3\pi/2) = -v > 0$ and there exist θ_1 and θ_2 such that $F(\theta) \geqq -v/2 > 0$ for $3\pi/2 < \theta_1 \leqq \theta \leqq \theta_2$. Hence for all $r = u + iv$, except for the case $u < 0$, $v = 0$, there exists a $M > 0$ and suitable θ_1 and θ_2 for which $F(\theta) \geqq M$, $\theta_1 \leqq \theta \leqq \theta_2$. This case requires only a minor modification which will be provided shortly. It is easy to see that for given θ_1 and θ_2 we can find $\delta > 0$ for which the regions $W_\varepsilon = \{z \mid \varepsilon \leqq \rho < \delta,\ \theta_1 \leqq \theta \leqq \theta_2\}$ lie entirely within the disc for $0 < \varepsilon < \delta$.

Now consider $||\phi_\pm||^2$:

$$\begin{aligned} \int_D \int |\phi_\pm(z)|^2 dx\, dy &\geqq \lim_{\varepsilon \to 0} \int_{W_\varepsilon} \int |\phi_\pm(z)|^2 dx\, dy \\ &= \lim_{\varepsilon \to 0} \int_{\theta_1}^{\theta_2} \int_\varepsilon^\delta \rho^{-3} e^{2F(\theta)/\rho} d\rho\, d\theta \\ &\geqq \lim_{\varepsilon \to 0} (\theta_2 - \theta_1) \int_\varepsilon^\delta e^{2M/\rho} \rho^{-3} d\rho . \end{aligned}$$

Since $\int_0^\delta e^{2M/\rho} \rho^{-3} d\rho$ diverges it follows that the $\phi_\pm$ are not square summable, provided r is not a negative number. When $r = u + iv = u < 0$ we merely agree to set $\theta = 0$ so that for $|z| < 1$ the argument of $z - \alpha$ is restricted to the interval $\pi/2 < \theta < 3\pi/2$. Then $F(\pi) = -u > 0$ and the argument is the same as before.

In case 3, $a^2 > 4|c|^2$, the coefficient of u' has distinct zeros at $\alpha = (-a + R)/2c$ and $\beta = (-a - R)/2c$ where $R = (a^2 - 4|c|^2)^{1/2} > 0$. For $a > 0$,

$$|\beta| = \frac{R + a}{2|c|} > \frac{a}{2|c|} > 1 ,$$

and therefore $|\alpha| < 1$. For $a < 0$,

$$|\alpha| = \frac{R - a}{2|c|} > \frac{|a|}{2|c|} > 1 ,$$

and therefore $|\beta| < 1$. Without loss of generality we assume $|\alpha| < 1$, and $|\beta| > 1$. For $|z| < |\alpha| < 1$, the functions ϕ_+ and ϕ_- given by

$$\phi_+(z) = (z - \alpha)^{-r}(z - \beta)^{-t}$$
$$\phi_-(z) = (z - \beta)^{-s}(z - \alpha)^{-u}$$

where $r = (R + b - a)/R - i/R$ and $s = (R + b - a)/R + i/R$, are fundamental solutions for $Lu = iu$ and $Lu = -iu$ respectively. Now suppose ψ is any nontrivial element of $\mathscr{H}$ which satisfies $Lu = \pm iu$. In particular ψ is analytic for $|z| < |\alpha| < 1$. From uniqueness results this implies that $\psi(z) = c\phi_\pm(z)$ for $|z| < |\alpha|$, where $c \neq 0$. By the identity theorem for analytic functions this implies $\psi(z) = c\phi_\pm(z)$ for $|z| < 1$, hence $\phi_\pm(z)$ is analytic in $|z| < 1$. But $\phi_\pm(z)$ has a singularity at $|\alpha| < 1$, therefore, the equations $Lu = \pm iu$ have no nontrivial solutions in $\mathscr{H}$.

3. In this section we obtain conditions on the coefficients of L which insure that for all λ every solution of $L\phi = \lambda\phi$ is in $\mathscr{H}$. If L is a formally symmetric operator satisfying these conditions the defect indices of the operator T_0 are equal to the order of L and T_0 has a self-adjoint extension in $\mathscr{H}$.

In [2] it was shown that if $L = \sum_{k=0}^{n} p_k D^k$ is formally symmetric then the p_i are polynomials of degree at most $n + i$. Regarding such L with polynomial coefficients we have

THEOREM 3.1. *Let* $L = \sum_{k=0}^{n} p_k D^k$ *where* $n \geqq 2$, $p_n(0) \neq 0$, *and* $p_k = \sum_{i=0}^{n+k} a_i(k) z^k$, *and*

$$A = |a_0(n)|^{-1} \sum_{i=1}^{2n} |a_i(n)| ,$$

$$\hat{B} = n(n + 1)/2, \quad \textit{and} \tag{3.1}$$

$$B = |a_0(n)|^{-1} \sum_{i=1}^{2n} |a_i(n)n[(n + 1)/2 - i] + a_{i-1}(n - 1)| .$$

If $A < 1$ or $A = 1$ and $B < \hat{B}$ then every solution of $L\phi = 0$ is in $\mathscr{H}$.

Proof. Since $p_n(0) = a_0(n) \neq 0$, every solution of $Lu = 0$ at the origin is analytic in some neighborhood of the origin. Let $\phi(z) = \sum_{j=0}^{\infty} b_j z^j$ be any such solution, we will show that there exists a positive constant K and positive integer p such that $|b_j| \leqq Kj^{-1/p}$ for j sufficiently large. Consequently the series $\sum_{j=0}^{\infty} |b_j|^2/(j+1)$ converges and ϕ belongs to $\mathscr{H}$.

We begin by obtaining a recurrsion formula for the b_j. Substituting $\phi(z) = \sum_{j=0}^{\infty} b_j z^j$ into the equation $L\phi(z) = 0$ we obtain

$$L\phi(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{n} \sum_{i=0}^{n+k} a_i(k)\pi_k(j - i + k)b_{j-i+k}z^j ,$$

where

$$\begin{aligned} \pi_k(\lambda) &= \lambda(\lambda - 1) \cdots (\lambda - k + 1) \qquad & k \leqq \lambda \\ &= 0 & k > \lambda . \end{aligned}$$

Hence $L\phi = 0$ if and only if the following relationship holds for all j.

$$\sum_{k=0}^{n} \sum_{i=0}^{n+k} a_i(k)\pi_k(j - i + k)b_{j-i+k} = 0 . \tag{3.2}$$

Hence,

$$\begin{aligned} &\sum_{k=0}^{n-1} \sum_{i=0}^{n+k} a_i(k)\pi_k(j - i + k)b_{j-i+k} \\ &+ \sum_{i=1}^{2n} a_i(n)\pi_n(j - i + n)b_{j-i+n} + a_0(n)\pi_n(j + n)b_{j+n} = 0 . \end{aligned}$$

Noting that the sums involve only the b_{j-n} thru b_{j+n-1} (where $j > n$) and $\pi_n(j + n)$ never vanishes we may solve for b_{j+n} to obtain

$$b_{j+n} = -(S_1 + S_2)/a_0(n)\pi_n(j + n) , \tag{3.3}$$

where

$$S_1 = \sum_{i=1}^{2n} a_i(n)\pi_n(j - i + n)b_{j-i+n} ,$$

and

$$S_2 = \sum_{k=0}^{n-1} \sum_{i=0}^{n+k} a_i(k)\pi_k(j - i + k)b_{j-i+k} ,$$

for $j > n$.

We now investigate the nature of S_1 and S_2 as polynomials in j. It can be shown that $\pi_n(j + n - 1)$ is a polynomial of degree n in j,

$$\pi_n(j + n - i) = j^n + \left[\frac{n(n+1)}{2} - in\right]j^{n-1} + \cdots , \tag{3.4}$$

for $i = 1, \cdots, 2n$. Using (3.4) in (3.3) we obtain

$$\begin{aligned} S_1 &= j^n \sum_{i=1}^{2n} a_i(n) b_{j-i+n} \\ &\quad + j^{n-1} \sum_{i=1}^{2n} a_i(n) \left[\frac{n(n+1)}{2} - in\right] \\ &\quad + \text{lower powers of } j . \end{aligned} \tag{3.5}$$

Now consider S_2. Since $\pi_k(j - i + k)$ is a polynomial of degree k in j, an examination of (3.3) shows that S_2 is a polynomial of degree $n - 1$ in j, and that the only terms which contribute to the coefficient of j^{n-1} are those corresponding to $k = n - 1$. Hence

$$\begin{aligned} S_2 &= j^{n-1} \sum_{i=0}^{2n-1} a_i(n-1) b_{j-i+n-1} \\ &\quad + \text{lower powers of } j. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) we obtain

$$\begin{aligned} S_1 + S_2 &= j^n \sum_{i=1}^{2n} a_i(n) b_{j-i+n} \\ &\quad + j^{n-1} \sum_{i=1}^{2n} \left[a_i(n)\left(\frac{n(n+1)}{2} - in\right) + a_{i-1}(n-1)\right] b_{j-i+n} \\ &\quad + \cdots , \quad (j > n) . \end{aligned} \tag{3.7}$$

Since $\pi_n(j + n) = j^n + (n(n + 1))/2j^{n-1} + \cdots$, is always positive (3.3) yields

$$|b_{j+n}| = \frac{|S_1 + S_2|}{|a_0(n)|[j^n + \hat{B}j^{n-1} + \cdots]} . \tag{3.8}$$

We now estimate $|S_1 + S_2|$. Let $M(j) = \operatorname{Max}(|b_{j-n}|, \cdots, |b_{j+n-1}|)$, then it follows from (3.1) and (3.7) that $|S_1 + S_2| \leqq |a_0(n)|[M(j)Aj^n + M(j)Bj^{n-1} + \cdots]$. Hence

$$|b_{j+n}| \leqq \frac{Aj^n + Bj^{n-1} + \cdots}{j^n + \hat{B}j^{n-1} + \cdots} M(j) \tag{3.9}$$

for $j > n$, where A, B, and $\hat{B}$ are given by (3.1).

Consider the estimate (3.9) for $|b_{j+n}|$,

$$|b_{j+n}| \leqq Q(j)M(j) \qquad j > n , \tag{3.10}$$

where $Q(j) = (Aj^n + Bj^{n-1} + \cdots)/(j^n + \hat{B}j^{n-1} + \cdots)$. We note that for fixed ζ, $Q(j) \leqq 1 + \zeta j^{-1}$ for j sufficiently large if and only if $Aj^n +$

$Bj^{n-1} + \cdots \leqq j^n + (\hat{B} + \zeta)j^{n-1} + \cdots$. Hence if $A < 1$ or $A = 1$ and $B < \hat{B} + \zeta$ we have

$$Q(j) \leqq 1 + \zeta j^{-1} \tag{3.11}$$

for j sufficiently large. Now consider the expression

$$(1 + \zeta(j+1)^{-1})\ (j - n + 1)^{-1/p}\ ,$$

where $\zeta < 0$ and p a positive integer. It is not difficult to see that this is dominated by $(j + n + 1)^{-1/p}$ for j sufficiently large if and only if

$$j^{p+1} + (p + p\zeta + n + 1)j^p + \cdots \leqq j^{p+1} + (p - n + 1)j^p + \cdots ,$$

for j sufficiently large. Hence, we have

$$(1 + \zeta(j+1)^{-1})(j - n + 1)^{-1/p} \leqq (j + n + 1)^{-1/p} \tag{3.12}$$

for j sufficiently large if $p \geqq -2n\zeta^{-1}$.

We now show that there exists a positive constant K and positive integer p for which $|b_j| \leqq Kj^{-1/p}$, j sufficiently large. By hypothesis either $A < 1$ or $A = 1$ and $B < \hat{B}$. If $A < 1$ let $\zeta = -1$ and $p = 2n$, if $A = 1$, select ζ such that $B - \hat{B} < \zeta < 0$ and $p > -2n\zeta^{-1}$. For j sufficiently large, say $j > j_1$, (3.11) and (3.12) hold. Set

$$K = \max_{j \leqq j_1 + n} |b_j| j^{1/p}$$

so that $|b_j| \leqq Kj^{-1/p}$ for $j \leqq j_1 + n$. Using (3.10) and (3.11) it follows that

$$|b_{j_1+n+1}| \leqq (1 + \zeta(j_1 + 1)^{-1})M(j_1 + 1)\ ,$$

where

$$\begin{aligned} M(j_1 + 1) &= \operatorname{Max}\,(K(j_1 - n + 1)^{-1/p}, \cdots, K(j_1 + n)^{-1/p}) \\ &= K(j_1 - n + 1)^{-1/p}\ . \end{aligned}$$

Hence $|b_{j_1+n+1}| \leqq (1 + \zeta(j_1 + 1)^{-1})K(j_1 - n + 1)^{-1/p}$, and using (3.12) this yields

$$|b_{j_1+n+1}| \leqq K(j_1 + n + 1)^{-1/p}\ . \tag{3.13}$$

We now proceed inductively to establish

$$|b_{j_1+n+k}| \leqq K(j_1 + n + k)^{-1/p} \qquad k = 2, 3, \cdots . \tag{3.14}$$

Let $K_1 = \max_{j \leqq j_1+n+1} |b_j| j^{1/p}$, now $K_1 = \max\{K, |b_{j_1+n+1}|(j_1 + n + 1)^{1/p}|\} \leqq K$, making use of (3.13). Using (3.11) yields

$$|b_{j_1+n+2}| \leqq (1 + \zeta(j_1 + 2)^{-1})M(j_1 + 2)$$

where

$$\begin{aligned} M(j_1 + 2) &= \operatorname{Max}\,(K(j_i - n + 2)^{-1/p}, \cdots, K(j_1 + n + 1)^{-1/p}) \\ &= K(j_1 - n + 2)^{-1/p} \,. \end{aligned}$$

Using (3.12) it follows that

$$|b_{j_1+n+2}| \leqq K(j_1 + n + 2)^{-1/p} \,.$$

Continuing on in this manner we establish (3.14) and the theorem is proved.

We note that the conditions (3.1) of Theorem 3.1 involve only the coefficients of the polynomials p_n and p_{n-1}, hence if L satisfies the conditions of (3.1) so do the operators $L \pm i$. Hence we have established the following.

THEOREM 3.2. *Let L be a formally symmetric operator which satisfies* (3.1), *then the associated operator T_0 has defect indices $n_+ = n_- = n$.*

COROLLARY 3.3. *The operator $L = (c_1 z^4 + \bar{c}_1)d^2/dz^2 + (6c_1 z^3 + c_3 z^2 + a_2 z + \bar{c}_3)d/dz + (6c_1 z^2 + 2c_3 z + a_3)$, where a_3 and a_2 are real and $|c_1| > |c_3| + |a_2|/2$, has self-adjoint extensions.*

Proof. Applying the algorithm given in Theorem 2.3 of [2] the general second order formally symmetric operator has coefficients

$$\begin{aligned} p_2(z) &= c_1 z^4 + c_2 z^3 + a_1 z^2 + \bar{c}_2 z + \bar{c}_1 \\ p_1(z) &= 6c_1 z^3 + (c_3 + 3c_2)z^2 + a_2 z + \bar{c}_3 \\ p_0(z) &= 6c_1 z^2 + 2c_3 z + a_3 \,, \end{aligned}$$

where a_1, a_2, and a_3 are real.
Now $A = (|c_1| + 2|c_2| + |a_1|)/|c_1| \geqq 1$ and $A = 1$ if and only if $c_2 = a_1 = 0$. Now $\hat{B} = 3$ and $B = (|c_1| + |a_2| + 2|c_3|)/|c_1| < 3$ if and only if $|c_1| > |c_3| + |a_2|/2$. Hence the result follows from the previous theorem.

REFERENCES

1. N. Dunford and J. Schwartz, *Linear Operators*, Part II, Interscience Publishers, 1963.
2. A. L. Villone, *Self-adjoint differential operators*, Pacific J. Math., **35** (1970), 517-531.

Received September 15, 1972 and in revised form January 5, 1973. This work was supported in part by NSF Grant GP-3594.

CALIFORNIA STATE UNIVERSITY, SAN DIEGO