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ON THE LU QI-KENG CONJECTURE AND THE BERGMAN REPRESENTATIVE DOMAINS

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The main purpose of this paper is to give the affirmative solution for the Lu Qi-Keng conjecture in the case of bounded complete circular domains.

Some results are that Bell's proposition, which is related to the evaluation about the Bergman kernel functions of homogeneous complete circular domains, is extended to the case of bounded complete circular domains and that the Bergman representative functions with respect to $z_0(\neq 0)$ of any bounded and any homogeneous bounded complete circular domain with center at the origin are biholomorphic and isomorphic (biholomorphic in the narrow sense), respectively.

1. Introduction. The Bergman kernel function $k_D(z, \bar{t})$ of a domain D in C^n plays important roles in the theory of biholomorphic mappings because of its biholomorphic relative invariance (see (15)).

Lu Qi-Keng [10] introduced the conjecture, which is still open, that "The equation $k_D(z, \bar{t}) = 0$ has no solution in $D \times D^*$, if D in C^n is a bounded simply connected schlicht domain", where $D^* = \{\bar{t} \mid t \in D\}$ and the symbol $^-$ denotes the conjugate.

It is known that in general multiply connected schlicht domains do not satisfy the Lu Qi-Keng conjecture ([11] Theorems 1, 3; [12] Example). But in the case of simply connected schlicht domains in C^n $(n \geqq 2)$ it seems that we have no example which does not satisfy the conjecture. In C it is easily seen by the Riemann's mapping theorem that any simply connected schlicht domain satisfies the conjecture.

In § 2 we shall show that any bounded complete circular domain satisfies the Lu Qi-Keng conjecture, and by using this result extend the following proposition of Bell [1], which is recently proved:

PROPOSITION. *Let D be any homogeneous domain in C^n, n-dimensional complex space, such that if $z \in D$ and $\lambda \in C$ and $|\lambda| \leqq 1$ then $\lambda z \in D$* (i.e., *D is a complete circular domain with center at the origin*). *Then given a compact subset H of D there are constants $a_H > 0$ and $b_H < \infty$ such that for all $z \in D$ and $t \in H$*

$$a_H \leqq |k_D(z, \bar{t})| \leqq b_H$$

where k_D denotes the Bergman kernel function of the domain D.

Throughout this paper we shall call a holomorphic mapping $w =$

$f(z) \equiv {}^t(f_1(z), \cdots, f_n(z))$, $z \equiv {}^t(z_1, \cdots, z_n)$ (${}^t(\ )$: transpose), of a schlicht domain in C^n (i) a biholomorphism if $f(z)$ is holomorphic and the Jacobian $J_f(z) \neq 0$ in D, i.e., both f and the inverse f^{-1} are holomorphic and f is locally one-to-one (one-to-one in the sense of Riemann regions (see [3], p. 117)), (ii) an isomorphism if $f(z)$ is biholomorphic and globally one-to-one (both D and $f(D)$ are schlicht domain in C^n), and (iii) an automorphism if $f(z)$ is an isomorphism of D onto itself, respectively.

In § 3 we treat the representative domain introduced by S. Bergman [2] as a canonical domain of the biholomorphic equivalent class of a bounded schlicht domain in C^n corresponding to the unit disc in C in the Riemann's mapping theorem. The function which maps a domain D onto the representative domain of the biholomorphic equivalent class of D is called the representative function of D (see (10)). In general the representative function $w = w(z)$ of D with respect to a fixed point $z_0 \in D$ is singular at any point $z \in \{z \in D \mid k_D(z, \bar{z}_0) = 0\}$ and further is not locally one-to-one at any point $z \in \{z \in D \mid \text{Jacobian } J_w(z) = 0\}$. In these cases the representative domain does not belong to the biholomorphic equivalent class of D, and this weaken the representativity of the representative domain of the biholomorphic equivalent class of D.

We shall prove that the representative function of any bounded complete circular domain D with respect to $z_0 \in D$ is biholomorphic, and that in the case of homogeneous bounded complete circular domains the representative function is an isomorphism. We give an example in the case of Cartan domains. Finally, we treat the two sorts of m-representative domains of Maschler [7] and Kato [4] or Kikuchi [5] and give an example in the case of a unit disc.

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2. Lu Qi-Keng conjecture.

DEFINITION. A domain $D \subset C^n$ is called a Lu Qi-Keng domain if the equation

$$(1) \qquad k_D(z, \bar{t}) = 0$$

has no solution in $D \times D^*$ ([10], [11], [12]).

If (1) has no solution in $D \times E^*$, where $E^* = \{\bar{t} \mid t \in E \subset D\}$, it is said that $k_D(z, \bar{t})$ or D satisfies the Lu Qi-Keng conjecture for a subset E^* of D^*.

LEMMA 1. *Assume that the Bergman kernel function $k_D(z, \bar{t})$ of a bounded domain in C^n satisfies the Lu Qi-keng conjecture at a* *point* $\bar{t}_0 \in D^*$. *Let us set*

$$\Omega_\Delta = \{t \in D \mid k_D(z, \bar{t}) \neq 0 \quad for\ z \in \Delta\}\ ,$$

where Δ *is an arbitrary compact subset of* D. *Then* Ω_Δ *must be an open set in* D.

Proof. Let us take an arbitrary point $\tau \in \Omega_\Delta (\neq \emptyset)$. Then it follows that $k_D(z, \bar{\tau}) \neq 0$ for $z \in \Delta$. As it is well known that $k_D(z, \bar{t})$ is holomorphic with respect to $2n$ complex variables $(z, \bar{t}) \in D \times D^*$, there exists a neighborhood $U(z_\alpha, \bar{\tau}) = U(z_\alpha) \times U_{z_\alpha}(\bar{\tau})$ for each point $z_\alpha \in \Delta$ such that $k_D(z, \bar{t}) \neq 0$ for $(z, \bar{t}) \in U(z_\alpha, \bar{\tau})$, where $U(z_\alpha)$ and $U_{z_\alpha}(\bar{\tau})$ are neighborhoods of z_α and $\bar{\tau}$, respectively. Since $\bigcup_\alpha U(z_\alpha)$ $(z_\alpha \in \Delta)$ is an open covering of the compact set Δ, Δ has a finite covering $\bigcup_{i=1}^m U(z_i) \supset \Delta$, where $z_i \in \Delta$ $(i = 1, 2, \cdots, m)$. If we set $U(\bar{\tau}) = \bigcap_{i=1}^m U_{z_i}(\bar{\tau})$, $U(\bar{\tau})$ is an open neighborhood of $\bar{\tau}$ and $k_D(z, \bar{t}) \neq 0$ in $\Delta \times U(\bar{\tau})$. This shows that Ω_Δ is an open set in D.

REMARK 1. It is clear that $k_D(z, \bar{t}) \neq 0$ in $\bigcup_{i=1}^m U(z_i) \times U(\bar{\tau}) \supset \Delta \times U(\bar{\tau})$.

REMARK 2. A bounded complete circular domain D with center at t_0 is a Bergman minimal domain with the same center (see [6]). Therefore, $k_D(z, \bar{t})$ satisfies the Lu Qi-Keng conjecture at $\bar{t} = \bar{t}_0$, for a domain D is a Bergman minimal domain with center at t_0 if and only if

$$(2) \qquad k_D(z, \bar{t}_0) = k_D(t_0, \bar{t}_0) > 0 \quad \text{for } z \in D \ ([6])\ .$$

LEMMA 2. *Let* D *be a bounded complete circular domain with center at the origin, then* $\Omega_\Delta = D$, *that is,* $k_D(z, \bar{t}) \neq 0$ *for* $(z, \bar{t}) \in \Delta \times D^*$, *where* Δ *is an arbitrary compact subset of* D.

Proof. It is easily seen that D is starlike with respect to the origin (see Fuks [3], p. 45). If $\Omega_\Delta \neq D$, there exists a point $\omega \in D$ such that $k_D(\zeta, \bar{\omega}) = 0$ for some $\zeta \in \Delta$ and $k_D(z, \lambda\bar{\omega}) \neq 0$ for all $z \in \Delta$ and $\lambda \in [0, 1)$, because by Lemma 1 Ω_Δ is an open set including the origin and D is starlike. ω lies in $\partial\Omega_\Delta \cap D$ and does not lie in Ω_Δ.

In the case of circular domains with center at the origin, by the Cartan's theorem, the Bergman kernel function $k_D(z, \bar{t})$ of D has the form

$$(3) \qquad k_D(z, \bar{t}) = \sum_{k=0}^{\infty} \overline{\Phi_k(t)}\Phi_k(z) = \sum_{k=0}^{\infty} \overline{A_k t^k} A_k z^k\ ,$$

where $z^k = z \times \cdots \times z$, the symbol $\times$ denotes the Kronecker product here and after, and A_k is a $1 \times n^k$ constant coefficient matrix and $\{\Phi_0, \Phi_1(z), \Phi_2(z), \cdots\}$ ($\Phi_k(z) = A_k z^k$ ($k \geqq 1$): homogeneous polynomial of degree k and $\Phi_0 = A_0$: nonzero constant) is a complete orthonormal system of the Hilbert space of square integrable holomorphic functions on D.

As $\omega \in \partial\Omega_\Delta \cap D$ and $\omega \notin \Omega_\Delta$, we have $\lambda\bar{\omega} \in \Omega_\Delta^* = \{\bar{t} \mid t \in \Omega_\Delta\}$ for all $\lambda \in [0, 1)$. Let us consider a point $\lambda\bar{\omega}$ in place of $\bar{\tau}$ in the proof of Lemma 1 and $\bigcup_{i=1}^{m(\lambda)} U_\lambda(z_i)$ to be an open covering of Δ corresponding to $\lambda\bar{\omega}$. As we may take $1 - \lambda$ ($0 < \lambda < 1$) as small as we need, there exists $\lambda_0 \in (0, 1)$ such that $\lambda_0\omega \in \Omega_\Delta$ and $\zeta/\lambda_0 \in U_{\lambda_0}(z_i)$ for some i, because ζ (a critical point in Δ for ω) belongs to $U_{\lambda_0}(z_i) \cap \Delta$ for some i and $U_{\lambda_0}(z_i)$ is open. By Remark 1, for this λ_0 $k_D(\zeta/\lambda_0, \lambda_0\bar{\omega}) \neq 0$ holds. On the other hand, using the idea of D. Bell [1], we have by (3)

$$k_D(\zeta/\lambda_0, \lambda_0\bar{\omega}) = \sum_{k=0}^{\infty} \lambda_0^k \overline{A_k \omega^k}(1/\lambda_0^k) A_k \zeta^k = k_D(\zeta, \bar{\omega}) = 0 ,$$

which is contradictory. This completes the proof.

THEOREM 1. *Any bounded complete circular domain with center at the origin is a Lu Qi-Keng domain.*

Proof. By Lemma 2, $k_D(z, \bar{t}) \neq 0$ for $(z, \bar{t}) \in \Delta \times D^*$. Thus we obtain $k_D(z, \bar{t}) \neq 0$ for $(z, \bar{t}) \in D \times D^*$, since Δ is an arbitrary compact subset of D.

REMARK 3. Theorem 1 holds for any bounded complete Reinhardt circular domain, because Reinhardt circular domains are (Carathéodory) circular domains.

A homogeneous domain D for which $k_D(z, \bar{t})$ satisfies the Lu Qi-Keng conjecture at a point $\bar{t}_0 \in D^*$ is a Lu Oi-Keng domain by the homogeneity of D and the biholomorphic relative invariance (see (15)) of the Bergman kernel function (c.f. [1]). Therefore, any homogeneous complete circular domain D is a Lu Qi-Keng domain, because D satisfies the Lu Qi-Keng conjecture at the center by Remark 2.

Theorem 1 leads to the following extension of the Bell's proposition.

COROLLARY 1. *The Bell's proposition valid for any bounded complete circular domain D in C^n.*

Proof. In the proof of the Bell's proposition it is essential that a domain D is a Lu Qi-Keng domain and a complete circular one. From Theorem 1 D is a Lu Qi-Keng domain. Therefore, by using the same procedure of the proof of the Bell's proposition, for any bounded complete circular domain the result is obtained.

3. Biholomorphicity of representative functions. For convenience, first we note some notations and differential formulas. I_k shows the identity matrix of order k (k: positive integer), and the symbol $*$ denotes the transposed conjugate. $\partial/\partial z$ and $\partial/\partial z^*$ denote the differential operators:

$$\partial/\partial z \equiv (\partial/\partial z_1, \cdots, \partial/\partial z_n) \quad \text{and} \quad \partial/\partial z^* \equiv {}^t(\partial/\partial \bar{z}_1, \cdots, \partial/\partial \bar{z}_n) . \tag{5}$$

Let holomorphic functions A, B, and b of several complex variables $z \equiv {}^t(z_1, \cdots, z_n)$ and $\bar{z} \equiv {}^t(\bar{z}_1, \cdots, \bar{z}_n)$ be $(k \times l)$, $(l \times m)$ and $(n \times 1)$ matrices, respectively. Then the following differential formulas can be ascertained:

$$\partial(AB)/\partial z \equiv (\partial/\partial z) \times (AB) = (\partial A/\partial z)(I_n \times B) + A(\partial B/\partial z) , \tag{6}$$

$$\partial(A \times b)/\partial z \equiv (\partial/\partial z) \times (A \times b) = (\partial A/\partial z) \times b + A \times (\partial b/\partial z) \tag{7}$$

and

$$\partial A/\partial z \equiv (\partial/\partial z) \times A = (\partial A/\partial w)((dw/dz) \times I_l) , \qquad dw = (dw/dz)dz , \tag{8}$$

where $A = A(w, \bar{w})$ and $w = w(z) \equiv {}^t(w_1(z), \cdots, w_n(z))$ is a holomorphic vector function of z and $dz \equiv {}^t(dz_1, \cdots, dz_n)$ (see [4], [5], [9]).

If there exists a primitive function $F(z)$ of a function $A(z)$ such that $dF(z)/dz = A(z)$, $\int_{z_0}^{z} A(z)dz = F(z) - F(z_0)$ is called the integral of $A(z)$ from z_0 to z, where pathes of integral are homotopic in the domain.

It is known that the function $w(z)$ which maps a domain D onto the representative domain D_w of the biholomorphic equivalent class of D under the initial conditions

$$w(z_0) = \tau \quad \text{and} \quad dw(z_0)/dz = I_n , \qquad z_0 \in D , \tag{9}$$

is given in the matrix representation by making use of the Bergman kernel function as follows:

$$w(z) = T_D^{-1}(z_0, \bar{z}_0) \int_{z_0}^{z} T_D(z, \bar{z}_0) dz + \tau , \tag{10}$$

where $w(z) \equiv {}^t(w_1(z), \cdots, w_n(z))$, $z \equiv {}^t(z_1, \cdots, z_n) \in D$ and

$$T_D(z, \bar{t}) \equiv \partial^2 \log k_D(z, \bar{t})/\partial t^* \partial z \quad \text{(see [9], [13])} . \tag{11}$$

We call τ the center of the representative domain of D. $T_D(z, \bar{z})$ is the Bergman metric tensor and positive definite and $T_D(z, \bar{t})$ is holomorphic with respect to $(z, \bar{t}) \in D \times D^*$ if D is a Lu Qi-Keng domain.

If D is a bounded complete circular domain with center at the origin, D itself is the representative domain with the same center ([6]). It is known that a domain D is the representative domain with center at $t_0 \in D$ if and only if

$$T_D(z, \bar{t}_0) = T_D(t_0, \bar{t}_0) \quad \text{for} \quad z \in D \ ([9], [13]) . \tag{12}$$

Hence for a bounded complete circular domain D with center at the origin $\det T_D(z, 0) = \det T_D(0, 0) \neq 0$ (precisely > 0) for $z \in D$.

LEMMA 3. *Let D be a bounded complete circular domain with center at the origin, then*

$$\Lambda = \{t \in D \mid \det T_D(z, \bar{t}) \neq 0 \ for \ z \in \Delta\} ,$$

where Δ is an arbitrary compact subset of D, is an open set.

Proof. From (11), by calculations we have

$$T_D(z, \bar{t}) = [k_D(z, \bar{t})\partial^2 k_D(z, \bar{t})/\partial t^* \partial z - \partial k_D(z, \bar{t})/\partial t^* \partial k_D(z, \bar{t})/\partial z]/k_D^2(z, \bar{t}) . \tag{13}$$

(13) shows that $\det T_D(z, \bar{t})$ is holomorphic with respect to $(z, \bar{t}) \in D \times D^*$, because by Theorem 1 D is a Lu Qi-Keng demain, namely, $k_D(z, \bar{t}) \neq 0$ in $D \times D^*$. On the other hand, $\det T_D(z, 0) = \det T_D(0, 0) > 0$ for $z \in D$. Thus, by a proof similar to the proof of Lemma 1, we have the result.

LEMMA 4. *Let D be a bounded complete circular domain with center at the origin, then $\Lambda = D$, that is,* $\det T_D(z, \bar{t}) \neq 0$ *for* $(z, \bar{t}) \in D \times D^*$.

Proof. By the matrix differentiations (6) and (7) we have

$$\partial A_k z^k/\partial z = A_k \partial z^k/\partial z = A_k(I_n \times z^{k-1} + z \times I_n \times z^{k-2} + \cdots + z^{k-1} \times I_n)$$

and $\partial \overline{A_k t^k}/\partial t^* = (\partial A_k t^k/\partial t)^*$. Therefore, from (3) we have

$$\begin{aligned} \partial^2 k_D(z, \bar{t})/\partial t^* \partial z \,|_{z=z_1/\lambda, t=\lambda t_1} &= \partial^2 k_D(z_1/\lambda, \lambda \bar{t}_1)/\partial t^* \partial z \\ &= \partial^2 k_D(z_1, \bar{t}_1)/\partial t^* \partial z , \end{aligned}$$

$$\partial k_D(z, \bar{t})/\partial z \,|_{z=z_1/\lambda, t=\lambda t_1} = \partial k_D(z_1/\lambda, \lambda \bar{t}_1)/\partial z = \lambda \partial k_D(z_1, \bar{t}_1)/\partial z$$

and

$$\partial k_D(z, \bar{t})/\partial t^* \,|_{z=z_1/\lambda, t=\lambda t_1} = (1/\lambda)\partial k_D(z_1, \bar{t}_1)/\partial t^* ,$$

for the expansion (3) converges uniformly. From (13) and the above relations we arrive at

$$T_D(z_1/\lambda, \lambda \bar{t}_1) = T_D(z_1, \bar{t}_1) \tag{14}$$

for $1 - \lambda$ $(0 < \lambda < 1)$ as small as we need. Noting that Λ is an open set including the origin, by the similar way as in the proof of Lemma 2, we obtain $\Lambda = D$.

THEOREM 2. *The representative function of any bounded complete circular domain with center at the origin under the initial conditions* (9) *is biholomorphic.*

Proof. As Δ in Lemma 3 is an arbitrary compact subset of D, then $\det T_D(z, \bar{t}) \neq 0$ in $D \times D^*$ by Lemma 4. Therefore, for the representative function (10) for an arbitrary $z_0 \in D$ we have

$$\det(dw(z)/dz) = \det T_D(z, \bar{z}_0)/\det T_D(z_0, \bar{z}_0) \neq 0 \quad \text{for} \quad z \in D ,$$

which shows that the representative function (10) under the initial conditions (9) is locally one-to-one in D. On the other hand, Theorem 1 shows that (10) is holomorphic in D. Thus the representative function is biholomorphic in D with respect to an arbitrary point $z_0 \in D$.

REMARK 4. Theorems 1 and 2 are valid for domains in the sence of Riemann regions (which become simply connected Kaehler manifolds in the present case) belonging to the biholomorphic equivalent class and in particular for domains belonging to the isomorphic equivalent class of a bounded complete circular domain by virtue of the relative invariances:

$$k_D(z, \bar{t}) = k_E(X(z), \overline{X(t)})\, \overline{\det(dX(t)/dz)}\, \det(dX(z)/dz) \tag{15}$$

and

$$T_D(z, \bar{t}) = (dX(t)/dz)^* T_E(X(z), \overline{X(t)})(dX(z)/dz) , \tag{16}$$

where D is a bounded schlicht domain in C^n, $X(z)$ is a biholomorphism and E denotes the image manifold of D under $X(z)$ ([2]).

THEOREM 3. *The representative function* (10) *of any homogeneous complete circular domain D with center at the origin under the initial conditions* (9) *is an isomorphism.*

Proof. For the representative function (10) with respect to an arbitrary point $z_0 \in D$, let us assume $\eta = \eta(z)$ is a holomorphic homogeneous transformation (automorphism) which maps z_0 to the origin (center of D), by (16) we have

$$T_D(z, \bar{z}_0) = (d\eta(z_0)/dz)^* T_D(\eta(z), 0)(d\eta(z)/dz) \quad \text{for} \quad z \in D ,$$

and the representative function

$$w(z) = T_D^{-1}(z_0, \bar{z}_0)\int_{z_0}^{z} T_D(z, \bar{z}_0)dz + \tau$$
$$= (d\eta(z_0)/dz)^{-1}T_D^{-1}(0, 0)\int_0^{\eta} T_D(\eta, 0)d\eta + \tau$$

by (16) and (8). Since D itself is a representative domain with center at the origin ([6]), $T_D(\eta, 0) = T_D(0, 0) > 0$ follows from (12). Thus we obtain

$$w = (d\eta(z_0)/dz)^{-1}\int_0^{\eta} d\eta + \tau = (d\eta(z_0)/dz)^{-1}\eta + \tau .$$

If $z_1 \neq z_2$, $z_1, z_2 \in D$, from the transitivity of $\eta(z)$ it follows that $\eta(z_1) \neq \eta(z_2)$. Let us set $w(z_1) = w_1$ and $w(z_2) = w_2$, then we have $w_2 - w_1 = (d\eta(z_0)/dz)^{-1}(\eta(z_2) - \eta(z_1)) \neq 0$, because $\det(d\eta(z_0)/dz) \neq 0$. This shows that the representative function $w = w(z)$ is globally one-to-one in D.

EXAMPLE 1. The classical Cartan domains (bounded irreducible symmetric domains) are holomorphically homogeneous complete circular domains, and therefore Lu Qi-Keng domains. Thus the representative functions of them are isomorphisms. For instance, the representative function of the domain $R(I) = \{z \mid I_n - z^*z > 0,\ z\colon (m \times n)$ matrix variable$\}$ (first type of the Cartan domains) under the condition $w(z_0) = 0$ $(\tau = 0)$ is given by

$$\tilde{w}(z) = T_{R(I)}^{-1}(z_0, \bar{z}_0)\int_{z_0}^{z} T_{R(I)}(z, \bar{z}_0)d\tilde{z} , \qquad z_0 \in R(I) , \tag{17}$$

where $z = (z_{ij})$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$, $\tilde{z} = {}^t(z_{11}, z_{12}, \cdots, z_{1n}, z_{21}, \cdots, z_{2n}, \cdots, z_{m1}, \cdots, z_{mn})$ and

$$T_{R(I)}(z, \bar{z}_0) = (m + n)^t\,(I_n - z_0^* z)^{-1} \times (I_m - zz_0^*)^{-1} .$$

By direct calculations we have

$$w(z) = (I_m - z_0 z_0^*)^{-1}(z - z_0)(I_n - z_0^* z)^{-1}(I_n - z_0^* z_0) \tag{18}$$

(see [8], p. 13).

Noting that $(A \times B)^{-1} = A^{-1} \times B^{-1}$, ${}^t(A^{-1}) = ({}^tA)^{-1}$, we have

$$\begin{aligned} d\tilde{w}(z)/d\tilde{z} &= T_{R(I)}^{-1}(z_0, \bar{z}_0)T_{R(I)}(z, \bar{z}_0) \\ &= \{{}^t(I_n - z_0^* z_0) \times (I_m - z_0 z_0^*)\}\{{}^t(I_n - z_0^* z)^{-1} \times (I_m - zz_0^*)^{-1}\} . \end{aligned} \tag{19}$$

Thus we have $\det(d\tilde{w}(z)/d\tilde{z}) \neq 0$, ∞ in $R(I)$, and the representative function (18) is globally one-to-one.

A holomorphic homogeneous transformation $\eta = \eta(z)$ $(\eta(z_0) = \tau)$ in

$R(I)$ is given by

$$(20)\qquad \begin{aligned}&(I_m - z_0z_0^*)^{-1/2}(z - z_0)(I_n - z_0^*z)^{-1}(I_n - z_0^*z_0)^{1/2}\\ &= U_0(I_m - \tau\tau^*)^{-1/2}(\eta - \tau)(I_n - \tau^*\eta)^{-1}(I_n - \tau^*\tau)^{1/2}U_1\ ,\end{aligned}$$

where U_0 and U_1 are constant unitary matrices, as is well known ([8]). The representative function (18) is very close to the holomorphic homogeneous transformation (20) for $\tau = 0$.

The Maschler's m-representative function ([7]) and the m-representative function of Kato [4] or Kikuchi [5] of D under the same initial conditions $(w(z_0), dw(z_0)/dz, d^2w(z_0)/dz^2, \cdots, d^mw(z_0)/dz^m) = (0, I_n, 0, \cdots, 0)$, $z_0 \in D$, are given as follows:

$$(21)\qquad w(z) = (0, I_n, 0, \cdots, 0)P_m(z, \bar{z}_0)/(1, 0, \cdots, 0)P_{m-1}(z, \bar{z}_0)\ ,$$

where

$$P_i(z, \bar{z}_0) = \begin{bmatrix} k_{00^*} & \cdots & k_{i0^*} \\ \vdots & & \vdots \\ k_{0i^*} & \cdots & k_{ii^*} \end{bmatrix}^{-1} \begin{bmatrix} k_{00^*}(z, \bar{z}_0) \\ \vdots \\ k_{0i^*}(z, \bar{z}_0) \end{bmatrix} ,$$

$$k_{ij^*} = \partial^{i+j}k_D(z, \bar{t})/\partial t^{*j}\partial z^i\,|_{z=t=z_0} \quad \text{and} \quad k_{0j^*}(z, \bar{z}_0) = \partial^j k_D(z, \bar{t})/\partial t^{*j}\,|_{t=z_0}\ ,$$

and

$$(22)\qquad w(z) = \int_{z_0}^{z} (I_n, 0, \cdots, 0) \begin{bmatrix} T_{00^*} & \cdots\cdots & T_{m-10^*} \\ \vdots & & \vdots \\ T_{0m-1^*} & \cdots & T_{m-1m-1^*} \end{bmatrix}^{-1} \begin{bmatrix} T_{00^*}(z, \bar{z}_0) \\ \vdots \\ T_{0m-1^*}(z, \bar{z}_0) \end{bmatrix} dz\ ,$$

where $T_{ij^*} = \partial^{i+j}T_D(z, \bar{t})/\partial t^{*j}\partial z^i\,|_{z=t=z_0}$ and $T_{0j^*}(z, \bar{z}_0) = \partial^j T_D(z, \bar{t})/\partial t^{*j}\,|_{t=z_0}$, respectively. Here the ith partial differentiation of a function is defined as in [4], [13]. When m is equal to 1, (21) and (22) coincide with the Bergman representative function.

COROLLARY 2. *The m-representative function* (22) ($m \geqq 2$) *of a bounded complete circular domain D with center at the origin is holomorphic in D if* $\det(T_{ij^*}) \neq 0$ *at* z_0.

Proof. By Theorem 1 and (13) $T_D(z, \bar{t})$ is holomorphic in $D \times D^*$ and $\partial^j T_D(z, \bar{t})/\partial t^{*j}$ $(j \geqq 1)$ are so. Therefore, if $\det(T_{ij^*}) \neq 0$, (22) exists and is holomorphic.

EXAMPLE 2. The 2-representative function (22) of a unit dise with respect to $z_0(\neq 0) \in D$ is constructed by using $k_D(z, \bar{z}_0) = 1/\pi(1 - \bar{z}_0z)^2$ and $T_D(z, \bar{z}_0) = 2/(1 - \bar{z}_0z)^2$ as follows:

$$(23)\qquad w(z) = (1 - |z_0|^2)(1 - \bar{z}_0u)u\ , \qquad u = (z - z_0)/(1 - \bar{z}_0z)\ .$$

The 2-representative function (23) (see [4], [5]) is holomorphic in D with respect to an arbitrary point $z_0 \in D$, but is not locally one-to-one for any z_0 in $1/2 < |z_0| < 1$, because at such z_0 $dw(z)/dz = (1 - |z_0|^2)^2 (1 + 2|z_0|^2 - 3\bar{z}_0 z)/(1 - \bar{z}_0 z)^3 = 0$ has a solution $z = (1 + 2|z_0|^2)/3\bar{z}_0$ in $|z| < 1$.

The Maschler's 2-representative function (21) (for $m = 2$) of a unit disc with respect to $z_0 \in D$ is given by

$$(24) \quad w(z) = (1 - |z_0|^2)(1 - 3\bar{z}_0 u)u/(1 - 2\bar{z}_0 u) , \quad u = (z - z_0)/(1 - \bar{z}_0 z) ,$$

([4]). It is easily seen that (24) is meromorphic for any z_0 in $1/2 < |z_0| < 1$. And further (24) is not in general locally one-to-one, because $dw(z)/dz = (1-|z_0|^2)^2(1 - 6\bar{z}_0 u + 6\bar{z}_0^2 u^2)/(1 + 2|z_0|^2 - 3\bar{z}_0 z)^2 = 0$ has a solution $z = (1 + (3 + \sqrt{3})|z_0|^2)/(4 + \sqrt{3})\bar{z}_0$ in $|z| < 1$ for z_0 in $1/(3 + \sqrt{3}) < |z_0| < 1$ and has two solutions $z = (1 + (3 \pm \sqrt{3})|z_0|^2)/(4 \pm \sqrt{3})\bar{z}_0$ in $|z| < 1$ for z_0 in $1/(3 - \sqrt{3}) < |z_0| < 1$. Indeed, from $dw(z)/dz = 0$ we have $\bar{z}_0 u = (\bar{z}_0 z - |z_0|^2)/(1 - \bar{z}_0 z) = 1/(3 \pm \sqrt{3})$, and noting that $|z| < 1$ we have the result.

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