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CHARACTERIZATIONS OF λ CONNECTED PLANE CONTINUA

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A continuum M is said to be λ connected if any two of its points can be joined by a hereditarily decomposable continuum in M. Here we characterize λ connected plane continua in terms of Jones' functions K and L.

A nondegenerate metric space that is both compact and connected is called a *continuum*. A continuum M is said to be *aposyndetic at a point p of M with respect to a point q of M* if there exists an open set U and a continuum H in M such that $p \in U \subset H \subset M - \{q\}$.

In [1], F. Burton Jones defines the functions K and L on a continuum M into the set of subsets of M as follows:

For each point x of M, the set $K(x)$ ($L(x)$) consists of all points y of M such that M is not aposyndetic at x (y) with respect to y (x).

Note that for each point x of M, the set $L(x)$ is connected and closed in M [1, Th. 3]. For any point x of M, the set $K(x)$ is closed [1, Th. 2] but may fail to be connected [2, Ex. 4], [3].

Suppose that M is a plane continuum. In this paper it is proved that the following three statements are equivalent.

I. M is λ connected.

II. For each point x of M, the set $K(x)$ does not contain an indecomposable continuum.

III. For each point x of M, every continuum in $L(x)$ is decomposable.

Throughout this paper E^2 is the Euclidean plane. For a given set S in E^2, we denote the closure and the boundary of S relative to E^2 by Cl S and Bd S respectively.

DEFINITION. Let M be a continuum in E^2. A subcontinuum L of M is said to be a *link* in M if L is either the boundary of a complementary domain of M or the limit of a convergent sequence of complementary domains of M.

It is known that a plane continuum is λ connected if and only if each of its links is hereditarily decomposable [5, Th. 2].

THEOREM 1. *Suppose that a continuum M in E^2 contains an indecomposable continuum I, that disjoint circular regions V and Z in E^2 meet I, that a point x belongs to $M -$ Cl $(V \cup Z)$, and that ε is a positive real number. Then there exist continua H and F in I, arc-segments R and T in V, and a point y of $I \cap Z$ such that* (1)

$H \cup F \cup R \cup T$ separates y from x in E^2, and (2) if D is the y-component of $E^2 - (H \cup F \cup R \cup T)$, then each point of D is within ε of I.

Proof. Define p and q to be points of $V \cap I$ that belong to distinct composants of I. Let $\{P_n\}$ and $\{Q_n\}$ be monotone descending sequences of circular regions in E^2 centered on and converging to p and q respectively such that $\mathrm{Cl}\, P_1 \cap \mathrm{Cl}\, Q_1 = \varnothing$ and $\mathrm{Cl}\,(P_1 \cup Q_1)$ is in V.

Suppose that for each positive integer n, only finitely many disjoint continua in $I - (P_n \cup Q_n)$ intersect $\mathrm{Bd}\, P_n$, $\mathrm{Bd}\, Q_n$, and Z. Since I has uncountably many composants, there exists a composant C of I such that for each n, no continuum in $C - (P_n \cup Q_n)$ meets $\mathrm{Bd}\, P_n$, $\mathrm{Bd}\, Q_n$, and Z. It follows that for each n, there is a continuum L_n in $C - (P_n \cup Q_n \cup Z)$ that meets both $\mathrm{Bd}\, P_n$ and $\mathrm{Bd}\, Q_n$. The limit of $\{L_n\}$ is a continuum in $I - Z$ that contains $\{p, q\}$. But since p and q belong to different composants of I and Z intersects I, this is a contradiction. Hence for some integer n, there exists an infinite collection W of disjoint continua in $I - (P_n \cup Q_n)$ such that each element of W meets $\mathrm{Bd}\, P_n$, $\mathrm{Bd}\, Q_n$, and Z.

There exists a sequence of distinct continua $\{H_i\}$ and two sequences of disjoint arc-segments $\{R_i\}$ and $\{T_i\}$ such that for each i,

(1) H_i is an element of W,

(2) R_i and T_i are in $\mathrm{Bd}\, P_n$ and $\mathrm{Bd}\, Q_n$ respectively,

(3) R_i and T_i each meets H_{2i} and no other element of $\{H_i\}$, and each has one endpoint in H_{2i-1} and the other endpoint in H_{2i+1}.

For each positive integer i, let y_i be a point of $H_{2i} \cap Z$ and define D_i to be the complementary domain of $H_{2i-1} \cup H_{2i+1} \cup R_i \cup T_i$ that contains y_i. Note that the elements of the sequence $\{D_i\}$ are disjoint domains in $E^2 - \mathrm{Cl}\,(P_n \cup Q_n)$. Since the union of the continuum $I \cup \mathrm{Cl}\,(P_n \cup Q_n)$ with its bounded complementary domains is a compact subset of E^2, for some i, every point of D_i is within ε of I and $H_{2i-1} \cup H_{2i+1} \cup R_i \cup T_i$ separates y_i from x in E^2.

THEOREM 2. *If M is a λ connected continuum in E^2, then for each point x of M, every continuum in the set $K(x)$ is decomposable.*

Proof. Assume that for some point x of M, the set $K(x)$ contains an indecomposable continuum I. We shall prove that this assumption implies the existence of a link in M that contains I; this will contradict the hypothesis of this theorem [5, Th. 2].

Let v and z be points of $M - \{x\}$ that belong to distinct composants of I. Define $\{V_i\}$ and $\{Z_i\}$ to be monotone descending sequences of circular regions in E^2 centered on and converging to v and z respectively such that $\mathrm{Cl}\, V_1 \cap \mathrm{Cl}\, Z_1 = \varnothing$ and $\mathrm{Cl}\,(V_1 \cup Z_1)$ is in $E^2 - \{x\}$.

First we show that for each positive integer i, there exists an arc A_i in $E^2 - M$ that goes from Bd V_i to Bd Z_i such that each point of A_i is within i^{-1} of I. By Theorem 1, for any given positive integer i, there exist continua H and F in I, arc-segments R and T in V_i, and a point y of $I \cap Z_i$ such that $H \cup F \cup R \cup T$ separates y from x in E^2 and each point of D (the y-component of $E^2 - (H \cup F \cup R \cup T)$) is within i^{-1} of I. Let U be a circular region containing x in E^2 whose closure misses $H \cup F \cup R \cup T$. Let G be a circular region containing y in E^2 whose closure is in $D \cap Z_i$. Since M is not aposyndetic at x with respect to y, the component of $M - G$ that contains x is not open relative to M at x. Hence there exist two components X and Y of $M - G$ that meet U. It follows that a simple closed curve J in $(E^2 - M) \cup G$ separates X from Y in E^2 [6, Th. 13, p. 170]. Note that J must intersect both G and U [6, Th. 50, p. 18]. Since $J \cap (M - G) = \varnothing$ and $H \cup F \cup R \cup T$ separates G from U in E^2, there is an arc-segment B in $(J \cap D) - M$ that has one endpoint in Bd G and the other endpoint in $R \cup T$. We define A_i to be an arc in $B - (V_i \cup Z_i)$ that goes from Bd V_i to Bd Z_i. Since A_i is in D, each of its points is within i^{-1} of I.

Note that since v and z do not belong to the same composant of I, the limit of each subsequence of $\{A_i\}$ is I. For each i, let Q_i be the complementary domain of M that contains A_i. If $\{Q_i\}$ does not have infinitely many distinct elements, then for some i, the link Bd Q_i in M contains I. Suppose that $\{Q_i\}$ has infinitely many distinct elements. Then some subsequence of $\{Q_i\}$ converges to a link in M [6, Th. 59, p. 24]. It follows that a link in M contains I. This contradicts the fact that M is λ connected [5, Th. 2]. Hence for each point x of M, every continuum in $K(x)$ is decomposable.

THEOREM 3. *Suppose that M is a continuum in E^2 and for each point x of M, every continuum in $K(x)$ is decomposable. Then for each point x of M, every continuum in $L(x)$ is decomposable.*

Proof. Assume that for some point x of M, there is an indecomposable continuum I in $L(x)$. We shall prove that from this assumption it follows that M is not aposyndetic at any point of I with respect to any other point of I. Hence for each point z of I, the set $K(z)$ in M contains I. This will contradict our hypothesis.

Suppose there exists a continuum E in M that does not contain I whose interior relative to M contains a point of I. There exist mutually exclusive circular regions V and Z in E^2 such that

(1) x does not belong to Cl $(V \cup Z)$,

(2) V and Z each meets I,

(3) E and V are disjoint,

(4) $M \cap Z$ is contained in E.

According to Theorem 1, there exist continua H and F in I, arc-segments R and T in V, and a point y of $I \cap Z$ such that $H \cup F \cup R \cup T$ separates y from x in E^2. Define D to be the y-component of $E^2 - (H \cup F \cup R \cup T)$. There exists a circular region G in E^2 containing y such that $\mathrm{Cl}\, G$ is in $D \cap Z$. Let U be a circular region in E^2 containing x whose closure misses $H \cup F \cup R \cup T$.

Since M is not aposyndetic at y with respect to x, the y-component of $M - U$ is not open relative to M at y. Hence $\mathrm{Bd}\, G - M$ contains an arc-segment A whose endpoints, p and q, lie in different components of $M - U$. There exists a simple closed curve J in $(E^2 - M) \cup U$ that separates p from q in E^2 such that $J \cap A$ is connected. Let B denote the component of $J - U$ that contains $J \cap A$. Since $H \cup F \cup R \cup T$ separates G from U in E^2 and B does not intersect $H \cup F$, it follows that both components of $B - A$ meet $R \cup T$. Evidently $B \cup V$ separates p from q in E^2 [6, Th. 32, p. 181]. But since E is a continuum in $E^2 - (B \cup V)$ that contains $\{p, q\}$, this is a contradiction. Hence each subcontinuum of M that contains a point of I in its interior relative to M contains I. This implies that for any point z of I, the set $K(z)$ in M contains I, which contradicts the hypothesis of this theorem. Hence for each point x of M, every continuum in $L(x)$ is decomposable.

THEOREM 4. *Suppose that for each point x of a plane continuum M, every continuum in $L(x)$ is decomposable. Then M is λ connected.*

Proof. Assume that M is not λ connected. It follows that some link in M contains an indecomposable continuum I [5, Th. 2]. By Theorem 1 in [4], each subcontinuum of M that contains a nonempty open subset of I contains I. But this implies that for each point x of I, the set $L(x)$ contains I, which is impossible. Hence M is λ connected.

THEOREM 5. *Suppose that M is a plane continuum. The following three statements are equivalent.*

I. *M is λ connected.*

II. *For each point x of M, every continuum in the set $K(x)$ is decomposable.*

III. *For each point x of M, every continuum in $L(x)$ is decomposable.*

Proof. This follows directly from Theorems 2, 3, and 4.

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