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ATOMS ON THE ROYDEN BOUNDARY

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Let R be a hyperbolic Riemann surface and P a nonnegative C^1-density on R. Every $\widetilde{PE}$-minimal function is shown to be $\widetilde{PD}$-minimal. Conversely $\widetilde{PD}$-minimal functions corresponding to atoms in a certain subset Δ_p of the Royden harmonic boundary are $\widetilde{PE}$-minimal. Points in Δ_P are atoms with respect to the PD-representing measure if and only if they are atoms with respect to the HD-representing measure.

Throughout this paper R denotes a hyperbolic Riemann surface. A positive function f in a family of real-valued functions X on R is called *X-minimal* if for every $g \in X$ with $f \geqq g \geqq 0$ there is a constant $c = c(g)$ such that $cf = g$. If Y is any family of functions on R, then the symbol $\tilde{Y}$ is used to denote the functions that are expressible as decreasing limits of sequences of nonnegative functions in Y. The space of harmonic functions with finite Dirichlet integrals over R, $\int_R du \wedge {}^*du < \infty$, is denoted by $HD(R)$ and for a nonnegative C^1-density P on R the space of Dirichlet finite $\left(\text{energy finite, } \int_R du \wedge {}^*du + u^2P < \infty\text{, resp.}\right)$ solutions of the equation $\Delta u = Pu$ on R is denoted by $PD(R)$ $(PE(R)$, resp.)

The study of the spaces $PE(R)$ and $PD(R)$ was initiated by M. Ozawa [9] and H. Royden [10] and recently revitalized by the idea of looking at them in terms of their boundary values on the Royden *harmonic* boundary (cf. [2] and [7]). Following M. Nakai [4] the more general classes $\widetilde{PE}(R)$ and $\widetilde{PD}(R)$ can also be characterized in terms of their boundary values.

One of the main concerns in the study of solutions of $\Delta u = Pu$ on Riemann surfaces is the "comparison theorems" between various spaces of solutions and harmonic function. The purpose of this paper is to give the precise relations between minimal functions in the classes $\widetilde{HD}(R)$, $\widetilde{PE}(R)$, and $\widetilde{PD}(R)$. The relation between the first two notions was given in [1].

Our main results appear in Nos. 7 and 11. Their proofs depend heavily on the results of several papers listed among the references. For the sake of convenience we quote them in Nos. 2 and 3. The results in Nos. 5, 6, and 8 are generalizations of results of M. Nakai for the case $P \equiv 0$. Comparison with the exact references given there and with the exposition in the monograph of Sario-Nakai [11] should clarify what is involved.

1. Throughout we shall use the notations of Sario-Nakai [11] which we list here. The Royden algebra of R is denoted by $M(R)$, the Royden compactification by R^* and the Royden boundary $\Gamma = \Gamma(R) = R^* \backslash R$. $M_\Delta(R)$ stands for the BD-closure of $M_0(R)$, the functions with compact support in $M(R)$ and the harmonic boundary $\Delta = \Delta(R)$ is the set of common zeros of functions in $M_\Delta(R)$.

If A is a subset of R we use the symbol ∂A for the boundary of A with respect to R, $\bar{A}$ for the closure of A in R^*, and bA for the set $(\bar{A} \backslash \overline{\partial A}) \cap \Gamma$. Thus $\bar{A} \backslash A = \overline{\partial A} \cup bA$. Also if B^* is a subset of R^*, then B will denote $B^* \cap R$.

A subregion G of R can be viewed as a Riemann surface in its own right. There is a unique mapping $j : G^* \to \bar{G}$ which is continuous, onto and leaves G invariant pointwise. Moreover, j restricted to $G \cup j^{-1}(bG)$ is a homeomorphism onto $G \cup bG$ (cf. [6, Proposition 7]). These facts will play an essential role later on.

2. We suppose that there is a nonnegative C^1-density P given on R and we consider the space $P(R)$ of solutions of $\Delta u = Pu$ on R. The subspace $\left\{ u \in P(R) \,\middle|\, \int_R du \wedge {}^* du < \infty \right\}$ of $P(R)$ will be denoted by $PD(R)$ and $PE(R) = \left\{ u \in PD(R) \,\middle|\, \int_R u^2 P < \infty \right\}$. When $P \equiv 0$, then $PD(R) = PE(R) = HD(R)$. We shall use P as a superscript in the symbol for quantities related to $PE(R)$ and P as a subscript in those related to $PD(R)$. The subspaces of bounded functions in $PD(R)$, $PE(R)$, and $HD(R)$ are denoted by $PBD(R)$, $PBE(R)$, and $HBD(R)$. Also when no confusion can arise we omit the reference to R.

We set

$$\Delta^P = \left\{ p \in \Delta \mid \text{there is a nbd } U^* \text{ of } p \text{ with} \int_U P < \infty \right\}$$

and

$$\Delta_P = \left\{ p \in \Delta \mid \text{there is a nbd } U^* \text{ of } p \text{ with} \iint_{U \times U} g_R(x, y) P(x) P(y) < \infty \right\}.$$

Here $g_R(\cdot, y)$ is the Green's function of R with singularity at y.

The set Δ^P was introduced in [2] and Δ_P by Nakai [7].

We now state several properties of Δ_P. The analogous ones for Δ^P are also valid and we leave it to the reader to formulate them. For every $u \in PD$, $u \mid \Delta \backslash \Delta_P = 0$ and $|u| \leq \sup_{\Delta_P} |u|$. Given $\varphi \in M(R)$ such that supp $\varphi \cap \Delta$ is compact and in Δ_P then there exists $u \in PD$ such that $u \mid \Delta = \varphi \mid \Delta$. Since all such φ's restricted to Δ_P are dense in the sup norm in the space $C_0(\Delta_P)$ of continuous function on Δ_P vanishing at infinity, it easily follows that for every $f \in C_0(\Delta_P)$ there is a $u \in P(R)$

such that u is continuous on $R \cup \Delta$ and $u \mid \Delta = f$. (Obviously we mean that $f \mid \Delta \backslash \Delta_P = 0$.) Furthermore, if we fix a point $z_0 \in R$, then there is a finite regular Borel measure μ_P on Δ_P such that $\int_{\Delta_p} u\, d\mu_P = u(z_0)$ for every $u \in PBD$. Using the result of Nakai [7] that every $u \in PD$ is the CD-limit of a sequence $\{u_n\} \subset PBD$ with the boundary values of $\{u_n\}$ on Δ also converging monotonically to those of u, we obtain the same formula for every $u \in PD$. We extend μ_P to Δ by setting $\mu_P(\Delta \backslash \Delta_P) = 0$ and call μ_P *the PD-representing measure for R with center* z_0.

Now denote by μ_P^z the above measure constructed for an arbitrary point $z \in R$. Consider an arbitrary $f \in C_0(\Delta_P)$. Then the above remark to the effect that f is the uniform limit of boundary values of functions in PD shows that $\int_\Delta f\, d\mu_P^z$ considered as a function of z is in $P(R)$. Thus the Harnack inequality gives that μ_P^z and μ_P are mutually absolutely continuous on Δ and hence there is a nonnegative function $K_P(z, \cdot) \in L^1(\mu_P)$ such that for every $u \in PD(R)$ and every $z \in R$, $u(z) = \int_\Delta K_P(z, \cdot) u\, d\mu_P$. The kernel $K_P(z, p)$ can also be chosen to have the following properties: $K_P(z_0, \cdot) = 1$, $K_P(\cdot, p) = 0$ if $p \in \Delta \backslash \Delta_P$ and for each $p \in \Delta_P$, $K_P(\cdot, p) \in P(R)$. It also follows that for any $f \in L^1(\mu_P)$, the function $u(z) = \int_\Delta K_P(z, \cdot) f\, d\mu_P \in P(R)$. If in addition f is bounded, $f \mid \Delta \backslash \Delta_P = 0$, $f \mid \Delta_P$ vanishes at infinity and f is continuous at a point $p \in \Delta_P$, then $\lim_{z \to p} u(z) = f(p)$ and $\lim_{z \to q} u(z) = 0$ for every $q \in \Delta \backslash \Delta_P$. Finally if s is a subsolution on R and $s \in M(R)$, then $s(z_0) \leqq \int_\Delta s\, d\mu_P$.

3. Some other results that we quote here for future reference are as follows. If $u \in P(R)$, u is bounded from below and $\underline{\lim}_{z \to p} u(z) \geqq 0$ for every $p \in \Delta$, then $u \geqq 0$. This result follows easily from the fact that for any compact $E \subset \Gamma \backslash \Delta$ there is a nonnegative superharmonic function s (and hence supersolution) such that s takes on the boundary values ∞ on E and 0 on Δ continuously (cf. [4]).

Suppose G is a subregion of R with

$$(*) \qquad \iint_{G \times G} g_R(z, w) P(z) P(w) < \infty .$$

Then there exists a positive isometric isomorphism $T: PBD(G) \to HBD(G)$ which is onto. Explicitly, $Tu = u + \tau u$ and $\tau u(z) = \int_G g_G(z, \cdot) u P$. If f is a nonnegative measurable function on G which is bounded by a function in $PBD(G)$, then $\tau f \in M_\Delta(G)$. If in addition ∂G is analytic, then τf vanishes continuously on ∂G (cf. [3, Theorem 7C, Theorem 10E, Theorem 11D]).

One of the consequences of these results is that for any $f \in M(R)$

there is a $u \in PD(G)$ such that u agrees with f on $\Delta(G)$, u has continuous boundary values f on ∂G. This follows because there is a function in $HD(G)$ with these properties. Using the mapping j it can also be seen that $u - f \mid bG \cap \Delta(R) = 0$.

4. The fact that $\Delta^P \subset \Delta_P$, which can be verified by a direct computation, is a consequence of the results quoted in No. 3. Indeed for a point $p \in \Delta$ there is a $u \in PD$ (resp. PE) with $u(p) \neq 0$ if and only if $p \in \Delta_P$ (resp. $p \in \Delta^P$). But trivially $PE \subset PD$.

Note that $\int_\Delta u\, d\mu_P^z = \int_\Delta u\, d\mu_z^P$ for every $u \in PE$ and hence $\mu_P^z(B) = \mu_z^P(B)$ for every Borel set $B \subset \Delta^P$. Thus we can state the

THEOREM. *For μ^P-almost every $p \in \Delta^P$, the kernel functions of PD and PE agree on R.*

We need only recall that for every $f \in C_0(\Delta^P)$, $\int_\Delta K_P(z, \cdot) f\, d\mu_P = \int_\Delta f\, d\mu_P^z = \int_\Delta f\, d\mu_z^P = \int_\Delta K^P(z, \cdot) f\, d\mu^P = \int_\Delta K^P(z, \cdot) f\, d\mu_P$. Pick a countable dense set of points $\{z_n\}$ in R. Then for every n, $K_P(z_n, \cdot) = K^P(z_n, \cdot)$ on Δ_P except for a set E_n with $\mu^P(E_n) = 0$. Let $E = \bigcup_1^\infty E_n$. By the continuity of the kernel functions we obtain $K_P(\cdot, p) = K^P(\cdot, p)$ for $p \in \Delta^P \backslash E$.

5. A function u belongs to $\widetilde{PD}$ by definition if it is the limit of a sequence $\{u_n\} \subset PD$ with $u_n \geqq u_{n+1} \geqq 0$. Since PD is a sublattice of $P(R)$, it is easily seen that $u \in \widetilde{PD}$ if and only if $u(z) = \inf\{v(z) \mid v \in PD, v \geqq u\}$. Thus we are led to consider the class $U(\Delta_P)$ of functions on Δ defined by $f \in U(\Delta_P)$ if $f(p) = \inf\{v(p) \mid v \in PD, v \mid \Delta \geqq f\}$. By interchanging infimum and integration we see that $u \in \widetilde{PD}$ if and only if there is an $f \in U(\Delta_P)$ such that $u(z) = \int_\Delta K_P(z, \cdot) f\, d\mu_P$.

For a real-valued function ψ on R we define $\bar{\psi}$ on Δ by $\bar{\psi}(p) = \overline{\lim}_{z \to p} \psi(z)$, for every $p \in \Delta$.

THEOREM. *Suppose $u(z) = \int_\Delta K_P(z, \cdot) f\, d\mu_P \in \widetilde{PD}$. Then $\bar{u} \leqq f$ and $\bar{u} = f\, \mu_P -$ a.e.*

For $P = 0$, this is due to Nakai (cf. [4, Theorem 3.3]).

For the proof let $v \in PD$ with $v \mid \Delta \geqq f$. Then $u \leqq v$ and hence $\bar{u} \leqq v \mid \Delta$. Since $f \in U(\Delta_P)$, we can conclude $\bar{u} \leqq f$. For the second assertion take a sequence $\{F_n\}$ of compact sets in Δ_P with $\mu_P(\Delta_P) = \lim \mu_P(F_n)$ and note that it suffices to prove $\bar{u} = f\, \mu_P -$ a.e. on F_n, for each n. To this end fix n and set $F = F_n$. First assume that f is bounded and hence u is also bounded since $\bar{u} \leqq f$. Suppose that for some $\varepsilon > 0$, there is a compact set $E \subset F$ such that $\bar{u}(p) < f(p) - \varepsilon$ for every $p \in E$. Denote by χ_E the characteristic function of E and set $w(z) = \varepsilon \int_\Delta K_P(z, \cdot)\chi_E \, d\mu_P$. Then $0 \leqq w \leqq \varepsilon$ on R and by the remarks made in No. 2 for every $p \in \Delta \backslash E$, $\lim_{z \to p} w(z) = 0$. Thus for every $p \in \Delta \backslash E$, $\overline{\lim}_{z\to p} u(z) + w(z) \leqq \bar{u}(p) \leqq f(p)$ and every $p \in E$, $\overline{\lim}_{z \to p} u(z) + w(z) \leqq \bar{u}(p) + \varepsilon < f(p)$. Now take any $v \in PD$ such that $v \mid \Delta \geqq f$. For every $p \in \Delta$ we have $\underline{\lim}_{z\to p} v(z) - u(z) - w(z) \geqq 0$. Since $v - u - w$ is bounded from below we can now conclude that $v \geqq u + w$. Taking the infimum over all such v's gives $u(z_0) \geqq u(z_0) + w(z_0)$. This says that $w(z_0) = 0$ which implies that $\mu_P(E) = 0$.

If f is unbounded, then take for every positive integer k a function $v_k \in PD$ such that $0 \leqq v_k \leqq k$ and $v_k \mid F = k$. Note that since PD is a lattice, $f \cap v_k$ (the pointwise infimum of f and v_k) is in $U(\Delta_P)$ and thus $u_k(z) = \int_\Delta K_P(z, \cdot)(f \cap v_k) d\mu_P \in \widetilde{PD}$. Therefore, $\bar{u}_k = f \cap v_k \; \mu_P$ – a.e. on F. Since $\bar{u} \geqq \bar{u}_k$ we obtain by letting $k \to \infty$ that $\bar{u} \geqq f \; \mu_P$ – a.e. on F, which completes the proof.

6. Before we turn to the problem of characterizing the $\widetilde{PD}$-minimal functions we make the following observation. The characteristic function χ_E of any compact set contained in Δ_P belongs to $U(\Delta_P)$. In fact for any $p \in \Delta_P \backslash E$, there is a nonnegative function $\varphi \in M(R)$ such that $\varphi(p) = 0$, $\varphi \mid E = 1$ and supp $\varphi \cap \Delta$ is compact and in Δ_P. Thus there is a nonnegative function $u \in PD$ such that $u \mid E = 1$ and $u(p) = 0$ and the assertion follows.

THEOREM. *If u is $\widetilde{PD}$-minimal on R then there exists a constant k and a point $p \in \Delta_P$ with $\mu_P(p) > 0$ such that $u = kK_P(\cdot, p)$ on R. If $p \in \Delta_P$ with $\mu_P(p) > 0$, then $K_P(\cdot, p)$ is $\widetilde{PD}$-minimal on R.*

(Cf. [4, Theorem 3.6].)

If u is $\widetilde{PD}$-minimal on R, then by Theorem 5, $u(z) = \int_\Delta K_P(z, \cdot)\bar{u} \, d\mu_P$. Set $E_n = \{\bar{u} \geqq 1/n\}$. Note that $\bar{u}$ is upper semicontinuous on Δ and hence E_n is compact in Δ. Since $\bar{u} \mid \Delta \backslash \Delta_P = 0$, $E_n \subset \Delta_P$. By definition of minimality $u > 0$ and consequently $\mu_P(\{\bar{u} > 0\}) > 0$. Therefore, we may choose an integer n such that $\mu_P(E_n) > 0$. Set $E = E_n$ and $w(z) = \int_\Delta K_P(z, \cdot)\chi_E \, d\mu_P \in \widetilde{PD}$. Since $u \geqq (1/n)w \geqq 0$, there is a constant c such that $cu = w$. By Theorem 5 we have that $\bar{w} = 1 \; \mu_P$ – a.e. on E and hence $1 = \sup_R w = \sup_R cu$. This implies that $c > 0$ and u is bounded.

Now let A be a compact subset of E with $\mu_P(E \backslash A) > 0$. Assume that $\mu_P(A) > 0$. Then set $v(z) = \int_\Delta K_P(z, \cdot)\chi_A \, d\mu_P$ and note that $v \in \widetilde{PD}$

and $u \geqq (1/n)v \geqq 0$. Thus there is a constant c_1 with $v = c_1 u$. As above it can be seen that $c_1 > 0$. Hence $c_1\bar{u} = 0\ \mu_P -$ a.e. on $\Delta_P \backslash A$. On the other hand, $E\backslash A \subset \Delta_P\backslash A$, $\mu_P(E\backslash A) > 0$ and $c\bar{u} \geqq 1\ \mu_P -$ a.e. on $E\backslash A$. This contradiction implies that $\mu_P(A) = 0$. This in turn implies that there is a point $p \in E$ with $\mu_P(E) = \mu_P(p)$. We therefore have $\mu_P(p)K(\cdot, p) = w = cu$.

For the proof of the second assertion assume $p \in \Delta_P$ and $\mu_P(p) > 0$. Then $\chi_P \in U(\Delta_P)$ and hence $\mu_P(p)K_P(z, p) = \int_\Delta K_P(z, \cdot)\chi_P\, d\mu_P \in \widetilde{PD}$. By Theorem 5 we have $\mu_P(p)\bar{K}_P(\cdot, p) = \chi_p\,\mu_P -$ a.e. If $v \in \widetilde{PD}$ with $K_P(\cdot, p) \geqq v \geqq 0$, then $\bar{v} \leqq \bar{K}_P(\cdot, p)$. Consequently $\bar{v} = \bar{v}(p)\chi_p$, $\mu_P -$ a.e. and we conclude that $v = \bar{v}(p)\,\mu_P(p)\,K_P(\cdot, p)$, i.e., $K_P(\cdot, p)$ is $\widetilde{PD}$-minimal.

7. Although for two arbitrary families of functions X and Y with $X \subset Y$ the notion of minimality in one has no bearing on minimality in the other we have the following corollaries to the above results of Nos. 4, 5, and 6.

THEOREM. *Every $\widetilde{PE}$-minimal function is a $\widetilde{PD}$-minimal function. A $\widetilde{PD}$-minimal function is $\widetilde{PE}$-minimal if and only if it vanishes continuously on $\Delta\backslash\Delta^P$.*

8. In order to describe the relationship between $\widetilde{HD}$-minimality and $\widetilde{PD}$-minimality we need the following considerations. Let G be a subregion of R with $bG \neq \emptyset$ and ∂G analytic. Denote by ν_P the PD-representing measure for G with center z_0 and by L_P the corresponding kernel. At this point it will be convenient to extend the definitions of ν_P and μ_P to all of $\Gamma(G)$ and $\Gamma(R)$ by setting them equal to zero on sets disjoint from $\Delta(G)$ and $\Delta(R)$.

THEOREM. *Suppose G is a subregion of R with $bG \neq \emptyset$, ∂G analytic and such that property (*) is satisfied. If B is a Borel subset of bG, then $\nu_P(j^{-1}(B)) > 0$ if and only if $\mu_P(B) > 0$, where ν_P and μ_P have their centers at the same $z_0 \in G$.*

The mapping j in the theorem was defined in No. 1. We present in Nos. 9 and 10 a simplified version of Nakai's proof for the case $P \equiv 0$ (cf. [6, Proposition 8]).

We begin by defining a measure σ_P on bG by setting $\sigma_P(U) = \nu_P(j^{-1}(U))$ for every open set U in bG. Note that σ_P is also a regular Borel measure and that it has the property that $\int_{bG} f\, d\sigma_P = \int_{j^{-1}(bG)} f\circ j\, d\nu_P$ for any nonnegative σ_P-measurable f. Take any $u \in PD(G)$ with continuous boundary values 0 on ∂G. Then $u\circ j$ vanishes on $j^{-1}(\overline{\partial G})$. But clearly the continuous extension u^* of u to G^* is equal to $u \circ j$. Consequently

$$u(z_0) = \int_{\Gamma(G)} u^* d\nu_P = \int_{j^{-1}(b(G))} u \circ j \, d\nu_P = \int_{bG} u \, d\sigma_P .$$

9. Note that our problem now is to prove $\sigma_P(B) > 0$ if and only if $\mu_P(B) > 0$. To this end we may assume B is compact, in view of the regularity of the measures. We also take $\{V_n^*\}$ a sequence of open sets in R^* such that $B \subset \bar{V}_{n+1}^* \subset V_n^* \subset G \cup bG$ and $\sigma_P(B) = \lim \sigma_P(V_n^* \cap \Gamma(R))$ and $\mu_P(B) = \lim \mu_P(V_n^* \cap \Gamma(R))$. Now we choose $f_n \in M(R)$ with $0 \leqq f_n \leqq 1, f_n \mid V_{n+1} = 1$ and supp $f_n \subset V_n$. The hypothesis on G (cf. No. 3), gives the existence of a function $t_n \in PD(G)$ with continuous boundary values 0 on ∂G and $t_n \mid bG \cap \Delta = f_n \mid bG \cap \Delta$. If we extend t_n to a function s_n on R by setting $s_n = 0$ on $R \backslash G$, then $s_n \in M(R)$ and s_n is a subsolution on R. Thus $s_n(z_0) \leqq \int_{\Gamma(R)} s_n d\mu_P$ and $s_n(z_0) = t_n(z_0) = \int_{bG} t_n d\sigma_P$. Since $\int_{\Gamma(R)} s_n \, d\mu_P \leqq \mu_P(V_n^* \cap \Gamma)$ and $\sigma_P(V_{n+1}^* \cap \Gamma) \leqq \int_{bG} t_n \, d\sigma_P$, we conclude by letting $n \to \infty$ that $\sigma_P(B) \leqq \mu_P(B)$.

10. For the converse assume that $\sigma_P(B) = 0$. Then $\lim t_n(z_0) = 0$ and hence by the Harnack principle t_n converges to 0 uniformly on compact subsets of G. The reflection principle allows us to conclude the same result on compact subsets of $G \cup \partial G$. This means that s_n converges to 0 uniformly on compact subsets of R.

Let $\{R_m\}_1^\infty$ be a regular exhaustion of R. Consider functions $u_{nm} \in C(R)$ such that $u_{nm} \mid R \backslash R_m = s_n$ and $u_{nm} \in P(R_m)$. Since s_n is a bounded nonnegative subsolution on R, the weak Dirichlet principle (cf. [7]) implies that there is a solution $u_n = BD\text{-}\lim_m u_{nm}$. Since $s_n - u_{nm} \in M_0(R)$, we also have $u_n - s_n \in M_\Delta(R)$, i.e., $u_n = s_n$ on Δ. Therefore, $\mu_P(B) = \lim \int u_n \, d\mu_P = \lim u_n(z_0)$.

On the other hand, we have $u_{1m} - u_{nm} \geqq s_1 - s_n$ since $s_1 - s_n$ is also a subsolution on R. This in turn implies that $u_{1,m+1} - u_{n,m+1} \geqq u_{1m} - u_{nm}$ and hence $u_1 - u_n \geqq u_{1m} - u_{nm}$. Note that for m so large that $z_0 \in R_m$ we have $\lim_n u_{nm}(z_0) = 0$ in view of the fact that $\{s_n\}$ converges uniformly to 0 on ∂R_m. Thus $\lim_n u_1(z_0) - u_n(z_0) \geqq u_{1m}(z_0)$ and consequently $\lim u_n(z_0) \leqq 0$. Since $u_n(z_0) \geqq 0$, we obtain $\lim u_n(z_0) = 0$ and the proof of Theorem 8 is complete.

11. We are ready to state our main result.

THEOREM. *A point $p \in \Delta_P(\Delta^P$ resp.) is an atom with respect to $\mu_P(\mu^P$ resp.) if and only if it is an atom with respect to μ_0.*

The statement for μ^P and Δ^P has been established by [1]. Since nonnegative solutions are subharmonic functions one can easily see that $\mu_0(B) \geqq \mu_P(B)$ for every Borel set $B \subset \Delta$. Thus if $p \in \Delta_P$ is an atom with respect to μ_P then it must be an atom with respect to μ_0.

Conversely suppose $p \in \Delta_P$ and $\mu_0(p) > 0$. Then by definition of Δ_P there is a neighborhood U^* of p with p with $\iint_{U \times U} g_R(x, y)P(x)P(y) < \infty$. By the well known result of Nakai [6, Proposition 9] we can find a neighborhood G^* of p with $G^* \subset U^*$, G a region in R and ∂G analytic. Note that $p \in bG$ and G satisfies condition (*). Thus $\nu_0(j^{-1}(p)) > 0$; that is, there is an atom with respect to ν_0 on the Royden boundary of G. Our task now is to show that this point is also an atom with respect to ν_P for then another application of Theorem 8 gives the desired result.

The isomorphism T described in No. 3 can be extended to a mapping on the bounded functions in $\widetilde{PD}(G)$. In fact, if $u \in \widetilde{PD}(G)$ and $u \leqq c$, then take $u_n \in PD(G)$ with $u_n \downarrow u$. Since $T^{-1}(c) \in PBD(G)$, $u \leqq T^{-1}(c)$ and $PD(G)$ is a sublattice of $P(G)$ we have that $u_n \wedge T^{-1}(c) \downarrow u$. Thus the set of bounded functions in $\widetilde{PD}(G)$ is exactly $\widetilde{PBD}(G)$. So for $u \in PBD(G)$ set $Tu = \lim Tu_n$ and note that $Tu \in \widetilde{HBD}(G)$. By the monotone convergence theorem T is again given by the formula $Tu = u + \tau u$ and hence is order preserving and commutes with multiplication by positive scalars. Also note that this extension maps $\widetilde{PBD}(G)$ onto $\widetilde{HBD}(G)$, which trivially are the bounded function in $\widetilde{HD}(G)$.

Since $\tau u \in M_\Delta(G)$ (cf. No. 3) for every $q \in \Delta(G)$ we have

$$(**) \qquad \overline{\lim}_{z \to q} Tu(z) = \overline{\lim}_{z \to q} u(z) .$$

Thus Theorem 5 shows that $Tu = 0$ if and only if $u = 0$. This in turn shows that T preserves minimal functions in $\widetilde{PBD}(G)$ and $\widetilde{HBD}(G)$. In view of the fact that all minimal functions in $\widetilde{PD}(G)$ and $\widetilde{HD}(G)$ are bounded we conclude that T preserves them.

Suppose that q_0 is the point in $\Delta(G)$ with $\nu_0(q_0) > 0$. Then Theorem 6 shows that $L_0(\cdot, q_0)$ is an $\widetilde{HD}$-minimal function on G. By the above remark and again by Theorem 6 there is a point $q_1 \in \Delta(G)$ such that $\nu_P(q_1) > 0$ and $TL_P(\cdot, q_1) = L_0(\cdot, q_0)$.

We trivially have $L_0(z, q_0) = 1/\nu_0(q_0)\int_{\Delta(G)} L_0(z, \cdot)\chi_{q_0}\, d\nu_0$. Since χ_{q_0} is continuous at every $q \in \Delta(G)$, $q \neq q_0$ we have $\lim_{z \to q} L_0(z, q_0) = 0$ for every $q \neq q_0$, (cf. No. 2). In view of the analogous property for $L_P(\cdot, q_1)$ and (**) we conclude that either $q_1 = q_0$ or $\overline{L_P(\cdot, q_1)} = 0$. But since $L_P(\cdot, q_1) > 0$, Theorem 5 excludes the latter alternative. Thus $\nu_P(q_0) > 0$.

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