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# THE ALGEBRA OF BOUNDED CONTINUOUS FUNCTIONS INTO A NONARCHIMEDEAN FIELD

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**Let $S$ be a topological space, $F$ a complete nonarchimedean rank 1 valued field, and $C^*(S, F)$ the Banach algebra of bounded, continuous, $F$-valued functions on $S$. Various topological conditions on $S$ and/or $F$ are shown to be equivalent, respectively, to each of the following: every maximal ideal of $C^*(S, F)$ is fixed; the only quotient field of $C^*(S, F)$ is $F$ itself; every homomorphism of $C^*(S, F)$ into $F$ is an evaluation at a point of $S$; the Stone-Weierstrass theorem holds for $C^*(S, F)$. It is also shown that a certain topological space derived from $S$ may be embedded in the space of maximal ideals of $C^*(S, F)$ with Gelfand topology, or in the space of homomorphisms of $C^*(S, F)$ into $F$.**

0. Introduction. Throughout this paper, $C^*(S, F)$ denotes the Banach algebra of bounded, continuous functions on a topological space $S$ into a complete nonarchimedean rank 1 valued field $F$. We introduce several stronger-than-usual topological separation properties, such as ultrahausdorff, ultraregular, and ultranormal; and several weaker-than-usual compactness properties, such as mildly compact, mildly countably compact, and mildly Lindelof. We then show that several key implications involving $C^*(S, F)$ become equivalences when the new topological properties replace their conventional counterparts.

In §1, we define and discuss these new topological properties, and relate them to the cofilters ("ouf-filtres") of van der Put [13]. In §2, we obtain a result on the metric structure of non-locally compact nonarchimedean Banach spaces.

In §3, we show that all maximal ideals of $C^*(S, F)$ are fixed if and only if $S$ is mildly compact (Theorem 15); and that $F$ is the only quotient field of $C^*(S, F)$ if and only if $F$ is locally compact or $S$ is mildly countably compact (Theorem 19). Using the result of §2, we also give necessary and/or sufficient conditions for the only homomorphisms of $C^*(S, F)$ into $F$ to be evaluations at points of $S$ (Theorems 20 and 21). We also show that the set of quasicomponents of $S$, appropriately topologized, is homeomorphic to the space of fixed maximal ideals of $C^*(S, F)$, with either of the Gelfand topologies defined by Shilkret [14] (Theorems 10 and 12).

In §4, we extend results of Kaplansky [7] and Chernoff, Rasala, and Waterhouse [3]: we introduce two versions of the Stone-Weierstrass property, and show that the stronger version in $C^*(S, F)$ is equivalent to mild compactness of $S$, and the weaker version is sufficient for mild countable compactness of $S$ (Theorems 22 and 23).

It is interesting to note that many of our results involve properties of $S$ only, and are independent of the choice of the nonarchimedean field $F$.

**1. Topology.** A *quasicomponent* of $S$ is a minimal nonempty intersection of sets clopen in $S$. The quasicomponents form a partition of $S$ into closed sets. Each quasicomponent is a union of components; if $S$ is compact and Hausdorff, the quasicomponents and components are identical [6].

Distinct points or sets in $S$ will be called *ultraseparated* if they are contained in disjoint clopen sets. $S$ will be called *ultrahausdorff*, or $UT2$, if distinct points are ultraseparated; equivalently, if every quasicomponent is a singleton. After Ellis [4], $S$ will be called *ultraregular*, or *UR*, if disjoints points and closed sets are ultraseparated; equivalently, if $S$ has a basis consisting of clopen sets. $S$ will be called *ultranormal*, or *UN*, if disjoint closed sets are ultraseparated.

$S$ is *totally disconnected*, or *TD*, if every component is a singleton. Hence, if $S$ is ultrahausdorff, it is totally disconnected; and if $S$ is compact, Hausdorff, and totally disconnected, then it is ultrahausdorff. We also note that, for a $T_1$ space, ultranormality implies ultraregularity, and ultraregularity implies the ultrachausdorff property. For a compact space, the ultrahausdorff property implies ultraregularity, and ultraregularity implies ultranormality.

LEMMA 1. *In an ultraregular space, every open or closed set is a union of quasicomponents.*

*Proof.* If $S$ is ultraregular, then every open set is a union of clopen sets and hence a union of quasicomponents. It follows that every closed set, being the complement of an open set, is also a union of quasicomponents.

LEMMA 2. *Let $G$ be a family of functions on a set $A$ into a topological space $B$, and let $A$ be topologized with the weak-$G$ topology. Then:*

(1) *If $B$ is ultraregular, $A$ is ultraregular.*

(2) *If $B$ is ultrahausdorff, and $G$ separates points of $A$, then $A$ is ultrahausdorff.*

*Proof.* (1) If $B$ is ultraregular, it has a clopen basis. The preimages, under the members of $G$, of these clopen sets form a clopen subbasis for $A$. Hence $A$ is ultraregular.

(2) Let $p$ and $q$ be distinct points of $A$. If $G$ separates points, then $g(p) \neq g(q)$ for some $g$ in $G$. If $B$ is ultrahausdorff, then $g(p)$ and $g(q)$ are contained in disjoint clopen sets $V$ and $W$ of $B$. Hence $g^{-1}(V)$ and $g^{-1}(W)$ are disjoint clopen neighborhoods of $p$ and $q$ in $A$. Thus, $A$ is ultrahausdorff.

THEOREM 1. *$S$ is ultraregular if and only if the topology on $S$ is the weak-$C^*(S, F)$ topology.*

*Proof.* If $S$ is ultraregular, it has a clopen basis. Since $C^*(S, F)$ contains all characteristic functions of clopen sets, it follows that these basis sets are weak-$C^*(S, F)$ clopen as well. Hence the two topologies are identical.

To prove the converse, we apply Lemma 2, part (1), setting $A = S$, $B = F$, and $G = C^*(S, F)$. Since $F$ is ultraregular, it follows that the weak-$C^*(S, F)$ topology on $S$ is ultraregular.

We will call $S$ *mildly compact*, or *MC*, if every clopen cover of $S$ has a finite subcover; *mildly countably compact* if every countable clopen cover has a finite subcover; and *mildly Lindelof* if every clopen cover has a countable subcover.

We mention several examples. The closed interval $[0, 1]$, with the points $1, 1/2, 1/3, \cdots$ deleted, is mildly compact but not compact. A countably infinite set with discrete topology is mildly Lindelof, but not mildly countably compact. The space of all countable ordinals is mildly countably compact, but not mildly Lindelof [5].

LEMMA 3. *$S$ is mildly countably compact if and only if every partition of $S$ into clopen sets is finite.*

*Proof.* If $S$ is not mildly countably compact, it has a clopen cover $\{A_i: i = 1, 2, 3, \cdots\}$ with no finite subcover. For each positive integer $n$, let $B_n = A_n - \bigcup \{A_i: 1 \leqq i < n\}$. Then the nonempty members of the family $\{B_n: n = 1, 2, 3, \cdots\}$ form an infinite clopen partition of $S$. The proof of the converse is direct.

THEOREM 2. (1) *An ultraregular, mildly compact space is compact.*
(2) *An ultraregular, mildly Lindelof space is Lindelof.*

*Proof.* If $S$ is ultraregular, it has a clopen basis. If $S$ is also mildly compact, then every covering of $S$ by members of this basis has a finite subcover. This last condition is sufficient for compactness [8]. The proof for mildly Lindelof spaces is similar.

The following diagrams of implications summarize some of our results. In these diagrams, *COMP* denotes "compact".

DIAGRAM 1. *For all spaces,*

$$\begin{array}{ccccccc}
TD,\ COMP & \Longleftarrow & UT2,\ COMP & \Longrightarrow & UR,\ COMP & \Longrightarrow & UN,\ COMP \\
\Downarrow & & \Downarrow & & \Updownarrow & & \Downarrow \\
TD,\ MC & \Longleftarrow & UT2,\ MC & & UR,\ MC & \Longrightarrow & UN,\ MC\,.
\end{array}$$

DIAGRAM 2. *For Hausdorff spaces,*

$$\begin{array}{ccccccc}
TD,\ COMP & \Longleftrightarrow & UT2,\ COMP & \Longleftrightarrow & UR,\ COMP & \Longleftrightarrow & UN,\ COMP \\
\Downarrow & & \Downarrow & & \Updownarrow & & \Updownarrow \\
TD,\ MC & \Longleftarrow & UT2,\ MC & \Longleftarrow & UR,\ MC & \Longleftrightarrow & UN,\ MC\,.
\end{array}$$

We note that all the implications of Diagram 2, except $TD$, $COMP \Rightarrow UT2$, $COMP$, hold for $T_1$ spaces as well.

The *quasicomponent quotient space* of $S$, denoted $Q(S)$, will be the space of quasicomponents of $S$ with the quotient topology [9]. For each point $s$ of $S$, $Q(s)$ will denote the quasicomponent containing $s$. If $P$ is a clopen set in $S$, and hence a union of quasicomponents, then $Q(P)$ is clopen in $Q(S)$. If $S$ is ultraregular, we also have, using Lemma 1: If $P$ is open, then $Q(P)$ is open; and if $P$ is closed, then $Q(P)$ is closed. The following theorem is now obvious:

THEOREM 3. (1) *$Q(S)$ is ultrahausdorff. The quotient mapping $Q: S \rightarrow Q(S)$ is a homeomorphism if and only if $S$ is ultrahausdorff.*

(2) *If $S$ is ultraregular, $Q(S)$ is ultraregular.*

(3) *If $S$ is compact, $Q(S)$ is compact.*

(4) *$Q(S)$ is mildly compact, or mildly countably compact, or mildly Lindelof, if and only if $S$ has the same property.*

The *ultraregular kernel* of $S$, denoted $K(S)$, will be the space whose points are the points of $S$ and whose topology is generated by the clopen sets of $S$. It is obvious that:

THEOREM 4. (1) *The topology of $K(S)$ is the weak-$C^*(S, F)$ topology (for* any *nonarchimedean field $F$).*

(2) *$K(S)$ is ultraregular. The topologies of $S$ and $K(S)$ are identical if and only if $S$ is ultraregular.*

(3) *A subset of $S$ is clopen in $K(S)$ if and only if it is clopen in $S$.*

(4) *$K(S)$ is ultrahausdorff if and only if $S$ is ultrahausdorff.*

(5) *If $S$ is compact, $K(S)$ is compact.*

(6) *$K(S)$ is mildly compact, or mildly countably compact, or mildly Lindelof, if and only if $S$ has the same property.*

We now show that the mappings $Q$ and $K$ commute.

THEOREM 5. *$K(Q(S))$ and $Q(K(S))$ are identical topological spaces.*

*Proof.* The points of both $K(Q(S))$ and $Q(K(S))$ are the quasicomponents of $S$. To show that the topologies are identical, we note that the following statements are equivalent:

$R$ is an open set in $K(Q(S))$.

$R$ is a union of clopen sets of $Q(S)$.

$Q^{-1}(R)$ is a union of clopen sets of $S$.

$Q^{-1}(R)$ is open in $K(S)$.

$R$ is open in $Q(K(S))$.

Henceforward, $QK(S)$ will denote the topological space of Theorem 5. We note that this space is both ultrahausdorff and ultraregular.

A filter on $S$ with a base consisting of clopen sets will be called a *cofilter*, and a maximal cofilter will be called an *ultracofilter*. An arbitrary filter $H$ will be called *fixed* (after van der Put, [13]) if it has nonempty intersection; if $M$ is a cardinal number, $H$ will be called *$M$-fixed* if every intersection of $M$ members of $H$ is nonempty. We will say that $H$ *recognizes* a partition $\{A_i: i \in I\}$ of $S$ if one of the sets $A_i$ is in $H$. We note that a cofilter on $S$ is an ultracofilter if and only if it recognizes all finite partitions of $S$ into clopen sets. It is obvious that:

LEMMA 4. *If an ultracofilter on $S$ is $M$-fixed, for some infinite cardinal $M$, then it recognizes all clopen partitions of $S$ of cardinality $M$.*

A partial converse to Lemma 4 is:

LEMMA 5. *If a cofilter $H$ on $S$ recognizes all countable clopen partitions of $S$, then $H$ is countably fixed.*

*Proof.* If $H$ is not countably fixed, then it contains a family $\{A_i: i = 1, 2, 3, \cdots\}$ of clopen sets with empty intersection. For every positive integer $n$, let $B_n = \bigcap \{A_i: 1 \leqq i \leqq n\}$; then $\{B_n: n = 1, 2, 3, \cdots\}$ is a family of clopen sets in $H$, ordered by exclusion, with empty intersection. Let $C_1 = S - B_1$, and for $n > 1$, let $C_n = B_{n-1} - B_n$. Then the family $\{C_n: n = 1, 2, 3, \cdots\}$ forms a clopen partition of $S$, but none of these sets is in $H$. Hence $H$ does not recognize all clopen partitions of $S$.

*Nonmeasurable cardinals* [5] may be characterized as follows: every countably fixed ultrafilter on a set of nonmeasurable cardinality is fixed. It is known that the nonmeasurable cardinals include $\aleph_0$, $\aleph_1$, $\aleph_2$, $\cdots$; they are closed under exponentiation, passage to a successor or to any smaller cardinal, and the supremum operation over a nonmeasurable index. The conjecture that all cardinals are nonmeasurable remains unproved; however, it is known that it can never be disproved.

LEMMA 6. *An ultracofilter $U$ on $S$ is countably fixed if and only if it recognizes all clopen partitions of $S$ of nonmeasurable cardinality.*

*Proof.* If $U$ recognizes all clopen partitions of $S$ of nonmeasurable cardinality, then, by Lemma 5, $U$ is countably fixed. Conversely, suppose $U$ is countably fixed and $\{A_i : i \in I\}$ is a clopen partition of $S$ of nonmeasurable cardinality. Then $U$ induces, via the quotient mapping, a countably fixed ultrafilter $U'$ on the family $\{A_i : i \in I\}$. Since this family is of nonmeasurable cardinality, $U'$ is fixed—that is, $U'$ contains a singleton $\{A_j\}$. Hence $A_j$ is in $U$, so $U$ recognizes the partition $\{A_i : i \in I\}$.

Taking the dual versions of our compactness definitions, and using Lemma 6, we easily have the following lemmas:

LEMMA 7. *$S$ is mildly compact if and only if every ultracofilter on $S$ is fixed.*

LEMMA 8. *The following are equivalent:*

(1) *$S$ is mildly countably compact.*

(2) *Every ultracofilter on $S$ is countably fixed.*

(3) *Every ultracofilter on $S$ recognizes all clopen partitions of $S$ of nonmeasurable cardinality.*

LEMMA 9. *The following are equivalent:*

(1) *$S$ is mildly Lindelof.*

(2) *Every countably fixed ultracofilter on $S$ is fixed.*

(3) *Every ultracofilter on $S$ which recognizes all clopen partitions of $S$ of nonmeasurable cardinality is fixed.*

## 2. The density of a nonarchimedean Banach space.

Let $X$ be a nonarchimedean Banach space over $F$. We will assume that $|F| \subseteqq ||X|| \subseteqq \mathrm{Cl}(|F|)$; i.e., (1) $X$ has a unit vector, and (2) if $F$ is discrete, then $||X|| = |F|$.

A sphere $T(x, d) = \{y \in X : ||y - x|| \leqq d\}$ in $X$ will be called a *closphere*; a sphere $W(x, d) = \{y \in X : ||y - x|| < d\}$ will be called an *osphere*. $V(X)$ will denote the closphere and subring $T(0, 1)$. All clospheres and ospheres are clopen sets; every point of a closphere or osphere is a center; and the clospheres, or ospheres, of any fixed radius form a partition of $X$ [1]. For any $x \in X$, $a \in F$, and $d > 0$, we have:

$$a \cdot T(x, d) = T(ax, |a| \cdot d) \qquad a \cdot W(x, d) = W(ax, |a| \cdot d) .$$

THEOREM 6. *If $X$ is locally compact, then every partition of a closphere in $X$ into clospheres of a fixed smaller radius is finite.*

*Proof.* If $X$ is locally compact, it contains a compact sphere $K$. Since every other sphere in $X$ is homeomorphic, by a translation and scalar multiplication, to a clopen subset of $K$, it follows that every sphere in $X$ is compact. The theorem follows.

The remainder of this section is devoted to proving the companion theorem:

THEOREM 7. *If $X$ is not locally compact, there exists an infinite cardinal $D(X)$ such that: if $d \in \mathrm{Cl}(|F|)$, then every partition of a closphere of radius $d$ in $X$ into clospheres of a fixed smaller radius is of cardinality $D(X)$.*

For $0 < d < 1$, let $r(X, d)$ denote the cardinality of the partition of $V(X)$ into clospheres of radius $d$. Obviously, if $0 < c < d < 1$, then $r(X, c) \geqq r(X, d)$.

LEMMA 10. *If $0 < c < 1$, $0 < d < 1$, and $d \in |F|$, then $r(X, cd) = r(X, c) \cdot r(X, d)$.*

*Proof.* Choose $a \in F$ with $|a| = d$. Since $V(X)$ contains $r(X, c)$ distinct clospheres of radius $c$, it follows that $T(0, d) = a$. $V(X)$ contains $r(X, c)$ distinct clospheres of radius $cd$. By translation, every closphere in $X$ of radius $d$ contains $r(X, c)$ distinct clospheres of radius $cd$; so $V(X)$, which contains $r(X, d)$ clospheres of radius $d$, must contain $r(X, c) \cdot r(X, d)$ clospheres of radius $cd$.

COROLLARY 11. *If $0 < d < 1$ and $d \in |F|$, then $r(X, d^n) = r(X, d)^n$ for any positive integer $n$.*

LEMMA 12. *If $X$ is not locally compact, and $0 < d < 1$, then $r(X, d)$ is infinite.*

*Proof.* Shilkret [14] has shown that if $X$ is not locally compact, then it is not discrete or has infinite residue space. If $X$ is not discrete, then for some $d > 0$ there is a family $\{x_n : n = 1, 2, 3, \cdots\}$ in $X$ such that $d < ||x_n|| < 1$ for all $n$ and $||x_n|| \neq ||x_m||$ for $n \neq m$. Hence the clospheres $\{T(x_n, d): n = 1, 2, 3, \cdots\}$ are distinct in $V(X)$, so $r(X, d)$ is infinite.

If $X$ has infinite residue space, then the members of the residue space form a partition of $V(X)$ into ospheres of radius 1. Since this partition is infinite, the equivalent or finer partition into clospheres of radius $d$ must also be infinite. Hence $r(X, d)$ is infinite.

LEMMA 13. *If $X$ is not locally compact, then the cardinal numbers $\{r(X, d): 0 < d < 1\}$ are all equal.*

*Proof.* (1) Suppose $X$ and $F$ are not discrete, and $0 < c < d < 1$. Choose $e \in |F|$ and a positive integer $n$ such that $d < e < 1$ and $e^n < c$. Applying Corollary 11 and Lemma 12, we have:

$$r(X, d) \geqq r(X, e) = r(X, e)^n = r(X, e^n) \geqq r(X, c) \geqq r(X, d) .$$

Hence $r(X, c) = r(X, d)$.

(2) Suppose $X$ and $F$ are discrete, with $\|X\| = |F|$, and $0 < d < 1$. Let $e < 1$ be a generator of the multiplicative group $|F - \{0\}|$, and let $n$ be the integer satisfying $e^n \leqq d < e^{n-1}$. Then every closphere of radius $d$ is a closphere of radius $e^n$; hence, by Corollary 11 and Lemma 12,

$$r(X, d) = r(X, e^n) = r(X, e)^n = r(X, e) .$$

For $X$ not locally compact, we now define $D(X)$, the *density* of $X$, to be $r(X, d)$ for any $d$ between 0 and 1. For $X$ locally compact, it will be convenient to define $D(X)$ to be 1.

*Proof of Theorem* 7. Suppose $X$ is not locally compact, $T(x, d)$ is a closphere in $X$ with $d \in \mathrm{Cl}(|F|)$, and $P$ is a partition of $T(x, d)$ into clospheres of radius $c < d$.

(1) If $d \in |F|$, then $T(x, d)$ is homeomorphic, by a translation and scalar multiplication, to $V(X)$; and this homeomorphism carries $P$, in a one-to-one fashion, onto the partition of $V(X)$ into clospheres of radius $c/d$. Hence, $\mathrm{card}(P) = r(X, c/d) = D(X)$.

(2) If $d \in \mathrm{Cl}(|F|)$, then there exist $e, f \in |F|$ such that $c < e < d < f$. Obviously, $T(x, e) \subseteqq T(x, d) \subseteqq T(x, f)$; and by part (1), $T(x, e)$ and $T(x, f)$ both contain precisely $D(X)$ clospheres of radius $c$. It follows that $T(x, d)$ contains $D(X)$ clospheres of radius $c$, so $\mathrm{card}(P) = D(X)$.

## 3. Maximal ideals of $C^*(S, F)$.

Let $X$ be a commutative non-archimedean Banach algebra over $F$, with identity $e$ of norm 1.

Each quotient field $F_M = X/M$, for $M$ a maximal ideal of $X$, is both a field extending $F$ (identifying each $b \in F$ with $b \cdot e + M \in F_M$), and a normed algebra over $F$ whose quotient norm extends the valuation on $F$ [11].

Each $x \in X$ gives rise to a function $x^*$ on $\mathfrak{M}$, the family of maximal ideals of $X$, given by $x^*(M) = x + M \in F_M(M \in \mathfrak{M})$. The family $X^* = \{x^*: x \in X\}$, with the supremun norm, is a normed algebra over $F$.

After Shilkret [15], we let $\mathfrak{M}' = \{M \in \mathfrak{M}: F_M = F\}$, and $X_0 = \{x \in X: x^*(M) \in F(M \in \mathfrak{M})\}$. On $\mathfrak{M}$, the *Gelfand topology* is the weak-$X_0^*$ topology; on $\mathfrak{M}'$, and all subsets of $\mathfrak{M}'$, the *strong Gelfand topology* is the weak-$X^*$ topology, and the *weak Gelfand topology* is the weak-$X_0^*$ topology.

THEOREM 8. (1) *All Gelfand topologies are ultraregular.* (2) *The strong Gelfand topology on* $\mathfrak{M}'$ *is ultrahausdorff.*

*Proof.* We apply Lemma 2, with $A = \mathfrak{M}'$ or $\mathfrak{M}$, $B = F$, and $G = X_0^*$ or $X^*$. Part (1) follows immediately; part (2) follows from the observation that $X^*$ separates points of $\mathfrak{M}'$.

For the remainder of this paper, $X$ will be the algebra $C^*(S, F)$. $T(b, d)$ and $W(b, d)$ will denote clospheres and ospheres in $F$. $\mathfrak{M}''$ will denote the family $\{M_s: s \in S\}$ of fixed maximal ideals of $X$, where $M_s = \{x \in X: x(s) = 0\}$. We note that $x^*(M_s) = x(s) \in F$ for all $x \in X$ and $M_s \in \mathfrak{M}''$; hence $\mathfrak{M}'' \subseteqq \mathfrak{M}' \subseteqq \mathfrak{M}$.

THEOREM 9. *For* $s, t \in S$: $M_s = M_t$ *if and only if* $Q(s) = Q(t)$.

*Proof.* If $M_s \neq M_t$, then $x(s) = 0 \neq x(t)$ for some $x \in X$. Let $C$ be a clopen neighborhood of 0 in $F$ excluding $x(t)$; then $x^{-1}(C)$ is a clopen set in $S$ containing $s$ but not $t$. Hence $Q(s) \neq Q(t)$.

Conversely, if $Q(s) \neq Q(t)$, there is a clopen set $K$ in $S$ containing $s$ but not $t$. Since the characteristic function of $K$ belongs to $M_t$ but not $M_s$, we have $M_s \neq M_t$.

COROLLARY 14. *Each member of* $X$ *is constant on quasicomponents of* $S$.

COROLLARY 15. *There is a natural one-to-one correspondence between* $QK(S)$ *and* $\mathfrak{M}''$, *given by* $Q(s) \to M_s(s \in S)$.

COROLLARY 16. *The natural mapping of* $S$ *onto* $\mathfrak{M}''$ *is a bijection if and only if* $S$ *is ultrahausdorff.*

LEMMA 17. $X_0$ *contains all characteristic functions in* $X$.

*Proof.* If $x \in X$ is a characteristic function, then $x^2 - x = 0$. Hence, for all $M \in \mathfrak{M}$, $x^*(M)^2 - x^*(M) = 0$; so $x^*(M) = 0$ or $1$. It follows that $x \in X_0$.

THEOREM 10. *The weak and strong Gelfand topologies on $\mathfrak{M}''$ are identical. This topology is ultrahausdorff and ultraregular.*

*Proof.* Let $(x^*)^{-1}(P)$ be a strong-Gelfand subbasic set in $\mathfrak{M}''$, where $P$ is an osphere in $F$, $x \in X$, and $x^*$ is regarded as a function on $\mathfrak{M}''$. Then
$(x^*)^{-1}(P) = \{M_s: x^*(M_s) \in P\} = \{M_s: x(s) \in P\} = \{M_s: s \in x^{-1}(P)\}$. Since $x^{-1}(P)$ is clopen in $S$, we have $x^{-1}(P) = y^{-1}(Q)$, where $y$ is the characteristic function of $x^{-1}(P)$, $Q$ is a clopen set in $F$ containing 1 but not 0, and $y^*$ is regarded as a function on $\mathfrak{M}''$. It follows that $(x^*)^{-1}(P) = (y^*)^{-1}(Q)$, a weak-Gelfand open set. Hence the two Gelfand topologies on $\mathfrak{M}''$ are identical; by Theorem 8, this topology is ultrahausdorff and ultraregular.

We can now speak of *the Gelfand topology on* $\mathfrak{M}''$ without ambiguity.

We now show that $C^*(S, F)$ is congruent to the algebra of bounded, continuous, $F$-valued functions on an ultrahausdorff, ultraregular space.

THEOREM 11. *$C^*(S, F)$ is congruent to $C^*(QK(S), F)$.*

*Proof.* Let $Q'$: $C^*(QK(S), F) \to C^*(S, F)$ be given by $Q'(x') = x' \circ Q$. Obviously, $Q'$ is a ring homomorphism and an isometry. To show that $Q'$ is onto $C^*(S, F)$, consider any $x \in C^*(S, F)$. Let $x' \in C^*(QK(S), F)$ be given by $x'(Q(s)) = x(s) (s \in S)$. By Corollary 14, $x'$ is well-defined; since $x$ is continuous on $S$, $x'$ is continuous on $QK(S)$; and obviously $Q'(x') = x$.

Thus, in general, we can only hope to recover the structure of $QK(S)$ from that of $X$. Only where $S$ is homeomorphic to $QK(S)$ — i.e., where $S$ is ultraregular and $T_1$—can we hope to recover $S$ itself.

Using the facts that the topology on $QK(S)$ is the weak-$C^*(QK(S), F)$ topology, and the topology on $\mathfrak{M}''$ is the weak-$C^*(S, F)^*$ topology, we easily have:

THEOREM 12. *The natural bijection of $QK(S)$ onto $\mathfrak{M}''$ is a homeomorphism.*

COROLLARY 18. *The natural mapping of $S$ onto $\mathfrak{M}''$ is continuous; it is a homeomorphism if and only if $S$ is ultraregular and $T_1$.*

We now establish a one-to-one correspondence between the closed ideals of $X$ and the cofilters on $S$.

For each $x \in X$, a *smallset* of $x$ will be a set

$$Sm(x, d) = \{s \in S: |x(s)| < d\} = x^{-1}(W(0, d)) \text{ for some } d > 0 .$$

Obviously, all smallsets are clopen. The *zero-set* $Z(x)$ will be the set

$x^{-1}(0)$. For any clopen set $L$ in $S$, $C(L)$ will denote the characteristic function of $L$.

For any proper ideal $I$ of $X$, let $G'(I)$ denote the family of all smallsets of members of $I$, and let $G''(I)$ denote the family of zero-sets of characteristic functions of $I$.

LEMMA 19. *$G'(I) = G''(I)$ for any ideal $I$ in $X$.*

*Proof.* Let $N = Sm(x, d) \in G'(I)$. Let $y \in X$ be given by $y(s) = 0(s \in N)$, $y(s) = x(s)^{-1}(s \notin N)$. Then $xy = C(S - N)$ is in $I$, so $N = Z(xy)$ is in $G''(I)$. Thus, $G'(I) \subseteqq G''(I)$. The reverse inclusion is trivial.

We can now show that $G'(I)$ generates a cofilter $G(I)$ on $S$. First, if $L, M \in G'(I)$, then $C(S - L), C(S - M) \in I$; hence $C(S - L \cap M) = C(S - L) + C(S - M) - C(S - L) \cdot C(S - M) \in I$; so $L \cap M \in G'(I)$. Second, $\phi = Z(e) \notin G'(I)$.

For any cofilter $G$ on $S$, let $I(G)$ denote the family of members of $X$ all of whose smallsets are in $G$.

LEMMA 20. *$I(G)$ is a closed ideal in $X$, for any cofilter $G$ on $S$.*

*Proof.* If $x, y \in I(G)$, then for any $d > 0$, $Sm(x + y, d) \in G$, since $Sm(x + y, d) \supseteqq Sm(x, d) \cap Sm(y, d) \in G$; hence $x + y \in I(G)$. If $x \in I(G)$ and $y \in X$, then for any $d > 0$, $Sm(xy, d) \in G$, since $Sm(xy, d) \supseteqq Sm(x, d\|y\|^{-1}) \in G$; hence $xy \in I(G)$, so $I(G)$ is an ideal. If $z \in \mathrm{Cl}\,(I(G))$, then for any $d > 0$, $\|z - x\| < d$ for some $x \in I(G)$, so $Sm(z, d) = Sm(x, d) \in G$; hence $z \in I(G)$, so $I(G)$ is closed.

LEMMA 21. (1) *$I(G(I)) = \mathrm{Cl}\,(I)$ for any ideal $I$ in $X$.* (2) *$G(I(G)) = G$ for any cofilter $G$ on $S$.*

*Proof.* (1) Let $x \in I(G(I))$. For any $d > 0$, $Sm(x, d) \in G(I)$; hence $Sm(x, d) = z(C(L))$ for some $C(L) \in I$; so $x \cdot C(L) \in I$ and $\|x - x \cdot C(L)\| < d$. Thus $x \in \mathrm{Cl}\,(I)$, so $I(G(I)) \subseteqq \mathrm{Cl}\,(I)$. The reverse inclusion follows from the fact that $I(G(I))$ is a closed ideal containing $I$.

(2) Let $L \in G(I(G))$. Then $L \supseteqq M$ for some $M \in G'(I(G))$; hence $C(S - M) \in I(G)$, so $M = Sm(C(S - M), 1/2) \in G$; so $L \in G$. Thus $G(I(G)) \subseteqq G$. The reverse inclusion is trivial.

The following theorem is now obvious.

THEOREM 13. (1) *There is a one-to-one correspondence between the proper closed ideals of $X$ and the cofilters on $S$, given by $I \rightarrow G(I)$, with inverse $G \rightarrow I(G)$.*

(2) *This correspondence carries the maximal ideals of $X$ onto*

*the ultracofilters on $S$, and the fixed maximal ideals onto the fixed ultracofilters.*

THEOREM 14. *$\mathfrak{M}''$ is Gelfand-dense in $\mathfrak{M}$, and weak-Gelfand dense in $\mathfrak{M}'$.*

*Proof.* Let $N \in \mathfrak{M}$, and let $V = \{M \in \mathfrak{M}: |x_i^*(M)| < d(1 \leqq i \leqq n)\}$ be a typical Gelfand-basic neighborhood of $N$, where $d > 0$ and $x_i \in X_0 \cap N(1 \leqq i \leqq n)$. Then the smallsets $\{Sm(x_i, d): 1 \leqq i \leqq n\}$ all belong to the ultracofilter $G(N)$; so the intersection of these smallsets is nonempty and contains a point $s$. Hence $|x_i^*(M_s)| = |x_i(s)| < d(1 \leqq i \leqq n)$, so $M_s \in V$. Thus $N$ is in the Gelfand-closure of $\mathfrak{M}''$, and $\mathfrak{M}''$ is Gelfand-dense in $\mathfrak{M}$. The proof that $\mathfrak{M}''$ is weak-Gelfand dense in $\mathfrak{M}'$ is similar.

COROLLARY 22. *The natural bijection of $QK(S)$ onto $\mathfrak{M}''$ carries $QK(S)$ onto a dense subspace of both $\mathfrak{M}$ and $\mathfrak{M}'$.*

From Lemma 7 and Theorem 13, we now have the key result:

THEOREM 15. *$\mathfrak{M}'' = \mathfrak{M}$ if and only if $S$ is mildly compact.*

The following corollary will be of use later.

COROLLARY 23. *Suppose $T$ is a mildly compact topological space, $I'$ is an ideal in $C^*(T, F)$, and the members of $I'$ do not all vanish at any point of $T$. Then $I' = C^*(T, F)$.*

Collecting results, we now nave the following theorem on the natural injection $B: Q(S) \to \mathfrak{M}$ and the mapping $B \circ Q: S \to \mathfrak{M}$.

THEOREM 16. (1) *$B$ is a homeomorphism if and only if $S$ is mildly compact.*

(2) *$B \circ Q$ is a bijection if and only if $S$ is mildly compact and ultrahausdorff.*

(3) *$B \circ Q$ is a homeomorphism if and only if $S$ is compact and ultrahausdorff.*

We note that, by Theorem 13, $\mathfrak{M} = \{I(U): U$ an ultracofilter on $S\}$.

THEOREM 17. *For any $x \in X$, $||x|| = ||x^*(I(U))||$ for some $I(U)$ in $\mathfrak{M}$; i.e., each member of $X$ realizes its norm on some maximal ideal.*

*Proof.* Let $x \in X$. For $d > 0$, let $K(d) = \{s \in S: |x(s)| > d\}$. The family $\{K(d): 0 < d < ||x||\}$ generates a cofilter on $S$, and is therefore contained in an ultracofilter $U$.

For $y \in I(U)$ and $0 < d < ||x||$, the sets $Sm(y, d)$ and $K(d)$ are both in $U$; hence some $s \in S$ belongs to $Sm(y, d) \cap K(d)$; so $|y(s)| < d$, $|x(s)| > d$, and $||x - y|| > d$. Therefore, $||x - y|| \geqq ||x||$ for all $y \in I(U)$, so $||x^*(I(U))|| = \inf\{||x - y||: y \in I(U)\} \geqq ||x||$. The reverse inequality is trivial.

THEOREM 18. *The following are equivalent:*

(1) *For any $x \in X$, $||x|| = |x(s)|$ for some $s \in S$; i.e., each member of $X$ realizes its norm at some point of $S$.*

(2) *$F$ is discrete, or $S$ is mildly countably compact.*

*Proof.* Suppose (2) is false. Then there is a clopen partition $\{A_i: i = 1, 2, 3, \cdots\}$ of $S$, and a bounded sequence $\{b_i: i = 1, 2, 3, \cdots\}$ in $F$ such that $\{|b_i|: i = 1, 2, 3, \cdots\}$ is strictly increasing. Let $x \in X$ be given by $x(s) = b_i(x \in A_i, i \geqq 1)$. Then for any $s \in S$, $|x(s)| < \sup\{|b_i|: i = 1, 2, 3, \cdots\} = ||x||$. The proof of the converse is similar.

For $x \in X$, and $U$ an ultracofilter on $S$, $x(U)$ will denote the filter on $F$ generated by the family $\{x(T): T \in U\}$.

LEMMA 24. *If $x \in X$, $U$ is an ultracofilter on $S$, and $b \in F$, then $x^*(I(U)) = b$ if and only if $x(U)$ converges to $b$.*

*Proof.* Suppose $x^*(I(U)) = b$. Then $(x - b \cdot e)^*(I(U)) = 0$, so $x - b \cdot e \in I(U)$. Hence, for any $d > 0$,

$$Sm(x - b \cdot e, d) = \{s \in S: |x(s) - b| < d\} = x^{-1}(W(b, d))$$

is in $U$, so $W(b, d)$ is in $x(U)$. Thus, every osphere containing $b$ is in $x(U)$, so $x(U)$ converges to $b$. The proof of the converse is similar.

COROLLARY 25. *A maximal ideal $I(U)$ is in $\mathfrak{M}'$ if and only if $x(U)$ converges for all $x \in X$.*

THEOREM 19. *$\mathfrak{M}' = \mathfrak{M}$ (i.e., $F$ is the only quotient field of $X$) if and only if $F$ is locally compact or $S$ is mildly countably compact.*

*Proof.* (1) Let $I(U) \in \mathfrak{M}$. For $x \in X$, and $d > 0$, let $\{T(b_i, d): i \in I\}$ be a partition of the closphere $T(0, ||x||)$ into clospheres of radius $d$. Then the nonempty members of the family $\{x^{-1}(T(b_i, d)): i \in I\}$ form a clopen partition of $S$. If $F$ is locally compact (in which case $I$ is finite) or $S$ is mildly countably compact, this partition must be finite;

hence one set $x^{-1}(T(b_j, d))$ is in $U$. Thus $T(b_j, d)$ is in $x(U)$, so $x(U)$ contains clospheres of arbitrarily small radius. Since $F$ is complete, $x(U)$ converges. By Corollary 25, $I(U) \in \mathfrak{M}'$; so $\mathfrak{M}' = \mathfrak{M}$.

(2) Suppose $F$ is not locally compact and $S$ is not mildly countably compact. Then for some $d > 0$, there is a bounded sequence $\{b_i: i = 1, 2, 3, \cdots\}$ in $F$ such that $|b_i - b_j| > d$ for $i \neq j$; and there is a countable clopen partition $\{A_i: i = 1, 2, 3, \cdots\}$ of $S$. The family $\{S - A_i: i = 1, 2, 3, \cdots\}$ generates a cofilter on $S$, and is therefore contained in an ultracofilter $U$. Let $x \in X$ be given by $x(s) = b_i (s \in A_i, i \geqq 1)$. Then $x(U)$ contains no closphere of radius $d$, since it contains the complement of every such closphere. Hence $x(U)$ does not converge in $F$, so $I(U)$ is not in $\mathfrak{M}'$. Thus $\mathfrak{M}' \neq \mathfrak{M}$.

LEMMA 26. *A maximal ideal $I(U)$ belongs to $\mathfrak{M}'$ if and only if $U$ recognizes all clopen partitions of $S$ of cardinality less than or equal to $D(F)$.*

*Proof.* We may assume that $F$ is not locally compact, for otherwise the lemma is trivial. Suppose that $U$ recognizes all clopen partitions of $S$ of cardinality less than or equal to $D(F)$, and let $x \in X$ and $0 < d < \|x\|$. Let $\{T(b_i, d): i \in I\}$ be a partition of $T(0, \|x\|)$; then card $(I) = D(F)$. The nonempty members of the family $\{x^{-1}(T(b_i, d)): i \in I\}$ form a clopen partition of $S$ of cardinality less than or equal to $D(F)$; hence $U$ contains a set $x^{-1}(T(b_j, d))$. Thus, $T(b_j, d)$ is in $x(U)$; $x(U)$ contains arbitrarily small spheres; $x(U)$ converges; and $I(U)$ is in $\mathfrak{M}'$. The proof of the converse is similar to part (2) of the proof of Theorem 19.

Recalling Lemma 6, we have:

COROLLARY 27. *If $D(F)$ is an infinite, nonmeasurable cardinal, then $I(U)$ is in $\mathfrak{M}'$ if and only if $U$ is countably fixed.*

Using Theorems 13, 15, and 19, Lemmas 9 and 26, and Corollary 27, we have the following theorems:

THEOREM 20. *$\mathfrak{M}'' = \mathfrak{M}'$ if and only if every cofilter on $S$ which recognizes all clopen partitions of $S$ of cardinality less than or equal to $D(F)$ is fixed.*

THEOREM 21. (1) *For $F$ locally compact: $\mathfrak{M}'' = \mathfrak{M}'$ if and only if $S$ is mildly compact.*

(2) *For $F$ not locally compact: If $S$ is mildly Lindelof, then $\mathfrak{M}'' = \mathfrak{M}'$.*

(3) *For $F$ not locally compact and $D(F)$ nonmeasurable: $\mathfrak{M}'' =$*

*$\mathfrak{M}'$ if and only if $S$ is mildly Lindelof.*

(4) *For $S$ mildly countably compact: $\mathfrak{M}'' = \mathfrak{M}'$ if and only if $S$ is mildly Lindelof.*

The question of whether $\mathfrak{M}'' = \mathfrak{M}'$ always implies that $S$ is mildly Lindelof remains open.

COROLLARY 28. *Suppose $F$ is not locally compact. Then:*

(1) *If $S$ is $T_1$, ultraregular, and mildly Lindelof, then $B \circ Q$ is a homeomorphism of $S$ onto $\mathfrak{M}'$.*

(2) *If $D(F)$ is nonmeasurable, and $B \circ Q$ carries $S$ homeomorphically onto $\mathfrak{M}'$, then $S$ is $T_1$, ultraregular, and mildly Lindelof.*

4. **Stone-Weierstrass properties.** We will say that $X$ has the *stong Stone-Weierstrass property* if every closed subalgebra which separates quasicomponents of $S$ is either $X$ itself or a fixed maximal ideal; and that $X$ has the *weak Stone-Weierstrass property* if the only closed subalgebra which separates quasicomponents and contains the constants is $X$ itself. This section is devoted to proving the following two theorems:

THEOREM 22. *$X$ has the strong Stone-Weierstrass property if and only if $S$ is mildly compact.*

THEOREM 23. *If $X$ has the weak Stone-Weierstrass property, then $S$ is mildly countably compact.*

We begin with a lemma of Kaplansky [7].

LEMMA 29. *If $D$ is a compact set in $F$, and $0 \neq a \in F$, then there is a polynomial $p(t)$ over $F$, without constant term, such that $p(a) = 1$ and $|p(b)| \leqq 1 (b \in D)$.*

*Proof.* We may assume that $a \in D$, for otherwise we can replace $D$ with $D \cup \{a\}$. Let $d = |a|^2/|D|$, where $|D| = \sup\{|b|: b \in D\}$. Then $d \leqq |a|$.

The set $D - T(0, |a|)$ is closed in $D$, hence compact; hence it has a finite partition $\{T(b_i, d) \cap D: 1 \leqq i \leqq n\}$ into clospheres of radius $d$. We may assume that $|b_1| \leqq |b_2| \leqq \cdots \leqq |b_n|$. We set $k(i) = 2^{i-1}(1 \leqq i \leqq n)$ and

$$p(t) = 1 - (1 - t/a) \cdot \prod_{i=1}^{n} (1 - t/b_i)^{k(i)} .$$

By straightforward computation, the lemma follows.

*Proof of Theorem* 22. (1) If $S$ is not mildly compact, then, by Theorem 15, $X$ has a nonfixed maximal ideal $I(U)$. If $Q$ and $R$ are distinct quasicomponents of $S$, then some clopen set $A \subseteqq S$ contains $Q$ but not $R$; either $A$ or $S - A$ is in $U$; and hence a characteristic function in $I(U)$ separates $Q$ and $R$. Thus, $I(U)$ is a closed subalgebra of $X$ which separates quasicomponents. Since $I(U)$ is neither $X$ itself nor a fixed maximal ideal, $X$ does not have the strong Stone-Weierstrass property.

(2) Suppose $S$ is mildly compact, and $Y$ is a closed subalgebra of $X$ which separates quasicomponents. We will prove that if $Y$ is contained in some fixed maximal ideal $M_s$, then $Y = M_s$; a similar proof shows that if $Y$ is not contained in any fixed ideal, then $Y = X$.

We therefore assume that $Y \subseteqq M_s$ for some $s \in S$. First, we contend that if $u, v \in S$ and $Q(s) \neq Q(u) \neq Q(v)$, then some $x_v \in Y$ satisfies: $x_v(u) = 1$, $x_v(v) = 0$, $||x_v|| = 1$.

To prove this, we note that some $y_1 \in Y$ separates $u$ and $v$; and some $y_2 \in Y$ does not vanish at $u$. Let $y = y_1 y_2 - y_1(v) \cdot y_2$; then $y \in Y$, $y(u) \neq 0$, and $y(v) = 0$. Let $a = y(u)$ and $D = y(S)$; then $D$ is mildly compact, hence compact. Let $p(t)$ be the resulting polynomial of Lemma 29, and let $x_v = p(y)$.

Second, we contend that if $V$ is a clopen set in $S$ containing $s$, $u$ is a point of $S$ outside $V$, and $d > 0$, then some $x \in Y$ satisfies: $x(u) = 1$, $|x(v)| \leqq d (v \in V)$, and $||x|| = 1$.

To prove this, we note that $V$ is mildly compact. For each $v \in V$, some $x_v \in Y$ satisfies: $x_v(u) = 1$, $x_v(v) = 0$, and $||x_v|| = 1$. For $v \in V$, let $W_v = \{w \in V: |x_v(w)| < d\}$; then the family $\{W_v: v \in V\}$ is a clopen cover of $V$ and has a finite subcover $\{W_{v(1)}, \cdots, W_{v(n)}\}$. Let $x = x_{v(1)} \cdot x_{v(2)} \cdots x_{v(n)}$.

Third, we contend that if $W$ is a clopen set in $S$ not containing $s$, then the characteristic function $C(W)$ is in $Y$.

To prove this, we note that if $0 < d < 1$ and $u \in W$, then some $x_u \in Y$ satisfies: $x_u(u) = 1$, $|x_u(v)| < d (v \in S - W)$, and $||x_u|| = 1$. Each set $W_u = \{w \in W: |x_u(w) - 1| < d\}$ is clopen in $W$; hence the family $\{W_u: u \in W\}$ is a clopen cover of $W$; and since $W$ is mildly compact, a finite subfamily $\{W_{u(1)}, \cdots, W_{u(n)}\}$ covers $W$. Let

$$x = e - (e - x_{u(1)}) \cdots (e - x_{u(n)}) ;$$

then $x \in Y$ and $||x - C(W)|| \leqq d$. Since $Y$ is closed, $C(W) \in Y$.

Finally, we show that $Y = M_s$. Let $x \in M_s$; for any $d > 0$, the preimages, under $x$, of the clospheres of radius $d$ in $F$ form a clopen partition of $S$. Since $S$ is mildly compact, this partition is finite: $S = \bigcup \{W_i: 1 \leqq i \leqq n\}$, where each $W_i = x^{-1}(T(a_i, d)) = \{u \in S: |x(u) - a_i| \leqq d\}$ for some $a_i \in F$. We may assume that $a_1 = 0$; i.e., that $s \in W_1$. Let $y = a_1 C(W_1) \cdots a_n C(W_n)$; then $y \in Y$ and $||x - y|| \leqq d$. Since $Y$

is closed, $x \in Y$.

*Proof of Theorem* 23. Suppose $S$ is not mildly countably compact. Then $S$ has an infinite clopen partition $\{T_i: i \in I\}$. Let $Y$ be the closed ideal in $X$ generated by the characteristic functions of these sets; and let $Z$ be the subalgebra $Y + F \cdot e$ of $X$. Then $Z$ is a closed subalgebra of $X$ which separates quasicomponents and contains the constants. However, every member of $Z$ must take values arbitrarily close to some constant on all but a finite number of the sets $\{T_i: i \in I\}$; so $Z$ is not equal to $X$ itself. Hence $X$ does not have the weak Stone-Weierstrass property.

We note that the question of whether the weak Stone-Weierstrass property is equivalent to mild compactness, or to mild countable compactness, or to some intermediate property, remains unresolved.

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