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COUNTEREXAMPLES IN THE BIHARMONIC CLASSIFICATION OF RIEMANNIAN 2-MANIFOLDS

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Crucial counterexamples in the biharmonic classification theory of Riemannian 2-manifolds have been deduced from certain general principles. The present note is methodological in nature: the aim is to supplement the theory by showing that very simple counterexamples can be directly constructed.

Whereas earlier work has been devoted to the class H^2 of nonharmonic biharmonic functions, here the class W of all biharmonic functions is discussed. This is of interest, since the classes O_{WB} and O_{WD} of Riemannian manifolds without (nonconstant) bounded or Dirichlet finite biharmonic functions are strictly contained in the corresponding classes O_{H^2B} and O_{H^2D}, as is seen by endowing the unit disk with a suitable conformal metric. Moreover, for W-functions the biharmonic equation need not be reduced to the Poisson equation but can be dealt with directly.

These aspects, however, are not essential. Our sole aim is to produce simple counterexamples. In particular, the function $\log\log(e^x+a)$ on a horizontal strip (Theorem 4) shows immediately that there are parabolic 2-manifolds which carry H^2D-functions. We also include some examples of 3-manifolds.

1. It is well known that there are no bounded harmonic functions on a parabolic manifold. In contrast, we shall show:

THEOREM 1. *There exist parabolic manifolds which carry nonconstant WB-functions.*

Proof. Consider in the complex (x, y)-plane the strip $\{-\infty < x < \infty; 0 \leqq y \leqq 2\pi\}$ with the lines $y = 0$ and $y = 2\pi$ identified by vertical translation so as to obtain a doubly connected Riemann surface S. The choice of the strip instead of the punctured plane is not essential, but it will slightly simplify the computation. Clearly $S \in O_G$, e.g., by virtue of the modular test (cf. [7]). On the Riemannian manifold $S_\lambda = (S, \lambda(z)\,|dz|)$ with $\lambda = e^x$, the function $u = \cos 2y$ is bounded biharmonic. In fact, $\Delta_\lambda u = e^{-2x}\Delta \cos 2y = -4\cos 2y \in H(S)$, where Δ_λ and Δ are the Laplace-Beltrami operators with respect to the metric $\lambda(z)|dz|$ and the Euclidean metric, and H stands for the class of harmonic functions. Thus $S_\lambda \in O_G - O_{WB}$.

2. THEOREM 2. *There exist hyperbolic manifolds which do not carry nonconstant WB-functions.*

Proof. We shall show that the Euclidean 3-space E^3 is such a manifold. Clearly $E^3 \notin O_G$. In order to prove that $E^3 \notin O_{WB}$, let $u \in WB(E^3)$. We recall that every biharmonic function in E^3 can be written as $h + r^2k$ with $h, k \in H(E^3)$ (cf. [1]), and any harmonic h can be expanded in orthogonal spherical harmonics $S_{nm}(\theta, \psi)$,

$$h = \sum_{n=0}^{\infty} r^n \sum_{m=1}^{2n+1} a_{nm} S_{nm}$$

(cf. [1] and [2]). Thus u has the expansion

$$u = \sum_{n=0}^{\infty} r^n \sum_{m=1}^{2n+1} (a_{nm} + b_{nm} r^2) S_{nm} .$$

We multiply both sides by $S_{nm} \sin\theta$, integrate with respect to θ and ψ, and conclude by the boundedness of u that $a_{nm} = 0$ for $n > 0$, all m; and $b_{nm} = 0$ for all (n, m). Thus $u = a_0$.

3. THEOREM 3. *There exist parabolic manifolds which do not carry nonconstant WB-functions.*

Proof. Let S^λ be the strip S_λ with a "cap" at $x = -\infty$, that is, we view S_λ as a simply connected parabolic manifold S^λ punctured at a point corresponding to $x = -\infty$. We assume that there is a $u \in WB(S^\lambda)$. Its restriction to S_λ has the expansion

$$u = \sum_{n=0}^{\infty} e^{nx}[(a_n + b_n e^{2x}) \cos ny + (c_n + d_n e^{2x}) \sin ny] .$$

We multiply both sides by $\cos ny + \sin ny$, integrate with respect to y, and conclude by the boundedness of u that $b_n = d_n = 0$ for all $n \geqq 0$. Hence $u = \sum_{n=0}^{\infty} e^{nx}(a_n \cos ny + c_n \sin ny)$. This is the restriction to S_λ of a harmonic function on S^λ, and we have $u = a_0$, hence the theorem.

That there exist hyperbolic manifolds which carry nonconstant WB-functions is obvious in view of the Euclidean disk.

4. We turn to the class D of functions with finite Dirichlet integrals $D(u) = \int du \wedge *du$.

THEOREM 4. *There exist parabolic manifolds which carry nonconstant WD-functions.*

Proof. We shall show that the function $u = \log\log(e^x + a)$ is in

WD on our parabolic strip S_λ with a suitable metric $\lambda(z)|dz|$. Here the constant $a > 1$ is so chosen that $a \log(1 + a) = 1$. The Euclidean Laplacian

$$\Delta u = \frac{e^x[e^x - a \log(e^x + a)]}{(e^x + a)^2[\log(e^x + a)]^2}$$

is of the same sign as x and has a positive derivative at $x = 0$. Thus $\Delta u/x$ is well defined and positive. Let $\lambda^2 = \Delta u/x$. On the manifold $S_\lambda = (S, (\Delta u/x)^{1/2}|dz|)$, we have $\Delta_\lambda u = x \in H(S)$, and therefore $u \in W$. Moreover, $D(u)$ is independent of the metric, and can be taken over S:

$$D(u) = \int_S \left(\frac{\partial u}{\partial x}\right)^2 dxdy = 2\pi \int_{-\infty}^{\infty} \left(\frac{e^x}{(e^x + a)\log(e^x + a)}\right)^2 dx$$

$$< 2\pi \int_{-\infty}^{\infty} \frac{(e^x + a)de^x}{(e^x + a)^2[\log(e^x + a)]^2} = -\frac{2\pi}{\log(e^x + a)}\Bigg|_{-\infty}^{\infty} < \infty .$$

5. The following trivial necessary condition is a modification of a test in [3]: If $u \in WD$, then $|(u, \Delta\varphi)| \leqq K\sqrt{D(\varphi)}$ for some constant K independent of φ and for all $\varphi \in C_0^\infty$. In fact, for $\varphi \in C_0^\infty$ with supp φ in some regular subregion Ω, $0 = \int_{\partial\Omega} u \wedge *d\varphi = \int_\Omega du \wedge *d\varphi - \int_\Omega u\Delta\varphi dV$ and $|(u, \Delta\varphi)| = \left|\int_\Omega du \wedge *d\varphi\right| = |D(u, \varphi)| \leqq \sqrt{D(u)}\,\sqrt{D(\varphi)} = K\sqrt{D(\varphi)}$.

THEOREM 5. *There exist hyperbolic manifolds which do not carry nonconstant WD-functions.*

Proof. We shall show that E^3 is such a manifold. Since $E^3 \notin O_G$, we only have to prove that $E^3 \in O_{WD}$. Let $u \in WD(E^3)$. Expand $\Delta u = h$ as in No. 2. Suppose $a_{nm} \neq 0$ for some (n, m). Let f be a fixed C_0 function on $[0, \infty)$ with supp $f \subset (0, 1)$, and set $\rho_t(r) = f(r - t)$, $\varphi_t(r, \theta, \psi) = \rho_t(r)S_{nm}(\theta, \psi)$. As $t \to \infty$,

$$|(u, \Delta\varphi_t)| = |(h, \varphi_t)| = \text{const}\int_t^{t+1} \rho_t(r)r^{n+2}dr = O(t^{n+2}) ,$$

$$D(\varphi_t) = \int_{E^3}\left[\left(\frac{\partial\varphi_t}{\partial r}\right)^2 + \frac{1}{r^2\sin^2\psi}\left(\frac{\partial\varphi_t}{\partial\theta}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\varphi_t}{\partial\psi}\right)^2\right]dV \sim O(t^2) ,$$

and $\sqrt{D(\varphi_t)} = O(t)$. We conclude that $a_{nm} = 0$ for all $n \geqq 0$. A fortiori $\Delta u = 0$, and $u \in HD(E^3)$. Since $E^3 \in O_{HD}$, we have $u = \text{const}$.

6. THEOREM 6. *There exist parabolic manifolds which do not carry nonconstant WD-functions.*

Proof. Let u be a WD-function on the "capped" strip S^λ of No.

3. The restriction $\Delta u \mid S_\lambda$ has the expansion $\Delta u = \sum_{n=0}^{\infty} e^{nx}(a_n \cos ny + b_n \sin ny)$. Suppose $a_n^2 + b_n^2 \neq 0$ for some n. Choose the testing function $\varphi_t(x, y) = \rho_t(x)(\cos ny + \sin ny)$, with $\rho_t(x)$ as before. As $t \to \infty$, $|(u, \Delta\varphi_t)| = |(\Delta u, \varphi_t)| = O(e^{(n+2)t})$ and $\sqrt{D(\varphi_t)} = O(1)$. Therefore, $a_n = b_n = 0$ for all n, and $u \in HD(S^\lambda)$. The theorem follows from $S^\lambda \in O_G \subset O_{HD}$.

That there exist hyperbolic manifolds which carry nonconstant WD-functions is obvious in view of the Euclidean disk.

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