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OPEN PROJECTIONS AND BOREL STRUCTURES FOR C^*-ALGEBRAS

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In this paper the relationships existing among the Boolean σ-algebra generated by the open central projections of the enveloping von Neumann algebra $\mathscr{B}$ of a C^*-algebra $\mathscr{A}$, the Borel structure induced by a natural topology on the quasi-spectrum of $\mathscr{A}$, and the type of $\mathscr{A}$ are discussed. The natural topology is the hull-kernel topology. It is shown that this topology is induced by the open central projections and is the quotient topology of the factor states of $\mathscr{A}$ (with the relativized w^*-topology) under the relation of quasi-equivalence. The Borel field is shown to be Borel isomorphic with the Boolean σ-algebra multiplied by the least upper bound of all minimal central projections. Finally, it is shown that $\mathscr{A}$ is GCR if and only if the Boolean σ-algebra (resp. algebra) contains all minimal projections in the center of $\mathscr{B}$, or equivalently, if and only if every point in the quasi-spectrum is a Borel set.

T. Digernes and the present author [10] showed that $\mathscr{A}$ is CCR if and only if the open projections are strongly dense in the center of $\mathscr{B}$. They also showed that the complete Boolean algebra generated by the open central projections is equal to the set of all central projections in $\mathscr{B}$ whenever $\mathscr{A}$ is GCR. Recently, T. Digernes [9] obtained the converse of this result for separable C^*-algebras.

2. **The Boolean algebra of open projections.** Let $\mathscr{B}$ be a von Neumann algebra with center $\mathscr{Z}$ and let $\mathscr{A}$ be a uniformly closed *-subalgebra of $\mathscr{B}$. A projections P in $\mathscr{Z}$ is said to be *open relative to* $\mathscr{A}$ if there is a two-sided ideal $\mathscr{J}$ in $\mathscr{A}$ whose strong closure is $\mathscr{B}P$. In the sequel *all ideals* (unless specifically excluded) will be assumed to be closed two-sided ideals. The definition corresponds to the definition of Akemann [1, Definition II.1] for C^*-algebras with identity. The set $\mathscr{P}(\mathscr{B}, \mathscr{A})$ of all open central projections of $\mathscr{B}$ relative to $\mathscr{A}$ contains 0,1 and the least upper bound (resp. greatest lower bound) of any (resp. any finite) subset [1, Proposition II.5, Theorem II.7].

Now let $\mathscr{B}$ be the enveloping von Neumann algebra of $\mathscr{A}$. The algebra $\mathscr{A}$ will be identified with its embedded image in $\mathscr{B}$. In this case the set $\mathscr{P}(\mathscr{B}, \mathscr{A})$ will be denoted simply by $\mathscr{P}$ and the projection in $\mathscr{P}$ will be called *open* projections. The smallest Boolean algebra (resp. σ-algebra) containing $\mathscr{P}$ will be denoted by $\langle\mathscr{P}\rangle$ (resp. $\langle\!\langle\mathscr{P}\rangle\!\rangle$).

Let $\hat{\mathscr{A}}$ be the set of all unitary equivalence classes of irreducible representations of $\mathscr{A}$. The set $\hat{\mathscr{A}}$ is called the *spectrum* of $\mathscr{A}$. For every irreducible representation ρ of $\mathscr{A}$ on the Hilbert space $H(\rho)$, let $[\rho]$ denote the class in $\hat{\mathscr{A}}$ of which ρ is the representative. If X is a subset of $\hat{\mathscr{A}}$, let $\mathscr{I}(X) = \cap \{\ker \tau \mid \tau \in X\}$. Here $\ker \tau$ is uniquely defined by $\ker \tau = \ker \rho$ for $\rho \in \tau$. Then setting $X^- = \{\tau \in \hat{\mathscr{A}} \mid \ker \tau \supset \mathscr{I}(X)\}$ for $X \neq \emptyset$ and $\emptyset^- = \emptyset$, we obtain a closure operation $\hat{\mathscr{A}}$. The topology defined by this closure operation is called the *hull-kernel* topology and is the family of subsets of $\hat{\mathscr{A}}$ given by

$$\{\{\tau \in \hat{\mathscr{A}} \mid \ker \tau \not\supset \mathscr{I}\} \mid \mathscr{I} \text{ is an ideal of } \mathscr{A}\}$$

(cf. [12, § 3]). Let ρ be a representation of $\mathscr{A}$ and let $\rho^{\sim}$ denote the unique extension of ρ to a σ-weakly continuous representation of $\mathscr{B}$ on $H(\rho)$ such that $\rho^{\sim}(\mathscr{B})$ is the σ-weak closure of $\rho(\mathscr{A})$. If $A \in \mathscr{Z}$ and $\tau \in \hat{\mathscr{A}}$ there is a unique scalar $A^{\sim}(\tau)$ such that $A^{\sim}(\tau)1_{H(\rho)} = \rho^{\sim}(A)$ for all $\rho \in \tau$. Here $1_{H(\rho)}$ is the identity operator on $H(\rho)$. With this notation, the hull-kernel topology of $\hat{\mathscr{A}}$ is given by $\{\{\tau \in \hat{\mathscr{A}} \mid P^{\sim}(\tau) = 1\} \mid P \in \mathscr{P}\}$.

Let $S_0(\hat{\mathscr{A}})$ (resp. $S(\hat{\mathscr{A}})$) denote the ring (resp. σ-ring) generated by the open subsets of $\hat{\mathscr{A}}$. Then there is a projection-valued measure γ of $S(\hat{\mathscr{A}})$ onto $\langle\!\langle \mathscr{P} \rangle\!\rangle$ such that $\gamma(\{\tau \in \hat{\mathscr{A}} \mid P^{\sim}(\tau) = 1\}) = P$ ([19, Theorem 1.9], cf. [12, 5.7.6]).

LEMMA 1. *Let $\mathscr{A}$ be a C^*-algebra, let $\mathscr{B}$ be the enveloping von Neumann algebra of $\mathscr{A}$ and let $\mathscr{P}$ be the set of all open projections of the center $\mathscr{Z}$ of $\mathscr{B}$. Let Q be a minimal projection of $\mathscr{Z}$ and let P_m be the least upper bounded of all minimal projections of $\mathscr{Z}$. Then $\mathscr{B}Q$ is a type I factor whenever Q is in the Boolean σ-algebra $\langle\!\langle \mathscr{P} \rangle\!\rangle P_m$.*

Proof.[1] If X_1 and X_2 are open subsets of $\hat{\mathscr{A}}$ with $X_1 \supset X_2$, then $\gamma(X_1 - X_2)^{\sim}(\tau) = \gamma(X_1)^{\sim}(\tau) - \gamma(X_2)^{\sim}(\tau) = 1$ for every $\tau \in X_1 - X_2$ and $\gamma(X_1 - X_2)^{\sim}(\tau) = 0$ for every $\tau \notin X_1 - X_2$. Since every set X in $S_0(\hat{\mathscr{A}})$ is the union of a finite number of mutually disjoint sets of the form $X_1 - X_2$ where X_1, X_2 are open in $\hat{\mathscr{A}}$ and $X_1 \supset X_2$, we see that $\gamma(X)^{\sim}(\tau) = 1$ if and only if $\tau \in X$. Since every element X in $S(\hat{\mathscr{A}})$ is the union of a monotonally increasing sequence of sets $\{X_n\}$ in $S_0(\hat{\mathscr{A}})$, we get that $\gamma(X)^{\sim}(\tau) = 1$ for every $\tau \in X$.

Now there is a set $X \in S(\hat{\mathscr{A}})$ with $\gamma(X)^{\sim} P_m = Q$. If $\tau \in X$ then $Q^{\sim}(\tau) = 1$, and so $\rho^{\sim}(Q) = 1$ for $\rho \in \tau$. This means that the kernel of $\rho^{\sim}$ is $\mathscr{B}(1 - Q)$. Since ρ is irreducible on $\mathscr{A}$ and since $\rho^{\sim}(\mathscr{B})$, which is

[1] This proof was suggested by the referee. My original proof was based on the results of [**10**].

isomorphic to $\mathscr{B}Q$, is equal to the weak closure of $\rho(\mathscr{A})$, we conclude that $\mathscr{B}Q$ is a type I factor.

The next result characterizes a GCR algebra in terms of the open central projections of its enveloping algebra.

THEOREM 2. *Let $\mathscr{A}$ be a C^*-algebra, let $\mathscr{B}$ be the enveloping von Neumann algebra of $\mathscr{A}$, and let $\mathscr{P}$ be the set of open projections of the center $\mathscr{Z}$ of $\mathscr{B}$. Then the following statements are equivalent:*
(1) *$\mathscr{A}$ is GCR;*
(2) *$\langle\mathscr{P}\rangle$ contains all minimal projections $\mathscr{Z}$; and*
(3) *$\langle\langle\mathscr{P}\rangle\rangle$ contains all minimal projections of $\mathscr{Z}$.*

Proof. (1) $\Rightarrow$ (2). We apply the fact that the set of open central projections in the enveloping von Neumann algebra of a CCR algebra is strongly dense in the set of central projection [10, Theorem 2].

There is a set $\{P_i \mid 0 \leqq i \leqq k\}$ of projections in $\mathscr{P}$ indexed by the ordinals such that (i) $P_0 = 0$, $P_k = 1$, (ii) $P_i < P_{i+1} (i < k)$, (iii) $\bigvee \{P_i \mid i < j\} = P_j$ if j is a limit ordinal with $j \leqq k$; and (iv) $\mathscr{B}_i = \mathscr{B}(P_{i+1} - P_i)$ is the strong closure of a CCR ideal $\mathscr{J}_i$ in $\mathscr{A}(1 - P_i)$ [10, proof of Theorem 3]. Let Q be a minimal projection in $\mathscr{Z}$. There is an ordinal $i < k$ such that $Q \leqq P_{i+1} - P_i$. Let $\mathscr{J}$ be the ideal in $\mathscr{A}$ given by $\mathscr{J} = \{A \in \mathscr{A} \mid AP_i = A\}$. Setting $\mathscr{J}' = \{A \in \mathscr{A} \mid A(1 - P_i) \in \mathscr{J}_i\}$, we obtain an ideal $\mathscr{J}'$ of $\mathscr{A}$ containing $\mathscr{J}$ such that $\mathscr{J}'/\mathscr{J}$ is isomorphic to $\mathscr{J}_i$. Let ρ be the unique extension of the representation $A + \mathscr{J} \rightarrow A(1 - P_i)$ of $\mathscr{J}'/\mathscr{J}$ onto $\mathscr{J}_i$ to a σ-weakly continuous representation of the enveloping von Neumann algebra $\mathscr{C}$ of $\mathscr{J}'/\mathscr{J}$ onto the strong closure $\mathscr{B}_i$ of $\mathscr{J}_i$ on the subspace of the Hilbert space of $\mathscr{B}$ corresponding to the projection $1 - P_i$ (cf. [12, 12.1.5]). Now, if $P \in \mathscr{P}(\mathscr{C}, \mathscr{J}'/\mathscr{J})$, we show that $\rho(P) + P_i$ is in $\mathscr{P}$. Indeed, there is an ideal $\mathscr{K}$ in $\mathscr{J}'/\mathscr{J}$ such that $\mathscr{C}P$ is the strong closure of $\mathscr{K}$ in $\mathscr{C}$. Let $\mathscr{K}'$ be an ideal in $\mathscr{J}'$ with $\mathscr{K}' \supset \mathscr{J}$ such that $\mathscr{K}'/\mathscr{J} = \mathscr{K}$. Then we have that the strong closure of $\mathscr{K}'(1 - P_i) = \rho(\mathscr{K})$ in $\mathscr{B}(1 - P_i)$ is equal to $\rho(\mathscr{C}P) = \mathscr{B}_i\rho(P) = \mathscr{B}\rho(P)$. This means that the strong closure of $\mathscr{K}'$ in $\mathscr{B}$ is equal to $\mathscr{B}(\rho(P) + P_i)$. Hence $\rho(P) + P_i$ is in P. Because $\mathscr{J}_i$ is CCR, the set $\mathscr{P}(\mathscr{C}, \mathscr{J}'/\mathscr{J})$ is strongly dense in the set of central projections of $\mathscr{C}$ [10, Theorem 2]. Recalling that ρ maps the center of $\mathscr{C}$ onto the center of $\mathscr{B}_i$ [14, III, § 5, Problem 7], we obtain a net $\{R_n\}$ of projections in $\mathscr{P}$ which majorizes P_i and is majorized by P_{i+1} and which converges strongly to $P_{i+1} - Q$. Since Q is a minimal projection, there is an n_0 such that $R_nQ = 0$ whenever $n \geqq n_0$. This means that the open projection $R = \bigvee \{R_n \mid n \geqq n_0\}$ is majorized by $P_{i+1} - Q$. But it is also clear that $P_{i+1} - Q \leqq R$. Hence, we get that $P_{i+1} - Q = R$ and consequently that $Q \in \langle \mathscr{P} \rangle$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (4). If $\mathscr{A}$ is not a *GCR* algebra, then $\mathscr{A}$ has a type III factor representation [24]. This means that there is a minimal projection $Q \in \mathscr{Z}$ such that $\mathscr{B}Q$ is a type III factor. This is impossible by Lemma 1. Hence $\mathscr{A}$ is a *GCR* algebra.

3. Borel structure on the quasi-spectrum. Throughout this section let $\mathscr{A}$ be a C^*-algebra, let $\mathscr{B}$ be the enveloping von Neumann algebra of $\mathscr{A}$, and let $\mathscr{P}$ be the set of open projections of the center $\mathscr{Z}$ of $\mathscr{B}$. The *weak* (resp. *strong*) topology of subalgebras of $\mathscr{B}$ will refer to the weak-operator (resp. strong-operator) topology. If ρ is a representation of $\mathscr{A}$ on a Hilbert space $H(\rho)$, let $\rho^{\sim}$ be the unique extension of ρ to a σ-weakly continuous representation of $\mathscr{B}$ on $H(\rho)$ so that the weak closure of $\rho(\mathscr{A})$ is equal to $\rho^{\sim}(\mathscr{B})$ [12, 12.1.5]. If ρ is nondegenerate (i.e., the identity of $H(\rho)$ lies in the weak closure of $\rho(\mathscr{A})$), then $\rho(\mathscr{B})$ is the von Neumann algebra generated by $\rho(\mathscr{A})$ [14, I, § 3, Theorem 2].

Now two nondegenerate representations ρ_1 and ρ_2 of $\mathscr{A}$ are said to be *quasi-equivalent* (notation: $\rho_1 \sim \rho_2$) if $\rho_1^{\sim}$ and $\rho_2^{\sim}$ have the same kernel. The relation of quasi-equivalence partitions the set of (nondegenerate) representations of $\mathscr{A}$ into quasi-equivalence classes. The class containing ρ is denoted by $[\rho]$. If $\rho_1 \in [\rho]$, then $\ker \rho = \ker \rho_1$ and thus for every class $[\rho]$, there is a uniquely associated ideal $\ker [\rho] = \ker \rho$ of $\mathscr{A}$. Furthermore, if ρ is a *factor representation* of $\mathscr{A}$ (i.e., $\rho^{\sim}(\mathscr{B})$ is a factor von Neumann algebra), then so is every ρ_1 in the class $[\rho]$ (cf. [12, § 5]).

Let $\hat{\mathscr{A}}$ be the set of all quasi-equivalence classes of factor representations. The set $\hat{\mathscr{A}}$ is called the *quasi-spectrum* of $\mathscr{A}$. If $A \in \mathscr{Z}$ and $\tau \in \hat{\mathscr{A}}$, then there is a unique scalar $A^{\sim}(\tau)$ such that $\rho^{\sim}(A) = A^{\sim}(\tau)1_{H(\rho)}$ for every $\rho \in \tau$. Here $1_{H(\rho)}$ is the identity operator on $H(\rho)$. So every $A \in \mathscr{Z}$ defines a complex-valued function $A^{\sim}$ on $\hat{\mathscr{A}}$ (cf. [7, § 4]). Now it is clear that the map $A \rightarrow A^{\sim}$ is a bounded *-homomorphism of $\mathscr{Z}$ into the C^*-algebra $F(\hat{\mathscr{A}})$ of bounded complex-valued functions on $\hat{\mathscr{A}}$. For each $\tau \in \hat{\mathscr{A}}$ there is a unique minimal projection of the algebra $\mathscr{Z}$ such that $Q^{\sim}(\tau) = 1$. Conversely, if Q is a minimal projection of $\mathscr{Z}$, there is a unique $\tau \in \hat{\mathscr{A}}$ such that $Q^{\sim}(\tau) = 1$. Thus there is a one-to-one map of the set of minimal projections of $\mathscr{Z}$ onto $\hat{\mathscr{A}}$. Therefore, if P_m denotes the least upper bound of all minimal projections in $\mathscr{Z}$, then $P_m^{\sim} = 1$. Furthermore, if $\mathscr{J}$ is an ideal of $\mathscr{A}$ and $P \in \mathscr{P}$ is such that $\mathscr{B}P$ is the strong closure of $\mathscr{J}$, then

$$\{\tau \in \hat{\mathscr{A}} \mid \ker \tau \not\supset \mathscr{J}\} = \{\tau \in \hat{\mathscr{A}} \mid P^{\sim}(\tau) = 1\}\,. \tag{1}$$

Now let $\tau \in \hat{\mathscr{A}}$. The ideal $\ker \tau$ is a *prime* ideal in the sense that $\ker \tau$ contains the intersection of two ideals $\mathscr{I}$ and $\mathscr{J}$ in $\mathscr{A}$ if and only if it contains one of them. Indeed, if $\rho \in [\tau]$ and $\rho(\mathscr{I}) \neq (0)$, then the strong closure of $\rho^\sim(\mathscr{I})$ is $\rho^\sim(\mathscr{B})$; otherwise, $\rho^\sim(\mathscr{B})$ would have a nontrivial center (cf. [11]). There is a net $\{A_n\}$ in $\mathscr{I}$ with $\lim \rho^\sim(A_n) = 1$ (strongly). Hence, for any $A \in \mathscr{J}$, we have that

$$\rho(A) = \rho^\sim(A) = \lim \rho^\sim(AA_n) = \lim \rho(AA_n) = 0 .$$

This means $\rho(\mathscr{J}) = (0)$. Thus $\ker \tau$ is a prime ideal. For any nonvoid subset X of $\hat{\mathscr{A}}$, we let $\mathscr{I}(X) = \cap \{\ker \tau \mid \tau \in X\}$ and we let

$$X^- = (\tau \in \hat{\mathscr{A}} \mid \ker \tau \supset I(X)\} .$$

Setting $\varnothing^- = \varnothing$, we get a unique topology on $\hat{\mathscr{A}}$, called the *hull-kernel topology*, such that the closure of a subset X of $\hat{\mathscr{A}}$ is X^- (cf. [12, 3.1]). The hull-kernel topology on $\hat{\mathscr{A}}$ generates a Borel structure $S(\hat{\mathscr{A}})$ on $\hat{\mathscr{A}}$.

Thus the construction of the hull-kernel topology for the quasi-spectrum is analogous to that of the hull-kernel topology of the spectrum. We shall see further parallels in Propositions 3 and 9. However, the greater size of the quasi-spectrum allows us to prove Theorem 11.

PROPOSITION 3. *Let $\mathscr{A}$ be a C^*-algebra, let $\mathscr{B}$ be the enveloping von Neumann algebra of $\mathscr{A}$, let $\mathscr{Z}$ be the center of $\mathscr{B}$, and let P_m be the least upper bound of all minimal projections in $\mathscr{Z}$. Let $\mathscr{C}$ the weak (-operator) sequential closure of the *-subalgebra of $\mathscr{Z}P_m$ generated by $\mathscr{P}P_m$. Then $\mathscr{C}$ is the C^*-algebra generated by $\langle\!\langle \mathscr{P} \rangle\!\rangle P_m$. Also there is an isomorphism λ of $\mathscr{C}$ onto the C^*-algebra $B(\hat{\mathscr{A}})$ of bounded $S(\hat{\mathscr{A}})$-Borel functions on the quasi-spectrum $\hat{\mathscr{A}}$ of $\mathscr{A}$ such that the image of $\langle\!\langle \mathscr{P} \rangle\!\rangle P_m$ is the set of all characteristic functions in $B(\hat{\mathscr{A}})$. Furthermore, the map λ is bi-continuous in the sense that $\{\lambda(C_n)\}$ converges pointwise to $\lambda(C)$ if and only if $\{C_n\}$ is a sequence in $\mathscr{C}$ that converges weakly to C.*

REMARK. On $\mathscr{Z}P_m$ the notions of strong and weak sequential convergence coincide.

Proof. The restriction λ of $A \to A^\sim$ to $\mathscr{C}$ is a *-homomorphism of $\mathscr{C}$ into $F(\hat{\mathscr{A}})$. If $\{C_n\}$ is a sequence in $\mathscr{C}$ that converges weakly to C, then $\{C_nQ\}$ converges uniformly to CQ for each minimal projection Q of $\mathscr{Z}$ and so $\lim \lambda(C_n) = \lambda(C)$ in the topology of pointwise convergence of $F(\hat{\mathscr{A}})$. Hence λ is continuous. If $\lambda(C) = 0$ for some $C \in \mathscr{C}$, then $C^\sim(\tau) = 0$ for all $\tau \in \hat{\mathscr{A}}$ and so $CQ = 0$ for all minimal projections

Q. This means $C = CP_m = 0$ and so λ is an isomorphism. Clearly, the inverse is continuous. We also have that

$$(2) \qquad \begin{aligned} \|\lambda(C)\| &= \text{lub}\{|\lambda(C)(\tau)| \mid \tau \in \hat{\mathscr{A}}\} \\ &= \text{lub}\{\|CQ\| \mid Q \text{ minimal}\} = \|C\| , \end{aligned}$$

for every $C \in \mathscr{C}$. Furthermore, the image of $\mathscr{P}P_m$ under λ is the set of all characteristic functions of open subsets of $\hat{\mathscr{A}}$ by relation (1). Hence λ maps the *-algebra generated by $\mathscr{P}P_m$ into $B(\hat{\mathscr{A}})$. By the continuity of λ and the norm preserving property (2), the map λ takes $\mathscr{C}$ into $B(\hat{\mathscr{A}})$.

Now we show that $\lambda(\mathscr{C})$ is sequentially closed in $B(\hat{\mathscr{A}})$. Let $\{C_n\}$ be a sequence in $\mathscr{C}$ such that $\{\lambda(C_n)\}$ converges pointwise to a function $f \in B(\hat{\mathscr{A}})$. Since λ is a *-isomorphism, we may assume that f and each C_n is self-adjoint. Now if C and D are self-adjoint in $\mathscr{C}$ there is a projection P in $\mathscr{C}$ with $PC + (1 - P)D = C \vee D$ in the lattice of self-adjoint elements in $\mathscr{Z}P_m$. In fact, the spectral projections $\{E(\alpha)\}$ and $\{F(\alpha)\}$ of C and D respectively are in $\mathscr{C}$. For example, let α be given and let g_n be the function of a real variable given by $g_n(t) = 0$ if $t \geqq \alpha$, $g_n(t) = 1$ if $t \leqq \alpha - n^{-1}$, and g_n linear on $[\alpha - n^{-1}, \alpha]$. Then $\{g_n(C)\}$ is a monotonally increasing sequence in $\mathscr{C}$ whose least upper bound is $E(\alpha)$. Let $\{r_n\}$ be an enumeration of the rationals. Then P is the least upper bound of the sequence of projections $\{F(r_m)(1 - E(r_n)) \mid r_m < r_n;\ n, m = 1, 2, \cdots\}$. Indeed, if Q is a minimal projection with $Q \leqq P$, then $Q \leqq F(r_m)(1 - E(r_n))$ for some $r_m < r_n$. This means that $Q \leqq F(r_m)$ and $Q \leqq 1 - E(r_n)$, and thus that $DQ \leqq r_m Q < r_n Q \leqq CQ$. Conversely, let Q be a minimal projection with $DQ < CQ$. Then there are r_m and r_n with $DQ < r_m Q < r_n Q < CQ$. This means that $Q \leqq F(r_m)(1 - E(r_n))$. Since P_m is the least upper bound of minimal projections, the projection P satisfies the requirements. We notice that $\lambda(C \vee D) = \lambda(CP) + \lambda((1 - P)D) = \lambda(C) \vee \lambda(D)$ since λ preserves order and since $\lambda(P)$ and $\lambda(1 - P)$ are characteristic functions of disjoint sets whose union is $\hat{\mathscr{A}}$. The analogous statements hold for $C \wedge D$. These facts allow us to assume that $\{C_n\}$ is bounded since we may replace each C_n by $C_n \wedge \|f\| P_m$. Now let $D_n = \bigvee \{C_k \mid k \geqq n\}$. We have that D_n lies in $\mathscr{C}$ since D_n is the strong limit of the sequence $\{\bigvee \{C_k \mid p \geqq k \geqq n\}\}$ in $\mathscr{C}$. Since λ is continuous, we get that

$$\lambda(D_n) = \lim \lambda(\bigvee \{C_k \mid p \geqq k \geqq n\}) = \bigvee \{\lambda(C_k) \mid k \geqq n\} .$$

By the same reasoning we get that

$$\lambda(\bigwedge D_n) = \bigwedge_n \bigvee \{\lambda(C_k) \mid k \geqq n\} .$$

Now $C = \bigwedge D_n \in \mathscr{C}$ and $f = \lim \lambda(C_k) = \lim\sup \lambda(C_k)$. Hence we have that $f = \lambda(C)$. This proves that λ maps $\mathscr{C}$ onto a sequentially closed subalgebra of $B(\hat{\mathscr{A}})$ containing the characteristic functions of all open sets. Hence $\lambda(\mathscr{C})$ maps onto $B(\hat{\mathscr{A}})$.

We now show $\lambda(\langle\!\langle \mathscr{P} \rangle\!\rangle P_m)$ is the set of all characteristic functions of Borel sets. However, a proof similar to the one we have already given shows that $\lambda(\langle\!\langle \mathscr{P} \rangle\!\rangle P_m)$ is a σ-complete Boolean algebra of characteristic functions. This Boolean algebra contains all characteristic functions of open sets and hence it coincides with the set of characteristic functions in $B(\hat{\mathscr{A}})$.

Finally, we show that $\mathscr{C}$ is the C^*-algebra $\mathscr{C}_0$ generated by $\langle\!\langle \mathscr{P} \rangle\!\rangle P_m$. Let $f \in B(\hat{\mathscr{A}})$ be real-valued and let n be a natural number. Then there is a partition $\{X_k \mid k = 0, \pm 1, \cdots, \pm n\}$ of $\hat{\mathscr{A}}$ into disjoint Borel sets such that each X_k is contained in the set

$$\{\tau \in \hat{\mathscr{A}} \mid kn^{-1} \| f \| \leqq f(\tau) \leqq (k+1)n^{-1} \| f \|\} .$$

If we set $g_k \in B(\hat{\mathscr{A}})$ equal to the characteristic function of X_k for every k, we get $\| \sum \alpha_k g_k - f \| \leqq n^{-1}$ for suitable scalars α_k. Because $\sum \alpha_k g_k \in \lambda(\mathscr{C}_0)$ and because λ is an isometry, we get that $f \in \lambda(\mathscr{C}_0)$. Due to the fact λ is a $*$-isomorphism, we get that $\mathscr{C}_0 = \mathscr{C}$.

For the spectrum of a C^*-algebra we have the following result.

PROPOSITION 4. *Let $\mathscr{A}$ be a C^*-algebra, let $\mathscr{B}$ be the enveloping von Neumann algebra of $\mathscr{A}$, and let $\mathscr{P}$ be the set of open projections of the center $\mathscr{Z}$ of $\mathscr{B}$. Let $\hat{\mathscr{A}}$ be the set of equivalence classes of irreducible representations of $\mathscr{A}$ with the hull-kernel topology. Then there is an isomorphism ϕ of the C^*-algebra $\mathscr{R}$ generated by $\langle\!\langle \mathscr{P} \rangle\!\rangle$ onto the algebra $B(\hat{\mathscr{A}})$ of all bounded complex-valued Borel functions on $\hat{\mathscr{A}}$ such that the image of $\langle\!\langle \mathscr{P} \rangle\!\rangle$ is the set of all characteristic functions in $B(\hat{\mathscr{A}})$. Furthermore, ϕ is continuous in the sense that $\{\phi(A_n)\}$ converges to $\phi(A)$ whenever $\{A_n\}$ is a sequence in $\mathscr{R}$ that converges strongly to A in $\mathscr{R}$.*

Proof. Let P_0 be the least upper bound of all minimal projections Q in $\mathscr{Z}$ such that $\mathscr{B}Q$ is type I. There is an isomorphism ψ of the smallest weakly sequentially closed $*$-subalgebra $\mathscr{D}$ of $\mathscr{Z}P_0$ containing $\langle\!\langle \mathscr{P} \rangle\!\rangle P_0$ onto $B(\hat{\mathscr{A}})$ such that $\langle\!\langle \mathscr{P} \rangle\!\rangle P_0$ maps onto the set of all characteristic functions of $B(\hat{\mathscr{A}})$. Also $\mathscr{D}$ is the C^*-algebra generated by $\langle\!\langle \mathscr{P} \rangle\!\rangle P_0$. This follows in the same way as Proposition 3.

We also have that the map $A \to AP_0$ is a homomorphism of $\mathscr{R}$ onto $\mathscr{D}$. Setting $\phi(A) = \psi(AP_0)$, we obtain a homomorphism of $\mathscr{R}$ onto $B(\hat{\mathscr{A}})$ that is continuous in the specified sense.

We show that ϕ is an isomorphism. There is a projection-valued operator γ defined on the Borel sets $S(\hat{\mathscr{A}})$ of $\hat{\mathscr{A}}$ such that $\gamma(\{\tau \in \hat{\mathscr{A}} \mid P^{\sim}(\tau) = 1\}) = P$ for every open projection P ([19, Theorem 1.9], cf. [12, 5.7.6]). Identifying the characteristic functions of $B(\hat{\mathscr{A}})$ with their supports, we get that $\gamma \cdot \psi(PP_0) = P$ for every $P \in \mathscr{P}$ and so $\gamma \cdot \phi(P) = P$ for every $P \in \mathscr{P}$. This means that $\gamma \cdot \phi(P) = P$ for every $P \in \langle\!\langle \mathscr{P} \rangle\!\rangle$. Now suppose $\phi(A) = 0$ for some $A \in \mathscr{R}$. Given $\varepsilon > 0$, there exist orthogonal projections $P_1, \cdots, P_n$ in $\langle\!\langle \mathscr{P} \rangle\!\rangle$ and positive scalars $\alpha_1, \cdots, \alpha_n$ such that $\| \sum \alpha_i P_i - A^*A \| < \varepsilon$. This means that $\| \sum \alpha_i \phi(P_i) \| < \varepsilon$. Since the $\phi(P_i)$ are disjoint characteristic functions, we have that $\phi(P_i) = 0$ for every i with $\alpha_i \geqq \varepsilon$. This means $P_i = \gamma \cdot \phi(P_i) = 0$ for all such i. Hence we have that $\| \sum \alpha_i P_i \| < \varepsilon$ and so that $\| A \|^2 = \| A^*A \| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have that $A = 0$. Hence ϕ is an isomorphism.

COROLLARY 5. *Let $\mathscr{A}$ be a C^*-algebra, let $\hat{\mathscr{A}}$ be the spectrum of $\mathscr{A}$, and let $\tilde{\mathscr{A}}$ be the quasi-spectrum of $\mathscr{A}$. Suppose that both $\hat{\mathscr{A}}$ and $\tilde{\mathscr{A}}$ have the hull-kernel topology. Then there is a pointwise continuous isomorphism of the algebra $B(\tilde{\mathscr{A}})$ of bounded Borel functions on $\tilde{\mathscr{A}}$ onto the algebra $B(\hat{\mathscr{A}})$ of bounded Borel functions on $\hat{\mathscr{A}}$.*

Proof. Let $\mathscr{P}$ be the set of open projections in the center $\mathscr{Z}$ of the enveloping von Neumann algebra $\mathscr{B}$ of $\mathscr{A}$. Let P_0 be the least upper bound of all minimal projections Q in $\mathscr{Z}$ such that $\mathscr{B}Q$ is type I and let P_m be the least upper bound of all minimal projections in $\mathscr{Z}$. Then the C^*-algebra $\mathscr{D}$ generated by $\langle\!\langle \mathscr{P} \rangle\!\rangle P_0$ is isomorphic to $B(\hat{\mathscr{A}})$ under a bi-continuous map for the strong and the pointwise topology (Proposition 4), and the C^*-algebra $\mathscr{C}$ generated by $\langle\!\langle \mathscr{P} \rangle\!\rangle P_m$ is isomorphic to $B(\tilde{\mathscr{A}})$ under a bi-continuous map for the strong and pointwise topology (Proposition 3). But the C^*-algebra $\mathscr{R}$ generated by $\langle\!\langle \mathscr{P} \rangle\!\rangle$ is isomorphic to $\mathscr{D}$ under the map $A \to AP_0$. Hence the map $A \to AP_0$ is an isomorphism of $\mathscr{C}$ onto $\mathscr{D}$. This isomorphism is certainly strongly continuous. Hence, there is a pointwise continuous isomorphism of $B(\tilde{\mathscr{A}})$ onto $B(\hat{\mathscr{A}})$.

REMARK. The set of bounded continuous complex-valued functions on $\hat{\mathscr{A}}$ has been described recently ([5], [13]). Due to the fact that $\hat{\mathscr{A}}$ need not be separated, the continuous functions do not approximate the Borel functions.

We describe a class of elements that lie in C^*-algebra $\mathscr{R}$ generated by $\langle\!\langle \mathscr{P} \rangle\!\rangle$. Let Z be the spectrum of $\mathscr{Z}$. For every $A \in \mathscr{B}$ and ζ in Z, let $A(\zeta)$ denote the image of A under the canonical map of $\mathscr{B}$ onto the algebra $\mathscr{B}$ reduced modulo the ideal generated by ζ. There is an element $\psi(A) \in \mathscr{Z}$ such that $\psi(A)^\wedge(\zeta) = \|A(\zeta)\|$ for all $\zeta \in Z$. Here $\psi(A)^\wedge(\zeta)$ is the Gelfand transform of $\psi(A)$ evaluated at ζ [18, Lemma 10].

PROPOSITION 6. *Let $\mathscr{A}$ be a C^*-algebra, let $\mathscr{B}$ be its enveloping von Neumann algebra, let $\mathscr{P}$ be the set of open projections of the center $\mathscr{Z}$ of $\mathscr{B}$, and let $\mathscr{E}$ be the uniformly closed $*$-subalgebra of $\mathscr{Z}$ generated by $\mathscr{P}$. Then, for every $A \in \mathscr{A}$, the element $\psi(A)$ lies in $\mathscr{E}$.*

Proof. Since $\mathscr{E}$ is a C^*-algebra and since $\psi(A) = \psi(A^*A)^{1/2}$, it is sufficient to show $\psi(A) \in \mathscr{E}$ for every A in $\mathscr{A}^+$. We have that there is a projection P in $\mathscr{Z}$ such that

$$\{\zeta \in Z \mid P^\wedge(\zeta) = 1\} = \text{clos}\,\{\zeta \in Z \mid \psi(A)^\wedge(\zeta) > 0\}$$

since Z is extremally disconnected. But it is clear that P is an open projection since $\mathscr{B}P$ is the strong closure of the principal ideal generated by A. Now, for any $\alpha > 0$, let f_α be the continuous function of a real-variable given by $f_\alpha(t) = 0$ if $t \leqq \alpha$ and $f_\alpha(t) = t - \alpha$ for $t > \alpha$. Then there is an open projection P with

$$\{\zeta \in Z \mid P^\wedge(\zeta) = 1\} = \text{clos}\,\{\zeta \in Z \mid \psi(f_\alpha(A))^\wedge(\zeta) > 0\}$$

and so

$$\{\zeta \in Z \mid P^\wedge(\zeta) = 1\} = \text{clos}\,\{\zeta \in Z \mid \psi(A)^\wedge(\zeta) > \alpha\}\,.$$

Now let n be a natural number. Let $P_k (k = 0, 1, \cdots, n-1)$ be the open projections given by

$$\{\zeta \in Z \mid P_k^\wedge(\zeta) = 1\} = \text{clos}\,\{\zeta \in Z \mid \psi(A)^\wedge(\zeta) > n^{-1}k\,\|A\|\}\,.$$

Let $Q_k = P_{k-1} - P_k$ for $1 \leqq k \leqq n-1$ and $Q_n = P_{n-1}$. Then we have that

$$\begin{aligned}&\|\psi(A) - \textstyle\sum n^{-1}k\,\|A\|\,Q_k\,\| \\ &\quad = \text{lub}\,\{|\,\psi(A)^\wedge(\zeta) - \textstyle\sum n^{-1}k\,\|A\|\,Q(\zeta)\,|\ \mid \zeta \in Z\} \leqq n^{-1}\,\|A\|\,.\end{aligned}$$

Hence, the element $\psi(A)$ is in $\mathscr{E}$.

For a separable C^*-algebra, we have a better result. We preserve the same notation as the preceding proposition.

COROLLARY 7. *Let $\mathscr{A}$ be a separable C^*-algebra, then the C^*-algebra $\mathscr{R}$ in $\mathscr{Z}$ generated by $\langle\!\langle \mathscr{P} \rangle\!\rangle$ is equal to the weak sequential closure of the C^*-algebra generated by $\{\psi(A) \mid A \in \mathscr{A}\}$.*

Proof. Let $P \in \mathscr{P}$ and let $\mathscr{J}$ be an ideal in $\mathscr{A}$ whose strong closure is $\mathscr{B}P$. The ideal $\mathscr{J}$ is a principal ideal generated by an element A of $\mathscr{A}$ [23, 6.5, Corollary]. This means that P is smallest projection in $\mathscr{Z}$ with $P\psi(A) = \psi(A)$. Hence P is in the weak sequential closure $\mathscr{R}_0$ of the C^*-algebra generated by $\psi(\mathscr{A})$. This proves that $\langle\langle \mathscr{P} \rangle\rangle$ and thus $\mathscr{R}$ is contained in $\mathscr{R}_0$.

Conversely, each element $\psi(A)$ is contained in $\mathscr{R}$ (Proposition 6). Let P_0 be the least upper bound of all projections Q in $\mathscr{A}$ such that $\mathscr{B}Q$ is a type I factor. The map $A \to AP_0$ of the weak sequential closure $\mathscr{A}^{\sim}$ of $\mathscr{A}$ in $\mathscr{B}$ is a weak sequentially continuous isomorphism onto the weak sequential closure of $\mathscr{A}P_0$ [6, Theorem 3.10]. Since $\mathscr{R}_0 \subset \mathscr{A}^{\sim}$ and $\mathscr{R} \subset \mathscr{A}^{\sim}$ and since $\mathscr{R}P_0 = \mathscr{D}$ is weakly sequentially closed (cf. Proposition 4), we may find, for each $A \in \mathscr{R}_0$, a $B \in \mathscr{R}$ such that $AP_0 = BP_0$. This means that $A = B$. Hence $\mathscr{R}_0 \subset \mathscr{R}$. Thus we get that $\mathscr{R} = \mathscr{R}_0$.

Now let $\mathscr{A}$ be a separable C^*-algebra and let $\mathscr{A}^{\sim}$ be the weak sequential closure of $\mathscr{A}$ in its enveloping algebra $\mathscr{B}$. The center $\mathscr{Z}(\mathscr{A}^{\sim})$ is contained in the center $\mathscr{Z}$ of $\mathscr{P}$. As is pointed out by E. B. Davies (cf. [6, p. 154] for the analogous statement for $\hat{\mathscr{A}}$) each open projection in $\mathscr{Z}$ is in $\mathscr{Z}(\mathscr{A}^{\sim})$. This means that $B(\hat{\mathscr{A}})$ is contained in the algebra $\{A^{\sim} \mid A \in \mathscr{Z}(\mathscr{A}^{\sim})\} \subset F(\hat{\mathscr{A}})$. Thus the Davies Borel structures on A (i.e., the weakest Borel structure such that all functions $\{A^{\sim} \mid A \in \mathscr{Z}(\mathscr{A}^{\sim})\}$ are Borel on $\hat{\mathscr{A}}$) is finer than the structure $S(\hat{\mathscr{A}})$ induced by the hull-kernel topology. In fact the Davies Borel structure separates points whereas the Borel structure $S(\hat{\mathscr{A}})$ does not in certain cases (for example, a separable uniformly hyperfinite C^*-algebra). The C^*-algebra $\mathscr{R}$ generated by the Boolean σ-algebra $\langle\langle \mathscr{P} \rangle\rangle$ is contained in $\mathscr{Z}(\mathscr{A}^{\sim})$. In order that $\mathscr{Z}(\mathscr{A}^{\sim}) = \mathscr{R}$, a necessary and sufficient condition is that the Davies and hull-kernel Borel structure on $\hat{\mathscr{A}}$ coincide. Now, if $\mathscr{A}$ is a GCR algebra, then all the Borel structures on $\hat{\mathscr{A}}$ coincide [12, 3.8.3] and so $\mathscr{Z}(\mathscr{A}^{\sim}) = \mathscr{R}$. We note that a special case of this result is mentioned by Glimm [19, p. 899]. Conversely, if the Davies and the hull-kernel Borel structure coincide on $\hat{\mathscr{A}}$, then $\mathscr{A}$ is GCR. Indeed, it is sufficient to show that two irreducible representations ρ_1 and ρ_2 with the same kernels are equivalent [20]. It is this result, which is unavailable in the nonseparable case, that Digernes [9] used to characterize a separable GCR algebra. We have that $P^{\sim}([\rho_1]) = P^{\sim}([\rho_2])$ for every open projection P in $\mathscr{Z}$. Indeed, if $\mathscr{J}$ is an ideal in $\mathscr{A}$ whose strong closure is $\mathscr{B}P$, then $P^{\sim}([\rho_i]) = 0$ if and only if $\mathscr{J}$ is contained in the kernel of ρ_i. But this means that $P^{\sim}([\rho_1]) = P^{\sim}([\rho_2])$ for all P in $\langle\langle \mathscr{P} \rangle\rangle$ and thus the Davies Borel structure fails to separate $[\rho_1]$ and $[\rho_2]$. This implies that $[\rho_1] = [\rho_2]$ [8, Theorem 2.9]. Hence the algebra $\mathscr{A}$ is GCR. It is to be noted that Effros [15] proved that A is GCR if and only if the Mackey and

Davies Borel structure coincides on $\hat{\mathscr{A}}$.

We now examine the hull-kernel topology of the quasi-spectrum more closely. We show that this topology is induced by the canonical mapping of the factor states into the quasi-spectrum.

Let $\mathscr{A}$ be a C^*-algebra and let f be a state of $\mathscr{A}$. Let $L(f)$ be left ideal of $\mathscr{A}$ given by $L(f) = \{A \in \mathscr{A} \mid f(A^*A) = 0\}$, let $H(f)$ be the completion of the residue class $\mathscr{A} - L(f)$ with the inner product $(A - L(f), B - L(f)) = f(B^*A)$, and let ρ_f be the (nondegerate) representations of $\mathscr{A}$ on the Hilbert space $H(f)$ induced by left multiplication of $\mathscr{A}$ on $\mathscr{A} - L(f)$. The representation ρ_f is called the *canonical representation* of $\mathscr{A}$ induced by f. There is a cyclic unit vector x_f under $\rho_f(\mathscr{A})$ for $H(f)$ (equal to $1 - L(f)$ if $\mathscr{A}$ has identity 1 or equal to $\lim A_n - L(f)$ if $\{A_n\}$ is an increasing approximate identity in the positive part of the unit sphere of $\mathscr{A}$ if $\mathscr{A}$ has no identity) such that $\omega_{x_f} \cdot \rho_f(A) = (\rho_f(A)x_f, x_f) = f(A)$ for all $A \in \mathscr{A}$. The state f is called a *factor* (or primary) state if ρ_f is a factor representation of $\mathscr{A}$. Let $\mathscr{F}(\mathscr{A})$ be the space of all factor states of $\mathscr{A}$ with its relativized w^*-topology. We write $f \sim g$ for f, g in $\mathscr{F}(\mathscr{A})$ to denote $\rho_f \sim \rho_g$.

Now suppose that $\mathscr{A}$ is a C^*-algebra without an identity. Then an identity 1 may be adjoined to $\mathscr{A}$ to obtain a C^*-algebra $\mathscr{A}_e$ with identity so that $\mathscr{A}$ is a maximal ideal of $\mathscr{A}_e$ (cf. [12, 1.2.3]). Each state f on $\mathscr{A}$ has a unique extension f_e to a state of $\mathscr{A}_e$ obtained by setting $f_e(1) = 1$. The Hilbert spaces $H(f)$ and $H(f_e)$ can be identified with each other so that ρ_{f_e} restricted to $\mathscr{A}$ is precisely ρ_f. Further more, the identity of $\mathscr{A}_e$ gets carried into the identity operator on $H(f)$ (cf. [12, 2.1.4]). Therefore, the state f_e is a factor state if and only if f is. Furthermore, if f and g are factor states of $\mathscr{A}$, then $f \sim g$ if and only if $f_e \sim g_e$. Now let f_0 be the unique factor state of $\mathscr{A}_e$ that vanishes on $\mathscr{A}$. If f be a factor state of $\mathscr{A}_e$ not equal to f_0, then the ideal $\rho_f(\mathscr{A})$ of $\rho_f(\mathscr{A}_e)$ is nonzero and therefore is strongly dense in $\rho_f(\mathscr{A}_e)$ (cf. [11]). For any $\varepsilon > 0$ there is a net $\{B_n\}$ in $\mathscr{A}$ with $\text{lub} \| B_n \| \leqq 1 + \varepsilon$ such that $\{\rho_f(B_n)\}$ converges strongly to the identity [22]. Hence, the restriction g of f to $\mathscr{A}$ has norm not less than $(1 + \varepsilon)^{-1}$ since

$$\begin{aligned} \| g \| &\geqq (1 + \varepsilon)^{-1} \limsup | g(B_n) | \\ &= (1 + \varepsilon)^{-1} \limsup | (\rho_f(B_n)x_f, x_f) | \geqq (1 + \varepsilon)^{-1} . \end{aligned}$$

Therefore, g is a factor state of $\mathscr{A}$ with $g_e = f$. This means that the map e of $\mathscr{F}(\mathscr{A})$ into $\mathscr{F}(\mathscr{A}_e)$ defined by $e(f) = f_e$ is a one-to-one map of $\mathscr{F}(\mathscr{A})$ onto $\mathscr{F}(\mathscr{A}_e) - \{f_0\} = \mathscr{F}'(\mathscr{A}_e)$.

It is clear that e is a continuous map $\mathscr{F}(\mathscr{A})$ into $\mathscr{F}(\mathscr{A}_e)$. Furthermore, if $\mathscr{V}$ is open in $\mathscr{F}(\mathscr{A})$, then $e(\mathscr{V})$ is relatively open in

$\mathscr{F}'(\mathscr{A}_e)$. Since $\mathscr{F}'(\mathscr{A}_e)$ is open in $\mathscr{F}(\mathscr{A}_e)$, we may conclude that $e(\mathscr{V})$ is open in $\mathscr{F}(\mathscr{A}_e)$. So the map e is also an open map.

We now prove that quasi-equivalence is an open relation in the space $\mathscr{F}(\mathscr{A})$ by showing the *saturation* $\mathscr{X}^\sim$ of an open subset $\mathscr{X}$ of $\mathscr{F}(\mathscr{A})$ given by $\mathscr{X}^\sim = \{f \in \mathscr{F}(A) \mid f \sim g \in \mathscr{X}\}$ is open.

LEMMA 8. *The saturation under the relation of quasi-equivalence of an open subset of the space of factor states of a C^*-algebra is open.*

Proof. Let $\mathscr{V}$ be an open subset of the space $\mathscr{F}(\mathscr{A})$ of factor states of the C^*-algebra $\mathscr{A}$. We assume that $\mathscr{A}$ has an identity, and later we remove this assumption. Let g be a factor state in the saturation $\mathscr{V}^\sim$ of $\mathscr{V}$. We construct a neighborhood $\mathscr{W}$ of g such that $\mathscr{W} \subset \mathscr{V}^\sim$. There is an element $h \in \mathscr{V}$ with $g \sim h$. There are elements $C_1, C_2, \cdots, C_n$ in $\mathscr{A}$ and a δ with $0 < \delta < 1$ such that

$$\{f \in \mathscr{F}(\mathscr{A}) \mid |f(C_i) - h(C_i)| < \delta, i = 1, \cdots, n\}$$

is contained in $\mathscr{V}$. Without loss of generality we may assume that $C_1 = 1$. Due to the fact that $g \sim h$, there is an isomorphism ϕ of the von Neumann algebra $\rho_g(\mathscr{A})''$ generated by $\rho_g(\mathscr{A})$ on $H(g)$ onto the von Neumann algebra $\rho_h(\mathscr{A})''$ generated by $\rho_h(\mathscr{A})$ on $H(h)$ such that $\phi(\rho_g(\mathscr{A})) = \rho_h(\mathscr{A})$ for every $A \in \mathscr{A}$ (cf. [12, § 5]). Since an isomorphism of von Neumann algebras is σ-weakly continuous, [14, I, § 4, Theorem 2, Corollary 1], the functional $\omega_{x_h} \cdot \phi$ is a σ-weakly continuous state of $\rho_g(\mathscr{A})''$ such that $\omega_{x_h} \cdot \phi \cdot \rho_g = h$. This means that there is a sequence $\{x_i\}$ in $H(g)$ such that $\sum \|x_i\|^2 < +\infty$ and such that $\sum \omega_{x_i} = \omega_{x_h} \cdot \phi$ on $\rho_g(\mathscr{A})''$ [14, I, § 3, Theorem 2]. Setting $\eta = \delta(6 \max\{\|C_i\| \mid 1 \leqq i \leqq n\})^{-1}$, we may find a natural number m such that

$$\|\textstyle\sum \{\omega_{x_i} \mid m + 1 \leqq i < +\infty\}\| < \eta \,. \tag{3}$$

Since each x_i lies in the closure of $\rho_g(\mathscr{A})x_g$, there are $A_1, A_2, \cdots, A_m$ in $\mathscr{A}$ such that the vectors $\rho_g(A_i)x_g = y_i$ in $H(g)$ satisfy

$$\|\omega_{x_i} - \omega_{y_i}\| < m^{-1}\eta \tag{4}$$

for $i = 1, \cdots, m$.

Now let $\varepsilon = m^{-1}\eta$. We show every f in the neighborhood $\mathscr{W}$ of g given by

$$\mathscr{W} = \{f \in \mathscr{F}(\mathscr{A}) \mid |f(A_i^* C_j A_i) - g(A_i^* C_j A_i)| < \varepsilon \quad \text{for all } i = 1, \cdots, m; j = 1, \cdots, n\}$$

is contained in $\mathscr{V}^\sim$. Setting f' equal to

$$f'(A) = \sum \{f(A_i^* A A_i) \mid 1 \leqq i \leqq m]$$

for all $A \in \mathscr{A}$, we obtain a positive functional on $\mathscr{A}$ whose norm is given by $\| f' \| = f'(1) = \sum f(A_i^* A_i)$. Because $C_1 = 1$, we get

$$| f'(1) - \sum g(A_i^* A_i) | \leqq \sum | f(A_i^* A_i) - g(A_i^* A_i) | < \eta .$$

But we have that

$$\begin{aligned} | \sum g(A_i^* A_i) - 1 | &= | \sum g(A_i^* A_i) - \sum \omega_{x_i}(1) | \\ &= | \sum \{\omega_{y_i}(1) \mid 1 \leqq i \leqq m\} - \sum \{\omega_{x_i}(1) \mid 1 \leqq i < +\infty\} | \\ &\leqq \sum \{| \| \omega_{y_i} \| - \| \omega_{x_i} \| | \mid 1 \leqq i \leqq m\} \\ &\quad + \| \sum \{\omega_{x_i} \mid m + 1 \leqq i < +\infty\} \| < 2\eta \end{aligned}$$

by relations (3) and (4). This means that

$$(5) \qquad | f'(1) - 1 | < 3\eta < 1 .$$

Hence, we have $f'(1) \neq 0$. Setting $f'' = f' / \| f' \|$, we obtain a state f'' of $\mathscr{A}$ such that $f'' \sim f$ ([4] and [12, 5.3.6]).

We shall now show that $f'' \in \mathscr{V}$. First we have that

$$| f'(C_i) | \leqq f'(1)^{1/2} f'(C_i^* C_i)^{1/2} \leqq f'(1) \| C_i \|$$

for all $i = 1, \cdots, m$. By relation (5) this yields

$$(6) \qquad \begin{aligned} | f'(C_i) - f''(C_i) | &= | 1 - f'(1) | f'(1)^{-1} | f'(C_i) | \\ &\leqq | 1 - f'(1) | \; \| C_i \| < \delta/2 , \end{aligned}$$

for every $i = 1, \cdots, n$. Furthermore, for all i, we get

$$(7) \qquad \begin{aligned} &| f'(C_i) - h(C_i) | \\ &\leqq \sum \{| f(A_j^* C_i A_j) - g(A_j^* C_i A_j) | \; | 1 \leqq j \leqq m\} \\ &\quad + \sum \{| \omega_{y_j}(\rho_g(C_i)) - \omega_{x_j}(\rho_g(C_i)) | \mid 1 \leqq j \leqq m\} \\ &\quad + | \sum \{\omega_{x_j}(\rho_g(C_i)) \mid m + 1 \leqq j < +\infty\} | \\ &< m\varepsilon + \eta \| C_i \| + \eta \| C_i \| \leqq \delta/2 \end{aligned}$$

by relations (3) and (4). Combining (6) and (7), we obtain

$$\begin{aligned} | f''(C_i) - h(C_i) | &\leqq | f''(C_i) - f'(C_i) | \\ &\quad + | f'(C_i) - h(C_i) | < \delta/2 + \delta/2 = \delta , \end{aligned}$$

for all $i = 1, \cdots, n$. This proves that $f'' \in \mathscr{V}$. Hence, the lemma is true for C^*-algebras with identity.

Suppose $\mathscr{A}$ is a C^*-algebra without identity. Let $\mathscr{A}_e$ be the C^*-algebra obtained from $\mathscr{A}$ by adjoining the identity. We use the notation developed in the paragraph preceding this lemma. If $\mathscr{V}$ is an open subset of $\mathscr{F}(\mathscr{A})$, then $e(\mathscr{V})$ is open in $\mathscr{F}(\mathscr{A}_e)$. But the saturation $e(\mathscr{V})^{\sim}$ of $e(\mathscr{V})$ in $\mathscr{F}(\mathscr{A}_e)$ is $e(\mathscr{V}^{\sim})$. By the first part of the proof $e(\mathscr{V})^{\sim}$ is open. Thus the set $\mathscr{V}^{\sim} = e^{-1}(e(\mathscr{V})^{\sim}) = e^{-1}(e(\mathscr{V}^{\sim}))$ is open in $\mathscr{F}(\mathscr{A})$.

PROPOSITION 9. *Let $\mathscr{A}$ be a C^*-algebra. The map $f \rightarrow [\rho_f]$ is a continuous open mapping of the space $\mathscr{F}(\mathscr{A})$ of factor states of $\mathscr{A}$ onto the quasi-spectrum $\hat{\mathscr{A}}$ of $\mathscr{A}$ with its hull-kernel topology.*

Proof. Let ϕ denote the map $f \rightarrow [\rho_f]$. Let ρ be any nondegenerate factor representation of $\mathscr{A}$ on a Hilbert space H. There is a unit vector $x \in H$ such that $f(A) = (\rho(A)x, x)$ is a state of $\mathscr{A}$. There is an isometric isomorphism U of $H(f)$ onto the invariant subspace $K =$ closure $\rho(\mathscr{A})x$ of H defined by $U(A - L(f)) = \rho(A)x$ that carries ρ_f onto the subrepresentation $\rho \restriction K$ of ρ. Since $[\rho \restriction K] = [\rho]$ [12, 5.3.5], we get that $[\rho_f] = [\rho]$. Hence, the image of ϕ is equal to $\hat{\mathscr{A}}$.

Now let $\{f_n\}$ be a net in $\mathscr{F}(\mathscr{A})$ that converges to f in the w^*-topology. Let X be an open subset of $\hat{\mathscr{A}}$ containing $[\rho_f]$. There is an ideal $\mathscr{J}$ in $\mathscr{A}$ with $X = \{\tau \in \hat{\mathscr{A}} \mid \ker \tau \not\supset \mathscr{J}\}$. This mean there is an $A \in \mathscr{J}$ such that $f(A) \neq 0$. There is an n_0 such that $f_n(A) \neq 0$ whenever $n \geqq n_0$. Hence, the classes $[\rho_{f_n}]$ are in X whenever $n \geqq n_0$. This means $\{[\rho_{f_n}]\}$ converges to $[\rho_f]$. Thus ϕ is continuous.

For the proof that ϕ is an open map, we consider two cases: (1) $\mathscr{A}$ has an identity, and (2) $\mathscr{A}$ has no identity. First assume $\mathscr{A}$ has an identity. Let $\mathscr{V}$ be an open subset of $\mathscr{F}(\mathscr{A})$. We prove $\phi(\mathscr{V})$ open in $\hat{\mathscr{A}}$. By Lemma 8, we may assume that $\mathscr{V}$ is saturated. The complement $\mathscr{W}$ of $\mathscr{V}$ in $\mathscr{F}(\mathscr{A})$ is also saturated. It is sufficient to show that $\phi(\mathscr{W})$ is closed in $\hat{\mathscr{A}}$ since $\phi(\mathscr{W}) = \hat{\mathscr{A}} - \phi(\mathscr{V})$. In fact, we shall show that $\phi(\mathscr{W}) = \{\tau \in \hat{\mathscr{A}} \mid \ker \tau \supset \mathscr{J}\}$, where $\mathscr{J} = \bigcap \{\ker \rho_f \mid f \in \mathscr{W}\}$. First it is clear that $\phi(\mathscr{W}) \subset \{\tau \in \hat{\mathscr{A}} \mid \ker \tau \supset \mathscr{J}\}$. Conversely, let f be a pure state in $\mathscr{F}(\mathscr{A})$ with $\ker \rho_f \supset \mathscr{J}$. Then there is a net $\{f_i\}$ in $\mathscr{W}$ and unit vectors $x_i \in H(f_i)$ for each i such that $f = \lim \omega_{x_i} \cdot \rho_{f_i}$ in the ω^*-topology ([16], cf. [12, 3.4.2 (ii)]). However, each state $g_i = \omega_{x_i} \cdot \rho_{f_i}$ is a factor state of $\mathscr{A}$ and is thus quasi-equivalent to f_i ([4] and [12, 5.3.5]). This means that $g_i \in \mathscr{W}$, and therefore, that the limit f of the net $\{g_i\}$ is in $\mathscr{W}$. Hence the set $\phi(\mathscr{W})$ contains $[\rho_f]$ whenever f is a pure state with $\ker \rho_f \supset \mathscr{J}$. Now let f be an arbitrary factor state of $\mathscr{A}$ with $\ker \rho_f \supset \mathscr{J}$. Then we have that $\mathscr{I} = \ker \rho_f$ is a prime ideal containing $\mathscr{J}$ (cf. introductory paragraphs of § 3). Let g be the state of the C^*-algebra $\mathscr{A}/\mathscr{I} = \mathscr{C}$ given by $g(A + \mathscr{I}) = f(A)$. Let $\mathscr{K}'$ be the maximal GCR ideal of $\mathscr{C}$. First we assume that $\mathscr{K}' = (0)$, i.e., $\mathscr{C}$ is an $NGCR$ algebra. Then the state space and the pure state space of $\mathscr{C}$ coincide [25, Theorem 2]. There is a net $\{g_i\}$ of pure states of $\mathscr{C}$ that converges in the w^*-topology to g. Setting $f_i(A) = g_i(A + \mathscr{J})$ for all $A \in \mathscr{A}$, we get a net $\{f_i\}$ of pure states in $\mathscr{A}$ that converges to f in the w^*-topology. Since each $f_i \in \mathscr{W}$ by the first part of the proof, we get $f \in \mathscr{W}$ and thus $[\rho_f] \in \phi(\mathscr{W})$. Now let $\mathscr{K}' \neq (0)$. We then have that the representation ρ_g of $\mathscr{C}$ is quasi-equivalent to an irreducible representation. Indeed, we have that $\rho_g(\mathscr{K}')x_g$ is dense in $H(g)$ since ρ_g is a factor representation of $\mathscr{C}$. But the von Neumann algebra $\rho_g(\mathscr{K}')''$ generated by $\rho_g(\mathscr{K}')$ on $H(g)$ is a type I algebra (cf. [12, 5.5.2]). This means that $\rho_g(\mathscr{K}')''$ has a nonzero abelian projection E. However, the projection E is also an abelian projection for the von Neumann algebra generated by $\rho_g(\mathscr{A}/\mathscr{J})$. Hence ρ_g is quasi-equivalent to an irreducible representation (cf. [12, 5.4.11]). Since the representation ρ of $\mathscr{A}$ defined by $\rho(A) = \rho_g(A + \mathscr{J})$ is unitarily equivalent to ρ_f, we see that ρ_f is quasi-equivalent to an irreducible representation. So there is a pure state h of $\mathscr{A}$ such that $h \sim f$. This means that $[\rho_f] = [\rho_h]$ is in $\phi(\mathscr{W})$. This completes the proof that $\phi(\mathscr{W})$ is closed. Hence, the map ϕ is an open map.

Now suppose that $\mathscr{A}$ does not have an identity. Let $\mathscr{A}_e$ be the C^*-algebra obtained from $\mathscr{A}$ by the adjunction of the identity. Let ϕ' be the map of $\mathscr{F}(\mathscr{A}_e)$ onto $\hat{\mathscr{A}_e}$ given by $\phi'(f) = [\rho_f]$. Let $\mathscr{V}$ be open in $\mathscr{F}(\mathscr{A})$. By using Lemma 8, we may assume that $\mathscr{V}$ is saturated. We have that $e(\mathscr{V})$ is an open saturated set in $\mathscr{F}(\mathscr{A}_e)$, whose image $\phi'(e(\mathscr{V}))$ is an open subset in $\hat{\mathscr{A}_e}$. There is an ideal $\mathscr{J}$ in $\mathscr{A}_e$ with $\phi'(e(\mathscr{V})) = \{\tau \in \hat{\mathscr{A}_e} \mid \ker \tau \not\supset \mathscr{J}\}$. We show that $\phi(\mathscr{V}) = \{\tau \in \hat{\mathscr{A}} \mid \ker \tau \not\supset \mathscr{J} \cap \mathscr{A}\}$. Indeed, let $f \in \mathscr{F}(\mathscr{A})$ and let $e(f) = g$. If $f \in \mathscr{V}$, then $\ker \rho_g \not\supset \mathscr{J}$ and so there is an $A \in \mathscr{J}$ with $g(A) \neq 0$. If $\{A_n\}$ is an increasing approximate identity in the unit sphere of $\mathscr{A}$, we have that $\lim f(A_nA) = \lim g(A_nA) = g(A)$ because $A_nA \in \mathscr{J} \cap \mathscr{A}$ for all n. This means that $\ker [\rho_f] \not\supset \mathscr{J} \cap \mathscr{A}$. Conversely, if $\ker [\rho_f] \not\supset \mathscr{J} \cap \mathscr{A}$, then $f(\mathscr{J} \cap \mathscr{A}) \neq 0$ and so $\ker [\rho_g] \not\supset \mathscr{J}$. There is an $h \in \mathscr{V}$ such that $e(h) \sim g$. This implies that $h \sim f$ and $[\rho_f] \in \phi(\mathscr{V})$. So $\phi(\mathscr{V}) = \{\tau \in \hat{\mathscr{A}} \mid \ker \tau \not\supset \mathscr{J} \cap \mathscr{A}\}$.

We can interpret Proposition 9 in terms of representations. An infinite dimensional Hilbert space H is said to have *sufficiently high dimension* for the factor states of $\mathscr{A}$, if there is a faithful representation ρ_0 of $\mathscr{A}$ on H such that, for any factor state f of $\mathscr{A}$, there is a unit vector $x \in H$ with $f = \omega_x \cdot \rho_0$. Now let H be a Hilbert space of sufficiently high dimension. (If $\mathscr{A}$ is separable, any infinite dimensional space has sufficiently high dimension.) Let CFac $(\mathscr{A}, H)$ be the family of all representations ρ on H for which there is a unit vector $x \in H$ such that $f = \omega_x \cdot \rho$ is a factor state and such that ρ vanishes on the orthogonal complement of the closure of the linear manifold

$\rho(\mathscr{A})x$. A topology may be defined on CFac $(\mathscr{A}, H)$ by allowing a net $\{\rho_n\}$ converge to ρ if and only if $\{\rho_n(A)\}$ converges to $\rho(A)$ in the strong topology on H for every $A \in \mathscr{A}$.

PROPOSITION 10. *Let $\mathscr{A}$ be a C^*-algebra, let H be a Hilbert space of sufficiently high dimension for the factor representations of $\mathscr{A}$. Let ψ be the map that carries each $\rho \in$ CFac $(\mathscr{A}, H)$ into its class $[\rho]$ in $\tilde{\mathscr{A}}$. Then ψ is a continuous open map of* CFac $(\mathscr{A}, H)$ *onto* $\tilde{\mathscr{A}}$.

Proof. It is clear that ϕ maps CFac $(\mathscr{A}, H)$ continuously onto $\tilde{\mathscr{A}}$.

We show that ψ is an open mapping. Let $\mathscr{U}$ be an open subset of CFac $(\mathscr{A}, H)$. Using virtually the same proof as K. Bichteler [3, Proposition 2.4(i)], we can find an open subset $\mathscr{V}$ of $\mathscr{F}(\mathscr{A})$ such that $\psi(\mathscr{U}) = \phi(\mathscr{V})$. However, we have shown that $\phi(\mathscr{V})$ is open in $\tilde{\mathscr{A}}$ (Proposition 9). Thus $\psi(\mathscr{U})$ is open in $\tilde{\mathscr{A}}$ and ψ is an open map.

REMARK. An infinite dimensional Hilbert space K is said to have sufficiently high dimension for the irreducible representations of $\mathscr{A}$ if there is a faithful representation ρ_0 of $\mathscr{A}$ on K such that, for every pure state f of $\mathscr{A}$, there is a unit vector $x \in K$ for which $f = \omega_x \cdot \rho_0$. A space H that has sufficiently high dimension for the factor representations certainly has sufficiently high dimension for the irreducible representations. Then let K have sufficiently high dimension for the irreducible representations. Let Irr $(\mathscr{A}, K)$ be the family of all representations ρ of $\mathscr{A}$ on K for which there is a unit vector x in K such that $\omega_x \cdot \rho$ is a pure state and ρ vanishes on the orthogonal complement of the closure of $\rho(\mathscr{A})x$. Then L. T. Gardner [17] proved $\rho \rightarrow [\rho]$ is a continuous open map of Irr $(\mathscr{A}, K)$ onto the spectrum of $\mathscr{A}$ (with the hull-kernel topology). Notice that Irr $(\mathscr{A}, H) \subset$ CFac $(\mathscr{A}, H)$.

We now characterize a *GCR* algebra in terms of the Borel structure on the quasi-spectrum.

THEOREM 11. *Let $\mathscr{A}$ be a C^*-algebra. The following are equivalent:*

(1) *$\mathscr{A}$ is a GCR algebra; and*

(2) *every point of the quasi-spectrum $\tilde{\mathscr{A}}$ of $\mathscr{A}$ is a Borel set in the Borel structure induced by the hull-kernel topology.*

Proof. (1) $\Rightarrow$ (2). If $\tau \in \tilde{\mathscr{A}}$, let Q be the unique minimal projection of the center $\mathscr{Z}$ of the enveloping von Neumann $\mathscr{B}$ algebra of $\mathscr{A}$ such that $Q^{\sim}(\tau) = 1$. By Theorem 2, the projection Q is in the Boolean algebra generated by the open central projections $\mathscr{P}$ of $\mathscr{B}$. By Proposition 3 we conclude that the characteristic function of the set

$\{\tau\}$ is in the algebra of bounded Borel function on $\hat{\mathscr{A}}$. Hence, the set $\{\tau\}$ is a Borel set of $\hat{\mathscr{A}}$.

$(2) \Rightarrow (1)$. Let Q be an arbitrary minimal projection in $\mathscr{Z}$. The image of Q under the map λ defined in Proposition 3 is the characteristic function of a point set in $\hat{\mathscr{A}}$. If P_m is the least upper bound of the minimal projection of $\mathscr{Z}$, then $Q \in \langle\langle \mathscr{P} \rangle\rangle P_m$ (Proposition 3). By Lemma 1 we have that $\mathscr{B}Q$ is type I. Because Q is arbitrary, the algebra $\mathscr{A}$ must be GCR [24].

Added May 1, 1973. For separable C^*-algebra $\mathscr{A}$, I have proved that the quotient Borel structure on $\hat{\mathscr{A}}$ induced by the map $f \rightarrow [\rho_f]$ of the factor states of $\mathscr{A}$ with the relativized w^*-topology into $\hat{\mathscr{A}}$ is the Mackey Borel structure of $\hat{\mathscr{A}}$.

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